CR11 - Graded exercise sheet - Week 6

Given a set $X \subseteq \mathbb{N}$, we denote by $[X]^n$ the collection of subsets of X of size n. A function $f : [\mathbb{N}]^n \to \mathbb{N}$ is *k-bounded* if for every $i \in \mathbb{N}$, there are at most k different $F \in [\mathbb{N}]$ such that $f(F) = i$. A finite set $H \subseteq \mathbb{N}$ is a *rainbow* for f if f restricted to $[H]^n$ is 1-bounded, that is, injective. The Rainbow Ramsey theorem for n-tuples and k-bounded colorings says that for every k-bounded function $f : [\mathbb{N}]^n \to \mathbb{N}$, there is an infinite rainbow.

Exercise 6.1: /4

Give an example of a 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ and of a finite rainbow $F \subseteq \mathbb{N}$ which cannot be extended into an infinite rainbow, that is, such that there is not infinite rainbow $H \supseteq F$.

Exercise 6.2: /16

Use the previous question to build a computable 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ with no infinite computable rainbow.

(Hint: Build a function f which satisfies for every e the following requirement $\mathcal{R}_e$: "If there are at least $2e+2$ many elements x such that $\Phi_e(x) \downarrow= 1$, then Φ_e cannot be extended into an infinite rainbow.")