# CR06 - Final exam

### Exercise 1: **

Let $A = \{\langle a, b\rangle : W_a \subseteq W_b\}$. What is the complexity of $A$ in the arithmetical hierarchy? Show that it is complete for its complexity.

### Solution 1:

$$A = \{\langle a, b\rangle : \forall n (n \notin W_a \vee n \in W_b)\}$$

Since $n \notin W_a$ is $\Pi^0_1$ and $n \in W_b$ is $\Sigma^0_1$, the overall formula is $\Pi^0_2$. Let us show that $A$ is $\Pi^0_2$-complete. Let $a$ be such that $W_a = \mathbb{N}$. Then $Tot = \{b : \langle a, b\rangle \in A\}$, so $Tot$ is many-one reducible to $A$. Since $Tot$ is $\Pi^0_2$-complete, then $A$ is $\Pi^0_2$-complete.

### Exercise 2: **

Let $A$ be a non-c.e. set. Use the finite extension method to prove that there exists a set $B$ which is hyperimmune relative to $A$, and such that $A$ is not $B$-c.e.

### Solution 2:

We want to satisfy the following requirements for every $e \in \mathbb{N}$:

- $\mathcal{R}_e$: $\exists x \Phi^A_e(x) \uparrow \vee \exists x \Phi^A_e(x) < p_B(x)$.
- $\mathcal{S}_e$: $W^B_e \neq A$.

If all the $\mathcal{R}$-requirements are satisfied, then $B$ is $A$-hyperimmune. If all the $\mathcal{S}$-requirements are satisfied, then $A$ is not $B$-c.e.

*Satisfying* $\mathcal{R}_e$. Assume $\sigma_n$ is defined. If $\Phi^A_e$ is partial, then simply take $\sigma_{n+1} = \sigma_n$. Otherwise, let $\sigma_{n+1} = \sigma_n 0000 \dots 0001$ where we add $\Phi^A_e(|\sigma_n|) + 1$ 0's in the string. This way, $p_B(|\sigma_n|) > \Phi^A_e(|\sigma_n|)$.

*Satisfying* $\mathcal{S}_e$. Assume $\sigma_n$ is defined. We have three cases:

- Case 1: there is some $k \notin A$ and some $\tau \succeq \sigma_n$ such that $k \in W^\tau_e$. Then let $\sigma_{n+1} = \tau$.
- Case 2: there is some $k \in A$ such that for every $\tau \succeq \sigma_n$, $k \notin W^\tau_e$. Then let $\sigma_{n+1} = \sigma_n$.
- Case 3: none of the cases above hold. Then $A = \{k : \exists \tau \succeq \sigma_n k \in W^\tau_e\}$ so $A$ is c.e, contradiction.

### Exercise 3: ***

Let $A$ be a non-c.e. set. Prove that every non-empty $\Pi^0_1$ class $\mathcal{C}$ has a member $X$ such that $A$ is not $X$-c.e.

### Solution 3:

We build an infinite decreasing sequence of non-empty $\Pi^0_1$ classes

$$\mathcal{C} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots$$

such that for every $e$, and every $X \in \mathcal{C}_{e+1}$, $W^X_e \neq A$. If we succeed, then for every $X \in \bigcap_n \mathcal{C}_n$, $A$ is not $X$-c.e. Since an infinite decreasing sequence of compacts is compact, $\bigcap_n \mathcal{C}_n \neq \emptyset$, so pick any $X \in \mathcal{C}_{e+1}$, $W^X_e \neq A$.

Construction: Assume $\mathcal{C}_e$ is defined. We have three cases:

- Case 1: there is some $n \notin A$ such that for every $X \in \mathcal{C}_e$, $n \in W_e^X$. Then let $\mathcal{C}_{e+1} = \mathcal{C}_e$.
- Case 2: there is some $n \in A$ and some $X \in \mathcal{C}_e$ such that $n \notin W_e^X$. Then let $\mathcal{C}_{e+1} = \{X \in \mathcal{C}_e : n \notin W_e^X\}$
- Case 3: none of the cases above hold. Then $A = \{n : \forall X \in \mathcal{C}_e n \in W_e^X\}$ so $A$ is c.e, contradiction.

## Cohesive sets

Let $\vec{R} = R_0, R_1, R_2, \dots$ be an infinite sequence of sets of integers. An infinite set $C \subseteq \mathbb{N}$ is *$\vec{R}$-cohesive* (or cohesive for $\vec{R}$) if for every $i \in \mathbb{N}$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$. Here, $X \subseteq^* Y$ is a notation to say that $X$ is included in $Y$ *up to finitely many elements*, that is, $|X \setminus Y| < \infty$.

### Exercise 4: *

Show that for every countable sequence of sets $\vec{R} = R_0, R_1, R_2, \dots$, there exists an infinite $\vec{R}$-cohesive set.

### Solution 4:

We build an infinite sequence of integers $a_0 < a_1 < \dots$ together with an infinite decreasing sequence of infinite sets $\mathbb{N} = A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ as follows: First, $a_0 = 0$ and $\mathbb{N} = A_0$. Assuming $a_n$ and $A_n$ are defined. Pick $a_{n+1} \in A_n$ such that $a_{n+1} > a_n$. Such element exists since $A_n$ is infinite. Then, if $A_n \cap R_n$ is infinite, let $A_{n+1} = A_n \cap R_n$. Otherwise, let $A_{n+1} = A_n \cap \overline{R}_n$.

Let $C = \{a_0 < a_1 < \dots\}$. We claim that $C$ is $\vec{R}$-cohesive. Indeed, given $n$,

$$\{a_{n+1}, a_{n+2}, \dots\} \subseteq A_{n+1}$$

and either $A_{n+1} \subseteq R_n$ or $A_{n+1} \subseteq \overline{R}_n$, so $C \subseteq^* R_n$ or $C \subseteq^* \overline{R}_n$.

### Exercise 5: *

Let $\vec{R} = R_0, R_1, \dots$ be the sequence of all computable sets. Show that no infinite $\vec{R}$-cohesive set is computable.

### Solution 5:

Suppose for the contradiction that $C = \{a_0 < a_1 < \dots\}$ is a computable $\vec{R}$-cohesive set. Then $D = \{a_{2n} : n \in \mathbb{N}\}$ is also computable. In particular, $C \cap D$ and $C \cap \overline{D}$ are both infinite. But since $C$ is cohesive for all the computable sets, $C \subseteq^* D$ or $C \subseteq^* \overline{D}$, contradiction.

### Exercise 6: **

Let $\vec{R} = R_0, R_1, \dots$ be the sequence of all computable sets. Show that every infinite $\vec{R}$-cohesive set is hyperimmune.

## Solution 6:

There are multiple ways to prove this, using either the definition of hyperimmunity in terms for c.e. arrays or in terms of functions. Deeply, they are the same. Let $C$ be an infinite $\vec{R}$-cohesive set.

Let $\{F_{f(n)} : n \in \mathbb{N}\}$ be a c.e. array. Let us show that $C \cap F_{f(n)} = \emptyset$ for some $n$.

We can define a computable function $g : \mathbb{N} \to \mathbb{N}$ such that (1) for every $n$, there is some $k$ such that $g(n) = f(k)$, and (2) for every $n$, $\max F_{g(n)} < \min F_{g(n+1)}$. In other words, $\{F_{g(n)} : n \in \mathbb{N}\}$ is a c.e. sub-array of $\{F_{f(n)} : n \in \mathbb{N}\}$ which is strictly increasing in a strong sense. Then the set $R = \bigcup_n F_{g(2n)}$ is computable. Since $C$ is $\vec{R}$-cohesive, $C \subseteq^* R$ or $C \subseteq^* \overline{R}$.

If $C \subseteq^* R$, then for any $n$ sufficiently large, $C \cap F_{g(2n+1)} = \emptyset$. If $C \subseteq^* \overline{R}$, then for any $n$ sufficiently large, $C \cap F_{g(2n)} = \emptyset$. In both cases, there is some $n$ such that $C \cap F_{f(n)} = \emptyset$.

Recall that a sequence of sets $R_0, R_1, \dots$ is *uniformly computable* if there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $i$, $\Phi_{f(i)} = R_i$. In other words, a sequence of sets is uniformly computable if there is a total computable function $\Phi_e$ such that for every $i, x \in \mathbb{N}$, $\Phi_e(\langle i, x \rangle) = 1$ iff $x \in R_i$.

From now on, unless specified, we fix an infinite sequence of uniformly computable sets $\vec{R} = R_0, R_1, R_2, \dots$. Given a string $\sigma \in 2^{<\mathbb{N}}$, we write

$$\vec{R}_\sigma = \bigcap_{\sigma(i)=0} \overline{R}_i \bigcap_{\sigma(i)=1} R_i \quad \text{and} \quad T_{\vec{R}} = \{\sigma \in 2^{<\mathbb{N}} : \vec{R}_\sigma \text{ is infinite } \}$$

## Exercise 7: **

Show that $T_{\vec{R}}$ is an infinite tree and that any path through $T_{\vec{R}}$ computes an $\vec{R}$-cohesive infinite set.

## Solution 7:

This is an effective variation of the construction of Exercise 4. Let $P$ be an infinite path through $T_{\vec{R}}$.

We build an infinite $P$-sequence of integers $a_0 < a_1 < \dots$ as follows: First, $a_0 = 0$. Assuming $a_n$ is defined, pick $a_{n+1} \in \vec{R}_\sigma$ larger than $a_n$, where $\sigma$ is the initial segment of $P$ of length $n+1$. Such an element exists since $\vec{R}_\sigma$ is infinite, as $P$ is a path through $T_{\vec{R}}$.

Let $C = \{a_0 < a_1 < \dots\}$. We claim that $C$ is $\vec{R}$-cohesive. Indeed, given $n$,

$$\{a_{n+1}, a_{n+2}, \dots\} \subseteq \vec{R}_\sigma$$

, where $\sigma$ is the initial segment of $P$ of length $n+1$. By definition of $\vec{R}_\sigma$, either $\sigma(n) = 0$ and $\vec{R}_\sigma \subseteq \overline{R}_n$, or $\sigma(n) = 1$ and $\vec{R}_\sigma \subseteq R_n$. In both cases $C \subseteq^* R_n$ or $C \subseteq^* \overline{R}_n$.

## Exercise 8: *

Show that any PA degree relative to $\emptyset'$ computes an infinite $\vec{R}$-cohesive set.

## Solution 8:

Let us show that $T_{\vec{R}}$ is $\emptyset'$-co-c.e. Indeed, the predicate "$\vec{R}_\sigma$ is infinite" is uniformly $\Pi^0_2$, so $T_{\vec{R}}$ is $\Pi^0_2$, hence $\emptyset'$-co-c.e.

We have seen that for every co-c.e. tree $T$, there is a computable tree $S$ such that $[S] = [T]$. By relativization, there is a $\emptyset'$-computable tree $S \subseteq 2^{<\mathbb{N}}$ such that $[S] = [T_{\vec{R}}]$.

Let $P$ be a PA degree relative to $\emptyset'$. In particular, $P$ computes an infinite path through $S$, hence through $T_{\vec{R}}$, so by the previous question, $P$ computes an infinite $\vec{R}$-cohesive set.

### Exercise 9: ***

Show that a set $X$ computes an infinite $\vec{R}$-cohesive set iff $X'$ computes a path through $T_{\vec{R}}$.

### Solution 9:

Suppose $X$ computes an infinite $\vec{R}$-cohesive set $C$. Let $T = \{\sigma \in 2^{<\mathbb{N}} : C \subseteq^* \vec{R}_\sigma\}$. Note that for every $n$, there is exactly one $\sigma \in T$ of length $n$. Moreover, $T$ is closed by prefix, and $T \subseteq T_{\vec{R}}$, so $T$ is the set of initial segments of a path $P$ through $T_{\vec{R}}$. Last, $T$ is $C'$-computable, since for every $n$, it suffices to search for some $\sigma$ of length $n$ and some $k \in \mathbb{N}$ such that $\forall x \in C\ (x > k \to x \in \vec{R}_\sigma)$. Then $P$ is $C'$-computable.

Suppose now $X'$ computes a path $P$ through $T_{\vec{R}}$. By Shoenfield's limit lemma, there is a $\Delta^0_2(X)$ approximation of $P$, that is, a uniformly $X$-computable sequence of sets $P_0, P_1, \dots$ such that for every $x$, $\lim_y P_y(x) = P(x)$. We construct a sequence of integers $a_0 < a_1 < \dots$ as follows. First, $a_0 = 0$. Assuming $a_n$ is defined, search for some $t > n$ and some $a_{n+1} \in \vec{R}_\sigma$ greater than $a_n$ such that $\sigma = P_t \restriction_{n+1}$. Such element must be found, since for any sufficiently large $t$, $P_t \restriction_{n+1} = P \restriction_{n+1}$, and then $\vec{R}_\sigma$ is infinite. Let $C = \{a_0, a_1, \dots\}$.

We claim that $C \subseteq^* \vec{R}_\tau$ for every $\tau \prec P$. Fix $\tau \prec P$ of length $k$. Let $t_0$ be such that for every $t > t_0$, $P_t \restriction_k = P \restriction_k$. Then for every $n > t_0$, $a_{n+1} \in \vec{R}_\sigma$ for some $\sigma$ and $t > n$ such that $\sigma = P_t \restriction_{n+1}$. Since $t > n > t_0$, then $P_t \restriction_k = P \restriction_k = \tau$, so $\tau \prec \sigma$. It follows that $a_{n+1} \in \vec{R}_\sigma \subseteq \vec{R}_\tau$.

### Exercise 10: *

Show that if there is no computable $\vec{R}$-cohesive set, then there is no low $\vec{R}$-cohesive set. Hint: use the previous questions.

### Solution 10:

Suppose $C$ is a low $\vec{R}$-cohesive set. Then by Exercise 9, $C'$ computes a path through $T_{\vec{R}}$. Since $C$ is low, then $C' \equiv_T \emptyset'$, so $T_{\vec{R}}$ has a $\emptyset'$-computable path, so still by Exercise 9, there is a computable infinite $\vec{R}$-cohesive set.

### Exercise 11: **

Show that there is a uniformly computable sequence of sets $\vec{R} = R_0, R_1, R_2, \dots$ such that $[T_{\vec{R}}]$ contains only $\text{DNC}_2$ functions relative to $\emptyset'$.

### Solution 11:

For every $n$, and $x$, let $R_n(x) = 1 - i$ if $\Phi_n^{\emptyset'_x}(n)[x] \downarrow = i$, and $R_n(x)$ is the parity of $x$ otherwise. Here, $\emptyset'_x$ is the approximation of the halting set at time $x$.

We claim that every $f \in [T_{\vec{R}}]$ is $\text{DNC}_2$. Indeed, suppose that $\Phi_n^{\emptyset'}(n) \downarrow = i$ for some $i < 2$. Let $s_0$ be the length of the oracle used for this computation. Let $s_1$ be sufficiently large such that $\emptyset'_{s_1} \restriction_{s_0} = \emptyset' \restriction_{s_0}$, and let $s_2 > s_1$ be larger than the time of computation of $\Phi_n^{\emptyset'}(n) \downarrow$. Then for every $x > s_2$, $\Phi_n^{\emptyset'_x}(n)[x] \downarrow = \Phi_n^{\emptyset'_n}(n) = i$.

Thus, for every string $\sigma$ of length larger than $n$, if $R_\sigma$ is infinite, then $\sigma(n) = 1 - i$. It follows that for every path $f \in [T_{\vec{R}}]$, $f(n) = 1 - i$, so $f(n) \neq \Phi_n^{\emptyset'}(n)$.

## Exercise 12: *

Show that there exists a uniformly computable sequence of sets $\vec{R} = R_0, R_1, R_2, \ldots$ such that computing an infinite $\vec{R}$-cohesive set is maximally difficult, in the sense that for every uniformly computable sequence of sets $\vec{S} = S_0, S_1, S_2, \ldots$, every infinite $\vec{R}$-cohesive set computes an infinite $\vec{S}$-cohesive set. Hint: just combine the previous questions.

## Solution 12:

By Exercise 11, there is a uniformly computable sequence of sets $\vec{R}$ such that $[T_{\vec{R}}]$ contains only $\text{DNC}_2$ functions relative to $\emptyset'$. We claim that computing an infinite $\vec{R}$-cohesive is maximally difficult. Let $\vec{s}$ be a uniformly computable sequence of sets and let $C$ be an $\vec{R}$-cohesive set. By Exercise 9, $C'$ computes a member of $[T_{\vec{R}}]$, hence $C'$ is of PA degree relative to $\emptyset'$. In particular, $C'$ computes a member of $[T_{\vec{S}}]$, so by Exercise 9, $C$ computes an infinite $\vec{S}$-cohesive set.