CR11 - Graded exercise sheet - Week 6

Given a set $X \subseteq \mathbb{N}$, we denote by $[X]^n$ the collection of subsets of X of size n. A function $f : [\mathbb{N}]^n \to \mathbb{N}$ is *k-bounded* if for every $i \in \mathbb{N}$, there are at most k different $F \in [\mathbb{N}]$ such that $f(F) = i$. A finite set $H \subseteq \mathbb{N}$ is a *rainbow* for f if f restricted to $[H]^n$ is 1-bounded, that is, injective. The Rainbow Ramsey theorem for n-tuples and k-bounded colorings says that for every k-bounded function $f : [\mathbb{N}]^n \to \mathbb{N}$, there is an infinite rainbow.

Exercise 6.1: /4

Give an example of a 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ and of a finite rainbow $F \subseteq \mathbb{N}$ which cannot be extended into an infinite rainbow, that is, such that there is not infinite rainbow $H \supseteq F$.

Solution 6.1:

Let $F = \{0, 1\}$, and let $f : [\mathbb{N}]^2 \to \mathbb{N}$ be defined for every $y \geq 2$ by $f(\{0, y\} = f(\{1, y\}) = 2y$, and for every other pair, $f(\{x, y\}) = 2\langle x, y\rangle + 1$. Then F cannot be extended into an infinite rainbow.

Exercise 6.2: /16

Use the previous question to build a computable 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ with no infinite computable rainbow.

(Hint: Build a function f which satisfies for every e the following requirement $\mathcal{R}_e$: "If there are at least $2e+2$ many elements x such that $\Phi_e(x) \downarrow= 1$, then Φ_e cannot be extended into an infinite rainbow.")

Solution 6.2:

We will try to satisfy the following requirements for each e : $\mathcal{R}_e$ "If there are at least $2e + 2$ many elements x such that $\Phi_e(x) \downarrow= 1$, then Φ_e cannot be extended into an infinite rainbow."

We build the coloring by stages. At stage s, we define $f(\{x, s\})$ for every $x < s$. At stage 0, f is nowhere defined. At stage s, for each $e < s$, proceed as follows: If $\Phi_0(x)[s] = 1$ for at least 2 different $x < s$, then pick x_0, x_1 such that $\Phi_0(x_0)[s] = 1$ and $\Phi_0(x_1)[s] = 1$, pick a fresh color $i \in \mathbb{N}$, and set $f(\{x_0, s\} = f(\{x_1, s\})$. If $\Phi_1(x)[s] = 1$ for at least 4 different $x < s$, then pick x_0, x_1 such that $\Phi_1(x_1)[s] = 1$ and $\Phi_1(x_1)[s] = 1$, such that $f(\{x_0, s\}$ and $f(\{x_1, s\})$ is not yet defined. Pick a fresh color $i \in \mathbb{N}$, and set $f(\{x_0, s\} = f(\{x_1, s\})$. And so on, up to Φ_{s-1}. Then, for any pair $f(\{x, s\})$ which is not yet defined, assign $f(\{x, s\})$ a fresh color, and go to the next stage $s + 1$.