CR11 - Graded exercise sheet - Week 4

A function $f : \mathbb{N} \to \{0,1\}$ is DNC_2 if for every $e \in \mathbb{N}$, $f(e) \neq \Phi_e(e)$. A function $f : \mathbb{N} \to \{0,1\}$ is $\text{DNC}_2(X)$ if for every $e \in \mathbb{N}$, $f(e) \neq \Phi_e^X(e)$.

Exercise 4.1: /5

Show that if f is $\text{DNC}_2(X)$, then $f \geq_T X$.

Solution 4.1:

To decide whether $n \in X$, construct an oracle machine Φ_e^X such that on every input x, $\Phi_e^X(e) \downarrow = 1 - X(n)$. Then $f(e) \neq \Phi_e^X(e)$, so $f(e) = X(n)$. Note that the procedure of transformation of n into e does not require to access the oracle X, since this is purely syntactical manipulation. Thus, only f was used as an oracle, so $X \leq_T f$.

Exercise 4.2: /8

Show that if X computes a DNC_2 function, then X computes a DNC_2 function f such that $f \equiv_T X$.

Solution 4.2:

Thanks to the padding lemma, we can computably find an infinite set $A = \{e_0 < e_1 < \dots\}$ of codes for the nowhere-defined machine. Beware, this does not mean that A contains *all* the codes of this machine, but we just need infinitely many of them. By definition of A, this means that a DNC_2 function can have any value over the domain A. Define $g(x) = f(x)$ if $x \notin A$, and $g(e_n) = X(n)$ otherwise. This is an X-computable procedure since $f \leq_T X$. Let us show that g is DNC_2 : If $x \notin A$, then $g(x) = f(x) =\neq \Phi_x(x)$, and if $x \in A$, $\Phi_x(x) \uparrow$, then $g(x) \neq \Phi_x(x)$. Last, $X \leq_T g$ since $X = \{n : g(e_n) = 1\}$.

A Turing degree $\mathbf{d}$ is *minimal* if the only degree strictly below it is $\mathbf{0}$, the degree of computable sets.

Exercise 4.3: /7

Show that $\{X \oplus Y : X \text{ is } \text{DNC}_2 \text{ and } Y \text{ is } \text{DNC}_2(X)\}$ is a Π^0_1 class. Deduce that there is no minimal PA degree.

Solution 4.3:

Let $\mathcal{C} = \{Z : \forall n R(Z \upharpoonright_n)\}$ where R is the following computable predicate :

$$R(\sigma \oplus \tau) \equiv \forall e < |\sigma| \Phi_e(e)[|\sigma|] \not\simeq \sigma(e) \wedge \forall e < |\tau| \Phi_e^\sigma(e)[|\tau|] \not\simeq \tau(e)$$

Here, $\Phi_e(e)[s] \not\simeq v$ is a notation for $\Phi_e(e)[s] \uparrow \vee \Phi_e(e)[s] \downarrow \neq v$.

We need to show that $X \oplus Y \in \mathcal{C}$ iff X is DNC_2 and Y is $\text{DNC}_2(X)$. The verification is left to the reader.

Let P be any set of PA degree. In particular, P computes a member of $\mathcal{C}$, so P computes a set $X \oplus Y$ such that X is DNC_2 and Y is $\text{DNC}_2(X)$. In other words, Y is of PA degree relative to X, and X is of PA degree. In particular, X is non-computable, and P is non-X-computable, so the degree of X is strictly between the degree of P and $\mathbf{0}$.