CR11 - Graded exercise sheet - Week 1

A *binary string* is a finite sequence $b_0b_1 \dots b_{k-1}$ of bits $b_i \in \{0,1\}$. The *concatenation* of two strings σ and τ is denoted by $\sigma\tau$. One can code binary strings into integers by multiple ways. For example, the string $b_0b_1 \dots b_{k-1}$ can be coded by $p_0^{b_0}p_1^{b_1} \dots p_{k-1}^{b_{k-1}}$, where p_i is the ith prime number. The actual representation of strings into integers does not matter, as long as the coding is computable. One can then define a set of strings to be computable if the corresponding set of integers is computable.

Exercise 1.1: /4

Given two sets of strings A and B, the *concatenation* of A and B is the set $AB = \{\sigma\tau : \sigma \in A \text{ and } \tau \in B\}$. Are the co-c.e. sets of strings closed under concatenation?

Solution 1.1:

By definition of AB, a string ρ is in AB iff there is a pair of strings σ, τ such that $\sigma \in A$, $\tau \in B$ and $\rho = \sigma\tau$. By the contrapositive, we can remove ρ from AB iff at some point, for every pair of strings σ, τ such that $\sigma\tau = \rho$, either σ has already left from A, or τ has left from B. Thus there is a computable procedure for enumerating the elements outside of AB, so AB is co-c.e.

A partial computable function Φ_e is *strictly partial* if it is not total. In other words, it is strictly partial if $\Phi_e(x)\uparrow$ for some $x \in \mathbb{N}$.

Exercise 1.2: /6

Show that the set $\mathsf{Part} = \{e : \Phi_e \text{ is strictly partial }\}$ is $\emptyset'$-c.e.
Hint: consider the set $A = \{(e,x) : \forall y \Phi_e(x)[y]\uparrow\}$.

Solution 1.2:

$$\mathsf{Part} = \{e : \exists x \forall y \Phi_e(x)[y]\uparrow\}$$

Consider the following set $A = \{(e,x) : \forall y \Phi_e(x)[y]\uparrow\}$. Let us show that $A \leq_T \emptyset'$.

Consider the total computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every e and x, $g(e,x)$ is the code of the program which on any input, searches for some y such that $\Phi_e(x)[y]\downarrow$. If found, then the program halts, otherwise, it loops forever. Then $(e,x) \in A$ iff $\forall y \Phi_e(x)[y]\uparrow$ iff $\Phi_{g(e,x)}(g(e,x))\uparrow$ iff $g(e,x) \notin \emptyset'$.

Since A is $\emptyset'$-computable, then it is $\emptyset'$-c.e., so by closure of the $\emptyset'$-c.e. sets under projection, Part is $\emptyset'$-c.e.

Recall that a set A is $\emptyset'$-c.e. if it is the domain of a partial $\emptyset'$-computable function.

Exercise 1.3: /10

Show that a set $A \subseteq \mathbb{N}$ is $\emptyset'$-c.e. iff there is a total computable function $f : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ such that for every x, $x \in A$ iff $\lim_y f(x,y) = 1$. Note that if $x \notin A$, then the limit $\lim_y f(x,y)$ does not necessarily exist.

Solution 1.3:

- Suppose first that $A \subseteq \mathbb{N}$ is $\emptyset'$-c.e. Let e be such that $A = \mathrm{dom}\,\Phi_e^{\emptyset'}$. Let $\emptyset'[0], \emptyset'[1], ...$ be a c.e. approximation of $\emptyset'$. Define $f(x,y) = 1$ iff $\Phi_e^{\emptyset'[y]}(x)[y]\downarrow$ and $\Phi_e^{\emptyset'[y+1]}(x)[y+1]\downarrow$ with the same use.

  - if $x \in A$, then $\Phi_e^{\emptyset'}(x)\downarrow$. Let σ be the use of this computation, and let t be such that σ is a prefix of $\emptyset'[t]$. Then for every $y > t$, $f(x,y) = 1$, hence $\lim_y f(x,y) = 1$.
  - if $\lim_y f(x,y) = 1$, there there is some threshold t such that for every $y > t$, $\Phi_e^{\emptyset'[y]}(x)[y]\downarrow$ and $\Phi_e^{\emptyset'[y+1]}(x)[y+1]\downarrow$ with the same use. It follows that the use of the computation is a true initial segment of $\emptyset'$, and therefore that $\Phi_e^{\emptyset'}(x)\downarrow$, so $x \in A$.

- Suppose now that there is a total computable function $f : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ such that for every x, $x \in A$ iff $\lim_y f(x,y) = 1$. Then $A = \{x : \exists t \forall y > t f(x,y) = 1\}$. Consider the set $B = \{(x,t) : \forall y > t f(x,y) = 1\}$. Let us show that $B \leq_T \emptyset'$.

  Consider the total computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every e and x, $g(e,x)$ is the code of the program which on any input, searches for some $y > t$ such that $f(x,y) \neq 1$. If found, then the program halts, otherwise, it loops forever. Then $(e,x) \in B$ iff $\forall y > t f(x,y) = 1$ iff $\Phi_{g(e,x)}(g(e,x))\uparrow$ iff $g(e,x) \notin \emptyset'$.

  Since B is $\emptyset'$-computable, then it is $\emptyset'$-c.e., so by closure of the $\emptyset'$-c.e. sets under projection, A is $\emptyset'$-c.e.