CR11 - Graded exercise sheet - Week 5

Let $\vec{R} = R_0, R_1, R_2, \dots$ be an infinite sequence of sets of integers. An infinite set $C \subseteq \mathbb{N}$ is *$\vec{R}$-cohesive* (or cohesive for $\vec{R}$) if for every $i \in \mathbb{N}$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$. Here, $X \subseteq^* Y$ is a notation to say that X is included in Y *up to finitely many elements*, that is, $|X \setminus Y| < \infty$.

Exercise 5.1: 6

Let $\vec{R} = R_0, R_1, \dots$ be the sequence of all computable sets. Show that every infinite $\vec{R}$-cohesive set is hyperimmune.

From now on, we fix an infinite sequence of uniformly computable sets $\vec{R} = R_0, R_1, R_2, \dots$.

The natural way to construct an $\vec{R}$-cohesive set is to first pick an element x_0, then let $A_0 = R_0$ or $A_0 = \overline{R}_0$, depending on which one is infinite, then pick an element $x_1 > x_0$ in A_0, then let $A_1 = A_0 \cap R_1$ or $A_1 = A_0 \cap \overline{R}_1$ depending on which one is infinite, then pick an element $x_2 > x_1$ in A_1, and so on.

Given a string $\sigma \in 2^{<\mathbb{N}}$, we write

$$\vec{R}_\sigma = \bigcap_{\sigma(i)=0} \overline{R}_i \bigcap_{\sigma(i)=1} R_i \quad \text{and} \quad T_{\vec{R}} = \{\sigma \in 2^{<\mathbb{N}} : \vec{R}_\sigma \text{ is infinite } \}$$

Intuitively, the paths of $T_{\vec{R}}$ are the valid decision sequences to build $\vec{R}$-cohesive sets.

Exercise 5.2: 14

Show that a set X computes an infinite $\vec{R}$-cohesive set iff X' computes a path through $T_{\vec{R}}$.

Hint: For the forward direction, it might easier to show that there is a $\Delta^0_2(X)$ path. For the reversal, by Schoenfield's limit lemma, X' computes a path through $T_{\vec{R}}$ iff X computes a stable function $f : \mathbb{N}^2 \to \{0,1\}$ whose limit function $\hat{f} : x \mapsto \lim_y f(x,y)$ is a path through $T_{\vec{R}}$.