CR06 - Final exam

Exercise 1: **

Let $A = \{\langle a, b\rangle : W_a \subseteq W_b\}$. What is the complexity of A in the arithmetical hierarchy? Show that it is complete for its complexity.

Exercise 2: **

Let A be a non-c.e. set. Use the finite extension method to prove that there exists a set B which is hyperimmune relative to A, and such that A is not B-c.e.

Exercise 3: ***

Let A be a non-c.e. set. Prove that every non-empty Π^0_1 class $\mathcal{C}$ has a member X such that A is not X-c.e.

Cohesive sets

Let $\vec{R} = R_0, R_1, R_2, \dots$ be an infinite sequence of sets of integers. An infinite set $C \subseteq \mathbb{N}$ is *$\vec{R}$-cohesive* (or cohesive for $\vec{R}$) if for every $i \in \mathbb{N}$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$. Here, $X \subseteq^* Y$ is a notation to say that X is included in Y *up to finitely many elements*, that is, $|X \setminus Y| < \infty$.

Exercise 4: *

Show that for every countable sequence of sets $\vec{R} = R_0, R_1, R_2, \dots$, there exists an infinite $\vec{R}$-cohesive set.

Exercise 5: *

Let $\vec{R} = R_0, R_1, \dots$ be the sequence of all computable sets. Show that no infinite $\vec{R}$-cohesive set is computable.

Exercise 6: **

Let $\vec{R} = R_0, R_1, \dots$ be the sequence of all computable sets. Show that every infinite $\vec{R}$-cohesive set is hyperimmune.

Recall that a sequence of sets $R_0, R_1, \dots$ is *uniformly computable* if there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every i, $\Phi_{f(i)} = R_i$. In other words, a sequence of sets is uniformly computable if there is a total computable function Φ_e such that for every $i, x \in \mathbb{N}$, $\Phi_e(\langle i, x\rangle) = 1$ iff $x \in R_i$.

From now on, unless specified, we fix an infinite sequence of uniformly computable sets $\vec{R} = R_0, R_1, R_2, \dots$. Given a string $\sigma \in 2^{<\mathbb{N}}$, we write

$$\vec{R}_\sigma = \bigcap_{\sigma(i)=0} \overline{R}_i \bigcap_{\sigma(i)=1} R_i \quad \text{and} \quad T_{\vec{R}} = \{\sigma \in 2^{<\mathbb{N}} : \vec{R}_\sigma \text{ is infinite }\}$$

Exercise 7: **

Show that $T_{\vec{R}}$ is an infinite tree and that any path through $T_{\vec{R}}$ computes an $\vec{R}$-cohesive infinite set.

Exercise 8: *

Show that any PA degree relative to $\emptyset'$ computes an infinite $\vec{R}$-cohesive set.

Exercise 9: ***

Show that a set X computes an infinite $\vec{R}$-cohesive set iff X' computes a path through $T_{\vec{R}}$.

Exercise 10: *

Show that if there is no computable $\vec{R}$-cohesive set, then there is no low $\vec{R}$-cohesive set. Hint: use the previous questions.

Exercise 11: **

Show that there is a uniformly computable sequence of sets $\vec{R} = R_0, R_1, R_2, \ldots$ such that $[T_{\vec{R}}]$ contains only DNC_2 functions relative to $\emptyset'$.

Exercise 12: *

Show that there exists a uniformly computable sequence of sets $\vec{R} = R_0, R_1, R_2, \ldots$ such that computing an infinite $\vec{R}$-cohesive set is maximally difficult, in the sense that for every uniformly computable sequence of sets $\vec{S} = S_0, S_1, S_2, \ldots$, every infinite $\vec{R}$-cohesive set computes an infinite $\vec{S}$-cohesive set. Hint: just combine the previous questions.