CR11 - Graded exercise sheet - Week 5

Let $\vec{R} = R_0, R_1, R_2, \dots$ be an infinite sequence of sets of integers. An infinite set $C \subseteq \mathbb{N}$ is *$\vec{R}$-cohesive* (or cohesive for $\vec{R}$) if for every $i \in \mathbb{N}$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$. Here, $X \subseteq^* Y$ is a notation to say that X is included in Y *up to finitely many elements*, that is, $|X \setminus Y| < \infty$.

Exercise 5.1: 6

Let $\vec{R} = R_0, R_1, \dots$ be the sequence of all computable sets. Show that every infinite $\vec{R}$-cohesive set is hyperimmune.

Solution 5.1:

There are multiple ways to prove this, using either the definition of hyperimmunity in terms for c.e. arrays or in terms of functions. Deeply, they are the same. Let C be an infinite $\vec{R}$-cohesive set.

Let $\{F_{f(n)} : n \in \mathbb{N}\}$ be a c.e. array. Let us show that $C \cap F_{f(n)} = \emptyset$ for some n.

We can define a computable function $g : \mathbb{N} \to \mathbb{N}$ such that (1) for every n, there is some k such that $g(n) = f(k)$, and (2) for every n, $\max F_{g(n)} < \min F_{g(n+1)}$. In other words, $\{F_{g(n)} : n \in \mathbb{N}\}$ is a c.e. sub-array of $\{F_{f(n)} : n \in \mathbb{N}\}$ which is strictly increasing in a strong sense. Then the set $R = \bigcup_n F_{g(2n)}$ is computable. Since C is $\vec{R}$-cohesive, $C \subseteq^* R$ or $C \subseteq^* \overline{R}$.

If $C \subseteq^* R$, then for any n sufficiently large, $C \cap F_{g(2n+1)} = \emptyset$. If $C \subseteq^* \overline{R}$, then for any n sufficiently large, $C \cap F_{g(2n)} = \emptyset$. In both cases, there is some n such that $C \cap F_{f(n)} = \emptyset$.

From now on, we fix an infinite sequence of uniformly computable sets $\vec{R} = R_0, R_1, R_2, \dots$.

The natural way to construct an $\vec{R}$-cohesive set is to first pick an element x_0, then let $A_0 = R_0$ or $A_0 = \overline{R}_0$, depending on which one is infinite, then pick an element $x_1 > x_0$ in A_0, then let $A_1 = A_0 \cap R_1$ or $A_1 = A_0 \cap \overline{R}_1$ depending on which one is infinite, then pick an element $x_2 > x_1$ in A_1, and so on.

Given a string $\sigma \in 2^{<\mathbb{N}}$, we write

$$\vec{R}_\sigma = \bigcap_{\sigma(i)=0} \overline{R}_i \bigcap_{\sigma(i)=1} R_i \quad \text{and} \quad T_{\vec{R}} = \{\sigma \in 2^{<\mathbb{N}} : \vec{R}_\sigma \text{ is infinite }\}$$

Intuitively, the paths of $T_{\vec{R}}$ are the valid decision sequences to build $\vec{R}$-cohesive sets.

Exercise 5.2: 14

Show that a set X computes an infinite $\vec{R}$-cohesive set iff X' computes a path through $T_{\vec{R}}$.

Hint: For the forward direction, it might easier to show that there is a $\Delta^0_2(X)$ path. For the reversal, by Schoenfield's limit lemma, X' computes a path through $T_{\vec{R}}$ iff X computes a stable function $f : \mathbb{N}^2 \to \{0, 1\}$ whose limit function $\hat{f} : x \mapsto \lim_y f(x, y)$ is a path through $T_{\vec{R}}$.

Solution 5.2:

Suppose X computes an infinite $\vec{R}$-cohesive set C. Let $T = \{\sigma \in 2^{<\mathbb{N}} : C \subseteq^* \vec{R}_\sigma\}$. Note that for every n, there is exactly one $\sigma \in T$ of length n. Moreover, T is closed by prefix, and $T \subseteq T_{\vec{R}}$, so T is the set of initial segments of a path P through $T_{\vec{R}}$. Last, T is C'-computable, since for every n, it suffices to search for some σ of length n and some $k \in \mathbb{N}$ such that $\forall x \in C\ (x > k \to x \in \vec{R}_\sigma)$. Then P is C'-computable.

Suppose now X' computes a path P through $T_{\vec{R}}$. By Shoenfield's limit lemma, there is a $\Delta^0_2(X)$ approximation of P, that is, a uniformly X-computable sequence of sets $P_0, P_1, \dots$ such that for every x, $\lim_y P_y(x) = P(x)$. We construct a sequence of integers $a_0 < a_1 < \dots$ as follows. First, $a_0 = 0$. Assuming a_n is defined, search for some $t > n$ and some $a_{n+1} \in \vec{R}_\sigma$ greater than a_n such that $\sigma = P_t \restriction_{n+1}$. Such element must be found, since for any sufficiently large t, $P_t \restriction_{n+1} = P \restriction_{n+1}$, and then $\vec{R}_\sigma$ is infinite. Let $C = \{a_0, a_1, \dots\}$.

We claim that $C \subseteq^* \vec{R}_\tau$ for every $\tau \prec P$. Fix $\tau \prec P$ of length k. Let t_0 be such that for every $t > t_0$, $P_t \restriction_k = P \restriction_k$. Then for every $n > t_0$, $a_{n+1} \in \vec{R}_\sigma$ for some σ and $t > n$ such that $\sigma = P_t \restriction_{n+1}$. Since $t > n > t_0$, then $P_t \restriction_k = P \restriction_k = \tau$, so $\tau \prec \sigma$. It follows that $a_{n+1} \in \vec{R}_\sigma \subseteq \vec{R}_\tau$.