CR11 - Graded exercise sheet - Week 2

Exercise 2.1: /10

Let A be a non-c.e. set. Use the finite extension method to prove that there exists a set B which is not A-c.e. and such that A is not B-c.e.

Solution 2.1:

Recall that $W_e^X = \operatorname{dom} \Phi_e^X$. We want to satisfy the following requirements for every $e \in \mathbb{N}$:

- $\mathcal{R}_e$: $W_e^A \neq B$.
- $\mathcal{S}_e$: $W_e^B \neq A$.

If all the $\mathcal{R}$-requirements are satisfied, then B is not A-c.e. If all the $\mathcal{S}$-requirements are satisfied, then A is not B-c.e.

Satisfying $\mathcal{R}_e$. Assume σ_n is defined. Let $x = |\sigma_n|$. If $\Phi_e^A(x) \uparrow$, then take $\sigma_{n+1} = \sigma_n 1$. Otherwise, let $\sigma_{n+1} = \sigma_n 0$.

Satisfying $\mathcal{S}_e$. Assume σ_n is defined. We have three cases:

- Case 1: there is some $k \notin A$ and some $\tau \succeq \sigma_n$ such that $k \in W_e^\tau$. Then let $\sigma_{n+1} = \tau$.
- Case 2: there is some $k \in A$ such that for every $\tau \succeq \sigma_n$, $k \notin W_e^\tau$. Then let $\sigma_{n+1} = \sigma_n$.
- Case 3: none of the cases above hold. Then $A = \{k : \exists \tau \succeq \sigma_n k \in W_e^\tau\}$ so A is c.e, contradiction.

Recall that $W_e = \operatorname{dom} \Phi_e$.

Exercise 2.2: /10

Show that $\texttt{Tot} = \{e : W_e = \mathbb{N}\}$ is Π_2^0-complete.

Solution 2.2:

First, let us show that $\texttt{Tot}$ is Π_2^0.

$$\texttt{Tot} = \{e : \forall x \exists t \Phi_e(x)[t] \downarrow\}$$

The predicate $P(e, x, t) \equiv \Phi_e(x)[t] \downarrow$ being computable, $\texttt{Tot}$ is Π_2^0.

Let $A = \{x : \forall y \exists z R(x, y, z)\}$ be a Π_2^0 set. Let us show that $A \leq_m \texttt{Tot}$. Let $f : \mathbb{N} \to \mathbb{N}$ be the computable function which to every n, associates the code $f(n)$ of the function $\Phi_{f(n)}$ defined as follows: On input y, $\Phi_{f(n)}$ searches for some z such that $R(n, y, z)$ holds. If found, $\Phi_{f(n)}(y) \downarrow$, otherwise $\Phi_{f(n)}(y) \uparrow$.

- If $n \in A$, then for every y, there is some z such that $R(n, y, z)$ holds, so $\Phi_{f(n)}$ is total, so $f(n) \in \texttt{Tot}$.
- If $n \notin A$, then there is some y such that for every z, $R(n, y, z)$ does not hold, so $\Phi_{f(n)}(y) \uparrow$, hence $f(n) \notin \texttt{Tot}$.