# CR11 - Graded exercise sheet - Week 3

### Exercise 3.1: /10

Build a *c.e.* infinite tree $T \subseteq 2^{<\mathbb{N}}$ such that every path computes $\emptyset'$.

### Solution 3.1:

Consider the following c.e. set :

$$T = \{\sigma \in 2^{<\mathbb{N}} : \exists k > |\sigma| \forall e < |\sigma| \Phi_e(e)[k] \downarrow \leftrightarrow \sigma(e) = 1\}$$

Let us show that $T$ is a tree, that is, $T$ is closed by prefix. Let $\sigma \in T$ and $\tau \prec \sigma$. Let $k$ witness that $\sigma \in T$. Then $k$ witness that $\tau \in T$.

Let us now show that $\emptyset'$ is the unique path of $T$. Therefore every path of $T$ computes $\emptyset'$. Let $P \in [T]$. Let $e \in \mathbb{N}$. Suppose first $\Phi_e(e) \uparrow$. Let $\sigma \prec P$ be longer than $e$. Then for every $k > |\sigma|$, $\Phi_e(e)[k] \uparrow$, so $\sigma(e) = 0$, hence $P(e) = 0$. Suppose now $\Phi_e(e) \downarrow$. Let $\sigma \prec P$ be such that $\Phi_e(e)[|\sigma|] \downarrow$. Then for every $k > |\sigma$, $\Phi_e(e)[k] \downarrow$, so $\sigma(e) = 1$, so $P(e) = 1$. Therefore $P(e) = 1$ iff $\Phi_e(e) \downarrow$. Therefore there is at most one path through $T$. One can easily see that $\emptyset'$ is a path through $T$.

A path $P$ in a tree $T \subseteq 2^{<\mathbb{N}}$ is *isolated* if there is some initial segment $\sigma \prec P$ such that $[\sigma] \cap [T] = \{P\}$. In other words, $P$ is isolated if there is some initial segment $\sigma \prec P$ such that for every $\tau \prec P$ with $|\sigma| \leq |\tau|$, exactly one of $\tau 0$ and $\tau 1$ has an infinite subtree below it. A path which is not isolated is a *limit point*.

### Exercise 3.2: /10

Let $T \subseteq 2^{<\mathbb{N}}$ be a computable tree such that $[T]$ has exactly one limit point $P$. Show that $P \leq_T \emptyset''$. Hint: try to define a $\emptyset''$-computable infinite subtree $S \subseteq T$ which removed all the isolated paths of $T$, so that $P$ becomes an isolated path of $S$.

### Solution 3.2:

A string $\sigma$ is *valid* if there is an extension $\tau \succeq \sigma$ such that the subtrees below $\tau 0$ and $\tau 1$ are infinite in $T$. In other words, a string $\sigma$ is valid if $[\sigma] \cap [T]$ contains at least 2 elements. Note that the predicate "the subtree below $\tau i$ is infinite" is $\Pi^0_1$, since it means that for every length $\ell$ greater than $|\tau|$, there is a node $\rho$ of length $\ell$ extending $\tau i$ and in $T$. Thus, being a valid string is $\Sigma^0_2$.

Let $S$ be the set of valid strings. Note that if $\sigma$ is valid, its prefixes are valid, and that $\sigma \in T$, so $S$ is a $\Sigma^0_2$ subtree of $T$.

Let us show that $[S]$ contains no isolated point of $[T]$. Indeed, if $X$ is an isolated point of $T$, there is some $\sigma \prec X$ such that $[\sigma] \cap [T]$ is a singleton, hence $\sigma$ is not valid, so $\sigma \notin S$.

Let us now show that $P$, the unique limit point of $T$, is in $S$. Indeed, for every $\sigma \prec P$, since $P$ is not an isolated point, $[\sigma] \cap [T]$ is not a singleton. Since $P \in [\sigma] \cap [T]$, then $[\sigma] \cap [T]$ must contain at least 2 elements, so $\sigma$ is valid, hence $\sigma \in S$. So any initial segment of $P$ is in $S$, so $P \in [S]$.

Since $[S] \subseteq [T]$ and any element of $T$ is either a limit point, or an isolated point, $[S] = \{P\}$. It follows that $P$ is an isolated path of $S$. By relativizing to $\emptyset''$, the proposition that for every computable tree, every isolated path is computable, $P$ is $\emptyset''$-computable.