Conservation theorems | 7

Prerequisites: Chapters 2 to 4

The importance of the combinatorial features of the forcing question extends to the proof-theoretic realm, especially for proving conservation theorems. In this setting, one usually starts with a model of a weak theory, and extends it to satisfy a stronger theory, while preserving some features of the original model. When working with models of weak arithmetic, the stake is to add new sets to the model while preserving induction. We shall see that Σ^0_n-induction can be preserved thanks to the existence of a Σ^0_n-preserving forcing question which is able to find a common extension witnessing a positive and a negative answer simultaneously.

In this chapter, we shall consider conservation theorems over RCA_0, a weak theory capturing computable mathematics. Thanks to the correspondence between computability and definability, we shall benefit from the framework of first-jump control to prove our main conservations theorems. However, the translation of computability-theoretic constructions to proof-theoretic ones requires a careful formalization, as many intuitive features of the integers are not necessarily true in models of weak arithmetic.

7.1 Context and motivation

At the end of the 19th century, the various paradoxes arising in the development of set theory led to a foundational crisis of mathematics. Mathematicians started to question the use of infinity in mathematics, partially due to the lack of ground to reality: with the discovery of the atom, and of the finiteness of the universe, infinity seemed to be a purely intellectual construction in which intuition failed. In the early 1920s, David Hilbert proposed a program as a solution to the foundational crisis, called *finitistic reductionism*. The goal was to show that every finitary statement proven by infinitary means, could also be proven finitarily. Thus, infinity would be a convenience language not affecting the truth value of finitary statements.[1]

1: There is an excellent article from Simpson [38] on the subject, presenting reverse mathematics as a partial realization of Hilbert's program.

Sadly, Gödel's incompleteness theorems showed the unrealizability of Hilbert's program in its full generality, as the consistency of Peano arithmetic is a finitary statement which is not provable by finitary means, but provable in set theory. Reverse mathematics can be considered as a partial realization of Hilbert's program, as it showed that many theorems of ordinary mathematics are provable over WKL_0, which is Π_2-conservative over primitive recursive arithmetic (PRA).[2] PRA is considered as capturing finitary mathematics (see Tait [39]), so any Π_2 theorem of WKL_0 can be proved by finitary means.

2: PRA is a system in the language of functions, capturing primitive recursive functions. Technically, the languages being different, saying that WKL_0 is Π_2-conservative over PRA requires some work in translating sentences from one language to the other. See Simpson [4, p. IX.3] for a formal development of the subject.

More generally, it is of foundational importance to understand the *first-order part* of a second-order theory, that is, the set of its first-order theorems. There exist two main methods to characterize the first-order part of a second-order theory T: either directly identify a first-order theory capturing the first-order part of T, or reduce the theory T to a weaker second-order theory for which the first-order part is already known. We shall mostly adopt the second approach, through Π^1_1-conservation.

Definition 7.1.1. Let T_0, T_1 be two theories of second-order arithmetic. A theory T_1 is Π^1_1-*conservative over* T_0 if every Π^1_1 sentence provable in T_1 is also provable in T_0. ◇

If furthermore T_1 implies T_0, then we say that T_1 is a Π^1_1-*conservative extension* of T_0. Proving that a theory T_1 is a Π^1_1-conservative extension of T_0 is a strong way of proving that T_1 and T_0 have the same first-order part. Indeed, the class of Π^1_1 sentences not only contains all the first-order sentences, but also every arithmetic sentence with second-order parameters.

Recall that a model of second-order arithmetic is of the form $\mathcal{M} = (M, S, +, \times, <, 0, 1)$ where $S \subseteq \mathcal{P}(M)$. A model $\mathcal{M}$ is *topped*[3] by a set $Y \in S$ if every $X \in S$ is $\Delta^0_1(Y)$-definable with parameters in M.[4]

Definition 7.1.2. A model $\mathcal{N} = (N, T, +^{\mathcal{N}}, \times^{\mathcal{N}}, <^{\mathcal{N}}, 0^{\mathcal{N}}, 1^{\mathcal{N}})$ is an ω-*extension*[5] of a model $\mathcal{M} = (M, S, +^{\mathcal{M}}, \times^{\mathcal{M}}, <^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$ if $\mathcal{N}$ and $\mathcal{M}$ differ only by their second-order part and $T \supseteq S$. In other words, $M = N$, and the basic operations coincide. ◇

We shall often omit the signature, and simply write $\mathcal{M} = (M, S)$ when there is no ambiguity. Proofs of Π^1_1-conservation are usually done through ω-extensions of countable models.

Proposition 7.1.3. Let T_0 and T_1 be two theories of second-order arithmetic. Suppose that every countable model $\mathcal{M} \models T_0$ can be ω-extended into a model $\mathcal{N} \models T_1$. Then T_1 is Π^1_1-conservative over T_0. ★

Proof. Let $\varphi \equiv \forall X \theta(X)$ be a Π^1_1 sentence, where θ is an arithmetic formula. Suppose that $T_0 \nvdash \varphi$. Then by Gödel's completeness theorem[6], there is a model of $T_0 \cup \{\neg\varphi\}$. By the downward Löwenheim–Skolem theorem[7], there is a countable such model $\mathcal{M} = (M, S) \models T_0 \cup \{\neg\varphi\}$. Let $X \in S$ be such that $\mathcal{M} \models \neg\theta(X)$. By assumption, there is an ω-extension $\mathcal{N} = (M, S_1) \models T_1$ of $\mathcal{M}$. Since $S_1 \supseteq S$, then $X \in S_1$. Moreover, since $\mathcal{N}$ is an ω-extension of $\mathcal{M}$, then $\mathcal{N} \models \neg\theta(X)$, so $\mathcal{N} \models \neg\varphi$. ■

In this chapter, we shall consider two base theories for T_0: RCA_0 and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. The techniques to prove Π^1_1-conservation over these two theories are pretty different, but both use a formalization of first-jump control.

7.2 Induction and collection

Before turning to the actual proofs of conservation, it is important to get familiar with some fundamental concepts of weak arithmetic. Classical mathematicians being used to work with full induction, it can be challenging to get an intuition on what constructions and theorems of mathematics remain valid over weak arithmetic. See Hájek and Pudlák [41] for a development of the basics of mathematics over increasingly strong axiomatic systems. The base system, RCA_0, is a restriction of the full second-order arithmetic on two axis:

- The *comprehension scheme* is restricted to Δ^0_1 predicates with parameters. By Post's theorem, this restrictions allows only the construction of sets computably from existing sets in the model. In ω-models, this ensures that the second-order part is a Turing ideal. The computability-theorist should already be familiar with this restriction.

3: Topped models should not be confused with top models, although there is a lot of beauty in models of weak arithmetic.

4: One can define a notion of Turing functional in weak models of arithmetic, and therefore define the Turing reduction. However, if the theory is too weak, the Turing reduction is not transitive. In order to have a Turing reduction $Y \leq_T X$ with a good behavior, one needs $(M, \{X\}) \models \mathsf{B}\Sigma^0_1$. See Groszek and Slaman [40].

5: The terminology might be confusing, as being an ω-extension has nothing to do with ω-models.

6: Recall that second-order arithmetic is a two-sorted first-order theory. A *Henkin structure* is a structure of second-order arithmetic in which the ownership relation $\in$ has its standard interpretation. Henkin proved that Gödel's completeness theorem also applies to Henkin tructures, that is, a second-order theory is *consistent* iff it admits a Henkin model.

7: The *downward Löwenheim-Skolem theorem* is a classical theorem from model theory, stating that for every structure $\mathcal{M}$ over a signature σ, and every infinite cardinal κ between $\operatorname{card}\mathcal{M}$ and $\operatorname{card}\sigma$, there is an elementary substructure of $\mathcal{M}$ of cardinal κ. In particular, the language of second-order arithmetic is countable, so consistency of a theory T implies the existence of a countable model of T.

- The *induction scheme* is restricted to Σ^0_1 formulas with parameters. This might be the less intuitive part, both in terms of consequences over the theory, and in terms of design choice. Indeed, why restrict induction to capture computable mathematics?

This section therefore focuses on the second restriction, and gives a brief overview on the impact of induction over the models of weak arithmetic. One can define a hierarchy of systems based on the complexity of formulas satisfying induction.

Definition 7.2.1. Given a class of formulas Γ, the Γ-*induction scheme* (written $\mathsf{I}\Gamma$) states, for every formula $\varphi(x) \in \Gamma$,

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\ \varphi(x)$$

We shall in particular be interested in the theories $\mathsf{I}\Sigma^0_n$ and $\mathsf{I}\Pi^0_n$.[8] Recall that Q denotes Robinson arithmetic (see Section 2.2). Most of our equivalences will be stated either over Q, $\mathsf{Q} + \mathsf{I}\Delta^0_0$ or $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$, where $\exp$ is the statement of the totality of the exponential.[9]

8: One should not confuse the arithmetic hierarchy on sets and on formulas. The former is a semantic notion, starting a the first level with computable predicates. The latter is a syntactic hierarchy, starting at the first level with *bounded arithmetic formulas*, that is, formulas with only quantifiers of the form $\forall x < t$ and $\exists x < t$ where t is a term. By a theorem of Gödel, the Σ^0_n sets are exactly the ones definable by a Σ^0_n formula, for $n \geq 1$, so the hierarchies coincide starting from level 1. On the other hand, some computable sets and even some primitive recursive sets are not definable by bounded arithmetic formulas.

Note that the hierarchies of Σ^0_n and Π^0_n formulas allow integer and set parameters, which is equivalent to quantify universally all free variables.

Proposition 7.2.2 (Paris and Kirby [42]). Fix $n \geq 1$. Then $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_n \leftrightarrow \mathsf{I}\Pi^0_n$. ★

Proof. We first prove $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_n \to \mathsf{I}\Pi^0_n$. Suppose that $\mathsf{I}\Sigma^0_n$ holds but $\mathsf{I}\Pi^0_n$ fails. Let $F(x)$ be a Π^0_n formula such that $F(0)$ and $\forall x(F(x) \to F(x+1))$, but $\neg F(a)$ for an integer $a > 0$. Let $G(y)$ be the formula $\exists x\ (a = x + y \wedge \neg F(x))$. Note that $G(y)$ is equivalent to a Σ^0_n formula. Moreover, $G(0)$ is true and $G(a)$ is false. Let y be such that $G(y)$ is true. In particular, there is an x such that $a = x + y$ and $\neg F(x)$. Since $F(0)$ holds, then $x > 0$ and $y < a$. Thus $a = (x-1)+(y+1)$ and by hypothesis, $\neg F(x) \to \neg F(x-1)$, therefore $G(y+1)$ is true. As $G(0)$ and $\forall y\ (G(y) \to G(y+1))$ and $\neg G(a)$, then $\mathsf{I}\Sigma^0_n$ fails.

We now prove $\mathsf{Q} \vdash \mathsf{I}\Pi^0_n \to \mathsf{I}\Sigma^0_n$. Suppose $\mathsf{I}\Pi^0_n$ holds but $\mathsf{I}\Sigma^0_n$ fails. Let $F(x)$ be a Σ^0_n formula such that $F(0)$ and $\forall x(F(x) \to F(x+1))$, but $\neg F(a)$ for an integer $a > 0$. Let $H(y)$ be the formula $\forall x\ (a = x + y \to \neg F(x))$. As before, $H(y)$ is equivalent to a Π^0_n formula. Additionally $H(0)$ is true and $H(a)$ is false. We also show $H(y) \to H(y+1)$. Then, $H(0)$ and $\forall y\ (H(y) \to H(y+1))$ and $\neg H(a)$, so $\mathsf{I}\Pi^0_n$ fails.[10] ■

9: Note that $\mathsf{Q} + \mathsf{I}\Sigma^0_1$, and *a fortiori* RCA_0, proves $\exp$, so all the implications of this section hold over RCA_0, and even over RCA^*_0, a weaker system that will be introduced in Section 7.4.

10: Note that in both directions, we used a formula with parameter a to witness failure of the other induction scheme. This is necessary, as the parameter-free versions of $\mathsf{I}\Sigma^0_n$ and $\mathsf{I}\Pi^0_n$ are not equivalent for $n \geq 1$. [43]

Exercise 7.2.3 (Hájek and Pudlák [41]). Given a class of formulas Γ, the Γ-*least principle* (written $\mathsf{L}\Gamma$) states, for every formula $\varphi(x) \in \Gamma$,

$$\exists x \varphi(x) \to \exists x(\varphi(x) \wedge \forall y < x \neg\varphi(y))$$

Show that $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_n \leftrightarrow \mathsf{L}\Pi^0_n$ and $\mathsf{Q} \vdash \mathsf{I}\Pi^0_n \leftrightarrow \mathsf{L}\Sigma^0_n$. ★

From a computability-theoretic viewpoint, bounded sets are finite and therefore trivially computable. In weak arithmetic on the other hand, not all bounded sets exist in the model, and their existence is closely related to the hierarchy of induction. A set $F \subseteq M$ is M-*coded* if it has a canonical code in M, that is, there is some $s \in M$ such that $s = \sum_{x \in F} 2^x$. Given $s \in M$, we write $\mathrm{Ack}(s)$ for the set coded by s.

11: These sets are also called *amenable* or *piecewise coded*. If $\mathcal{M} \models \mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$ then every set in S is M-regular.

Definition 7.2.4. Let $\mathcal{M} = (M, S)$ be a model. A set $A \subseteq M$ is *M-regular*[11] if every initial segment of A is M-coded. ◇

The following proposition states that the induction scheme is equivalent to a bounded version of the comprehension scheme. Therefore, restricting the induction corresponds to restricting the complexity of the finite sets in the model.

Proposition 7.2.5 (Hájek and Pudlák [41]). Fix $n \geq 1$. Then the following are equivalent over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$:

1. $\mathsf{I}\Sigma^0_n$;
2. Every Σ^0_n-definable set is regular. ★

Proof. Suppose first that every Σ^0_n-definable set is regular. Let φ be a Σ^0_n formula such that $\varphi(0)$ holds and $\forall x(\varphi(x) \to \varphi(x+1))$. Fix any $a \in \mathbb{N}$ and let $\sigma \in 2^{a+1}$ be the string defined by $\sigma(x) = 1$ iff $\varphi(x)$ holds. By regularity, σ exists. Let $\psi(x)$ be the Δ^0_0 formula defined by $\psi(x) \equiv (x \leq a \to \sigma(x) = 1)$. By $\mathsf{I}\Delta^0_0$, $\psi(x)$ holds for every x, so $\varphi(a)$ holds.

Suppose now $\mathsf{I}\Sigma^0_n$. Let φ be a Σ^0_n formula and $a \in \mathbb{N}$. Let $\psi(q)$ be the Π^0_n formula $(\forall x < a)(\varphi(x) \to x \in q)$, where $x \in q$ means that x belongs to the set canonically coded by q. Note that $2^a - 1$ is a canonical code for $\{x \in \mathbb{N} : x < a\}$, so $\psi(2^a - 1)$ holds. By $\mathsf{L}\Pi^0_n$ (which is equivalent to $\mathsf{I}\Sigma^0_n$ by Exercise 7.2.3), there is a least $q \in \mathbb{N}$ such that $\psi(q)$ holds. Then q is a canonical code of $\{x < a : \varphi(x)\}$. ■

The collection scheme is a principle equivalent to induction, but whose induced hierarchy is interleaved with the induction hiearchy. It plays a very important role in proving closure properties of levels of the arithmetic hierarchy.

Definition 7.2.6. Given a class of formulas Γ, the *Γ-collection scheme* (written $\mathsf{B}\Gamma$) states, for every formula $\varphi(x, y) \in \Gamma$,

$$\forall a[(\forall x < a \exists y \varphi(x, y)) \to \exists b \forall x < a \exists y < b \varphi(x, y)]$$

In other words, the collection scheme states that every bounded family of existential formulas admits a uniform existential bound. By contraction of quantifiers, $\mathsf{B}\Sigma^0_{n+1}$ is equivalent to $\mathsf{B}\Pi^0_n$.

Exercise 7.2.7 (Hájek and Pudlák [41]). Prove that $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_{n+1} \leftrightarrow \mathsf{B}\Pi^0_n$. ★

The following proposition is very useful for formulas manipulation:

Proposition 7.2.8 (Parsons [44]). Fix $n \geq 1$. Let $\varphi_0(x), \varphi_1(x), \varphi(x)$ be Σ^0_n (resp. Π^0_n) formulas. Then the following formulas are provably equivalent to a Σ^0_n (resp. Π^0_n) formula over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{B}\Sigma^0_n$:

(1) $\varphi_0(x) \wedge \varphi_1(x), \varphi_0(x) \vee \varphi_1(x)$;
(2) $\exists x < a \varphi(x), \forall x < a \varphi(x)$;
(3) $\exists x \varphi(x)$ (resp. $\forall x \varphi(x)$). ★

Proof. Say $\varphi_0(x) \equiv \exists y \theta_0(x,y)$, $\varphi_1(x) \equiv \exists y \theta_1(x,y)$ and $\varphi(x) \equiv \exists y \theta(x,y)$. The proof goes by induction, using the following equivalences:

$$\begin{array}{rcll}
\varphi_0(x) \wedge \varphi_1(x) & \leftrightarrow & \exists y \exists y_0, y_1 < y(\theta_0(x,y_0) \wedge \theta_1(x,y_1)) & (a)\\
\varphi_0(x) \vee \varphi_1(x) & \leftrightarrow & \exists y(\theta_0(x,y) \vee \theta_1(x,y)) & (b)\\
\exists x < a \varphi(x) & \leftrightarrow & \exists y \exists x < a \theta(x,y) & (c)\\
\forall x < a \varphi(x) & \leftrightarrow & \exists a \forall x < a \exists y < z \theta(x,y) & (d)\\
\exists x \theta(x) & \leftrightarrow & \exists z \exists x, y < z \theta(x,y) & (e)
\end{array}$$

Note that (a)(b)(c) and (e) are provable over $\mathsf{Q} + \mathsf{I}\Delta^0_0$, while (d) uses $\mathsf{B}\Sigma^0_n$. ■

The following theorem shows that the hierarchies of induction and collection are interleaved. Paris and Kirby [42] proved the following implications, which are both strict:

Theorem 7.2.9 (Paris and Kirby [42])
Fix $n \geq 1$.

1. $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_n \longrightarrow \mathsf{B}\Sigma^0_n$
2. $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_{n+1} \longrightarrow \mathsf{I}\Sigma^0_n$.

Actually, the levels of the collection hierarchy can be understood in terms of induction, using Δ^0_n predicates. Recall that for $n \geq 1$, Δ^0_n predicates do not form a syntactic class for formulas. Thankfully, one can extend the various schemes to Δ^0_n predicates using a syntactical trick.

Definition 7.2.10. Fix $n \geq 1$. The Δ^0_n-induction scheme (written $\mathsf{I}\Delta^0_n$) states, for every Σ^0_n formula $\varphi(x)$ and every Π^0_n formula $\psi(x)$:

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow [(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))) \rightarrow \forall x \varphi(x)]$$

The Δ^0_n-least principle ($\mathsf{L}\Delta^0_n$) is defined accordingly. By Gandy (see Slaman [45]), $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_n \leftrightarrow \mathsf{L}\Delta^0_n$. The proof of following theorem goes far beyond the scope of this book.

Theorem 7.2.11 (Slaman [45])
Fix $n \geq 1$.

- $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_n \longrightarrow \mathsf{I}\Delta^0_n$;
- $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp \vdash \mathsf{I}\Delta^0_n \longrightarrow \mathsf{B}\Sigma^0_n$.

Exercise 7.2.12 (Hájek and Pudlák [41]). Fix $n \geq 1$. Show that the following are equivalent over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$:

1. $\mathsf{I}\Delta^0_n$;
2. Every Δ^0_n-definable set is regular. ★

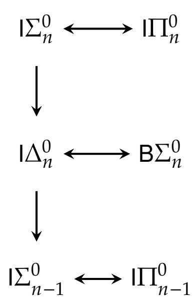

Figure 7.1: Induction hierarchy. Arrows stand for implications in $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$.

7.3 Conservation over RCA_0

The proof-theoretic strength of RCA_0 is relatively well understood. Its first-order part is $\mathsf{Q} + \mathsf{I}\Sigma_1$ [12], and it is a Π_2-conservative extension of PRA. In

12: We distinguish the class of Σ^0_n formulas in the language of second-order arithmetic from the class of Σ_n formulas in first-order arithmetic. In particular, in the former case, second-order parameters are allowed.

particular, every primitive recursive function is provably total over RCA_0, and every theorem of RCA_0 is finitistically reducible in the sense of Hilbert's program. Proving that a theory T is Π^1_1 conservative over RCA_0 is therefore a good way to show that T is finitistically reducible.

Given a model $\mathcal{M} = (M, S)$ and a set $G \subseteq M$, we denote by $\mathcal{M} \cup \{G\}$ and $\mathcal{M}[G]$ the ω-extensions whose second-order parts are $S \cup \{G\}$ and the $\Delta^0_1(\mathcal{M}, G)$-definable sets[13], respectively. The following exercise reflects the fact that every Σ^0_1-formula over $\mathcal{M}[G]$ is equivalent to a Σ^0_1-formula over $\mathcal{M} \cup \{G\}$.

13: Given a class of formulas Γ and a structure $\mathcal{M}$, we write $\Gamma(\mathcal{M})$ for the class of formulas with parameters in $\mathcal{M}$.

Exercise 7.3.1 (Friedman [46]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ and $G \subseteq M$ be such that $\mathcal{M} \cup \{G\} \models \mathsf{I}\Sigma^0_1$. Show that $\mathcal{M}[G] \models \mathsf{RCA}_0$. ★

Proposition 7.1.3 gives a general proof scheme to obtain conservation theorems between two second-order theories. One can prove a refined proposition in the particular case of conservation of Π^1_2 problems over RCA_0. Recall that a problem P is Π^1_2 if the relations $X \in \operatorname{dom} \mathsf{P}$ and $Y \in \mathsf{P}(X)$ are both arithmetically definable. The sentence $\forall X \in \operatorname{dom} \mathsf{P}\ \exists Y \in \mathsf{P}(X)$ is then Π^1_2.

Proposition 7.3.2. Let P be a Π^1_2 problem. Suppose that for every countable topped model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$, and every $X \subseteq M$ such that $\mathcal{M} \models X \in \operatorname{dom} \mathsf{P}$, there is a set $Y \subseteq M$ such that $\mathcal{M}[Y] \models \mathsf{RCA}_0 + (Y \in \mathsf{P}(X))$. Then $\mathsf{RCA}_0 + \mathsf{P}$ is Π^1_1-conservative over RCA_0.[14] ★

14: By Exercise 7.3.1, it is actually sufficient to require that

$$\mathcal{M} \cup \{Y\} \models \mathsf{I}\Sigma^0_1 + (Y \in \mathsf{P}(X))$$

Proof. Let $\varphi \equiv \forall Z \theta(Z)$ be a Π^1_1-sentence, where θ is an arithmetic formula. Suppose that $\mathsf{RCA}_0 \nvdash \varphi$. Then by Gödel's completeness theorem and the downward Löwenheim-Skolem theorem, there is a countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 \cup \{\neg\varphi\}$. Let $Z_0 \in S$ be such that $\mathcal{M} \models \neg\theta(Z_0)$. Let $\mathcal{M}_0 = (M, S_0)$, where S_0 be the set of Δ^0_1-definable sets over $(M, \{Z_0\})$. By Friedman [47], $\mathcal{M}_0 \models \mathsf{RCA}_0$, and by construction, $\mathcal{M}_0$ is topped by Z_0.

We define by external induction a countable sequence of sets $Z_0, Z_1, \dots$ and models $\mathcal{M}_0, \mathcal{M}_1, \dots$ such that for every $n \in \omega$,

1. $\mathcal{M}_n = (M, S_n) \models \mathsf{RCA}_0$ is topped by $Z_0 \oplus \dots \oplus Z_n$;
2. for every $X \in S_n$ such that $\mathcal{M}_n \models X \in \operatorname{dom} \mathsf{P}$, there is some $p \in \omega$ such that $\mathcal{M}_p \models Z_p \in \mathsf{P}(X)$.

Assuming $\mathcal{M}_n$ is defined and given some $X \in \mathcal{M}_n$ such that $\mathcal{M}_n \models X \in \operatorname{dom} \mathsf{P}$, by assumption, there is a set $Z_{n+1} \subseteq M$ such that $\mathcal{M}[Z_{n+1}] \models \mathsf{RCA}_0 + (Z_{n+1} \in \mathsf{P}(X)\}$. Let $\mathcal{M}_{n+1} = \mathcal{M}_n[Z_{n+1}]$. By construction, $\mathcal{M}_{n+1}$ is topped by $Z_0 \oplus \dots \oplus Z_{n+1}$.

Let $\mathcal{N} = (M, T)$ be defined by $T = \bigcup_n S_n$. Note that $\mathcal{N} \models \mathsf{RCA}_0$ since it is a union of models of RCA_0. By construction, $\mathcal{N}$ is an ω-extension of $\mathcal{M}$ and a model of P. Last, since $Z_0 \in T$ and θ is arithmetic $\mathcal{N} \models \neg\theta(Z_0)$, hence $\mathcal{N} \models \neg\varphi$. ■

The first-conservation theorem, due to Harrington (see Simpson [4]), is the most important one for its implications to Hilbert's program. Indeed, many theorems are provable by compactness arguments.

Theorem 7.3.3 (Harrington)
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ be a countable model and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $G \in [T]$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0$.

Proof. Consider the Jockusch-Soare forcing whose conditions are infinite trees $T_1 \subseteq T$ in S, partially ordered by inclusion. First of all, some simple facts such as the existence of extendible nodes of arbitrary length are not immediate in weak arithmetic. We prove a lemma stating that it is the case in models of RCA_0. Recall that a node σ is *extendible* in T_1 if the set of nodes in T_1 comparable with σ is infinite.

Lemma 7.3.4 (Fernandes et al. [48]). Let T_1 be a condition and $\ell \in M$. There is an extendible node $\sigma \in T_1$ of length ℓ.[15] ⋆

15: Note that the proof of this lemma only uses $\mathsf{Q} + \mathsf{B}\Sigma^0_1$.

Proof. Assume by contradiction that for every $\sigma \in 2^\ell$ the tree $\{\tau \in T_1 : \tau \text{ is comparable with } \sigma\}$ is M-bounded. Then

$$\forall \sigma \in 2^\ell \exists b \forall \tau \in 2^b, \sigma \prec \tau \to \tau \notin T_1$$

The formula $\forall \tau \in 2^b, \sigma \prec \tau \to \tau \notin T_1$ is Δ^0_0, so by $\mathsf{B}\Sigma^0_1$ (which holds in RCA_0 by Theorem 7.2.9), there is some $b \in M$ such that

$$\forall \sigma \in 2^\ell \exists c < b \forall \tau \in 2^c, \sigma \prec \tau \to \tau \notin T_1$$

This yields that T_1 is bounded by b, contradicting our assumption that T_1 is M-infinite.[16] ■

16: In general, the predicate "X is finite" is Σ^0_2, so if T_1 was an arbitrary set of strings, the existence of an extendible node would require $\mathsf{B}\Sigma^0_2$. Thanks to prefix closure, the predicate "T is finite" for a tree T is Σ^0_1 and $\mathsf{B}\Sigma^0_1$ is sufficient.

Thanks to Lemma 7.3.5, for every sufficiently generic filter $\mathcal{F}$, the class $\bigcap_{T_1 \in \mathcal{F}}[T_1]$ is a singleton $G_{\mathcal{F}}$. Indeed, for every condition T_1 and $\ell \in M$, letting σ be an extendible node in T_1 of length ℓ, the condition $T_2 = \{\tau \in T_1 : \tau \preceq \sigma \vee \sigma \prec \tau\}$ exists by Δ^0_0-comprehension and is a valid extension of T_1 forcing $\sigma \prec G$.

Exercise 3.3.7 defined a Σ^0_1-preserving forcing question for Jockusch-Soare forcing in a standard context. We re-define it and prove its properties in the context of weak arithmetic.

Given a condition T_1 and a Σ^0_1-formula (with parameters in $\mathcal{M}$) $\varphi(G) \equiv \exists y \psi(y, G{\upharpoonright}_y)$, let $T_1 \operatorname{?\vdash} \varphi(G)$ hold if there is some $\ell \in M$ such that for every $\sigma \in T$ such that $|\sigma| = \ell$, there is some $y < \ell$ such that $\psi(y, \sigma{\upharpoonright}y)$ holds. By Theorem 7.2.9, $\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_1$, so by Proposition 7.2.8, Σ^0_1-formulas are closed under bounded quantification. It follows that this relation is Σ^0_1. The following lemma shows that this is a forcing question in a strong sense, that is, if it holds, then the condition already forces the Σ^0_1 formula.

Lemma 7.3.5. Let T_1 be a condition and $\varphi(G)$ be a Σ^0_1 formula.

1. If $T_1 \operatorname{?\vdash} \varphi(G)$ then T_1 forces $\varphi(G)$;
2. If $T_1 \operatorname{?\nvdash} \varphi(G)$ then there is an extension $T_2 \subseteq T_1$ forcing $\neg\varphi(G)$. ⋆

Proof. Say $\varphi(G) \equiv \exists y \psi(y, G{\upharpoonright}_y)$.

1. Suppose $T_1 \operatorname{?\vdash} \varphi(G)$. Then we claim that for every $P \in [T_1]$, $\varphi(P)$ holds. Indeed, let $\ell \in M$ be such that for every $\sigma \in T$ such that $|\sigma| = \ell$, there is some $y < \ell$ such that $\psi(y, \sigma{\upharpoonright}y)$ holds. Fix some $P \in [T_1]$. Since $P{\upharpoonright}_\ell \in T$, there is some $y < \ell$ such that $\psi(y, P{\upharpoonright}_y)$ holds, so $\varphi(P)$ holds.
2. Suppose $T_1 \operatorname{?\nvdash} \varphi(G)$. Let $T_2 = \{\sigma \in T_1 : \forall y < |\sigma| \neg\psi(y, \sigma{\upharpoonright}_y\}$. By assumption, T_2 is an infinite subtree of T_1 and by Δ^0_0-comprehension it belongs to S. We claim that for every $P \in [T_2]$, $\neg\varphi(P)$ holds. Suppose for the contradiction that $\varphi(P)$ holds for some $P \in [T_2]$. Let $y \in M$ be

such that $\psi(y, P\upharpoonright_y)$ holds. Then $P\upharpoonright y+1 \notin T_2$, contradiction. So T_2 forces $\neg\varphi(G)$. ■

It follows from Lemma 7.3.5 that if $\varphi(G)$ and $\psi(G)$ are two Σ^0_1-formulas such that T_1 ?⊢ $\varphi(G)$ and T_1 ?⊬ $\psi(G)$, then there is an extension $T_2 \subseteq T_1$ forcing $\varphi(G) \wedge \neg\psi(G)$. The following lemma shows that if $\mathcal{F}$ is sufficiently generic, then $\mathcal{M} \cup \{G_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$.

Lemma 7.3.6. Let T_1 be a condition and $\varphi(x, X)$ be a Σ^0_1 formula such that T_1 forces $\neg\varphi(b, G)$ for some $b \in M$. Then there is an extension $T_2 \subseteq T_1$ and some $a \in M$ such that T_2 forces $\neg\varphi(a, G)$, and if $a > 0$, then T_2 forces $\varphi(a-1, G)$.[17] ★

17: Note that the proof of Lemma 7.3.6 uses essentially two properties of the forcing question: the fact that it is Σ^0_1-preserving, and its ability to find a simultaneous witness extension to a positive and a negative answer.

Proof. Let $A = \{x \in M : T_1 \text{ ?⊢ } \varphi(x, G)\}$. Since the forcing question is Σ^0_1-preserving, the set A is $\Sigma^0_1(\mathcal{M})$. Moreover, T_1 forces $\neg\varphi(b, G)$, so by Lemma 7.3.5, T_1 ?⊬ $\varphi(b, G)$, hence $b \notin A$. Since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$, and $A \neq M$, there is some $a \in M$ such that $a \notin A$, and if $a > 0$, then $a - 1 \in A$. By Lemma 7.3.5, there is an extension $T_2 \subseteq T_1$ forcing $\neg\varphi(a, G)$. Moreover, if $a > 0$, then since $a - 1 \in A$, by Lemma 7.3.5, T_1 forces $\varphi(a-1, G)$, hence so does T_2. This completes the proof of Lemma 7.3.6. ■

We are now ready to prove Theorem 7.3.3. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. By Lemma 7.3.4, there is a unique set $G \in \bigcap_{T_1 \in \mathcal{F}} [T_1]$. In particular, $G \in [T]$. By Lemma 7.3.6, $\mathcal{M} \cup \{G\} \models \mathsf{I}\Sigma^0_1$, so by Exercise 7.3.1, $\mathcal{M}[G] \models \mathsf{RCA}_0$. This completes the proof of Theorem 7.3.3. ■

Corollary 7.3.7 (Harrington)
WKL_0 *is a* Π^1_1*-conservative extension of* RCA_0.

Proof. Immediate by Theorem 7.3.3 and Proposition 7.3.2. ■

Recall that by Theorem 3.2.4, every set can become Δ^0_2 relative to a cone avoiding degree. This can be interpreted as saying that cone avoidance for Δ^0_2 instances and strong cone avoidance are equivalent. A formalization due to Towsner [49] of the notion of forcing yields a conservation theorem over RCA_0, saying informally that from the viewpoint of RCA_0, Δ^0_2 sets are indistiguishable from arbitrary sets.

Theorem 7.3.8 (Toswner [49])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ *be a countable model and* $A \subseteq M$ *be an arbitrary set. There is a set* $G \subseteq M$ *such that* A *is* $\Delta^0_2(G)$ *and* $\mathcal{M}[G] \models \mathsf{RCA}_0$.

Proof. Based on Shoenfield's limit lemma [7], we will construct a stable function $f : \mathbb{N}^2 \to 2$ such that for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists and equals $A(x)$. We are therefore going to build directly the function f by forcing, and let G be the graph of f.

The idea is to use the notion of forcing from Theorem 3.2.4, however there is a technical difficulty: Assume A is not regular, and fix $a \in M$ such that $A\upharpoonright a$ does not belong to M. Then, the condition $(\emptyset, a)$ has no extension (g, b) in $\mathcal{M}$ with $\{0, \dots, a\} \times \{0\} \subseteq \operatorname{dom} g$. Worse, the set of extensions of $(\emptyset, a)$ is not

Δ^0_1-definable with parameters in $\mathcal{M}$. Thankfully, the model being countable, one can lock non-uniformly a standard number of columns for each condition, and still obtain a stable function.

Consider the notion of forcing whose *conditions* arex pairs (g, I), such that

- $g \subseteq M^2 \to \{0, 1\}$ is a partial function with two parameters whose domain is M-finite, representing an initial segment of the function f that we are building.
- $I \subseteq M$ is a set of "locked" columns with $\operatorname{card} I \in \omega$, meaning that from now on, when we extend the domain of g with a new pair (x, y), if $x \in I$ then $g(x, y) = A(x)$.

The *interpretation* $[g, I]$ of a condition (g, I) is the class of all partial or total functions $h \subseteq M^2 \to 2$ such that

(1) $g \subseteq h$, i.e. $\operatorname{dom} g \subseteq \operatorname{dom} h$ and for all $(x, y) \in \operatorname{dom} g$, $g(x, y) = h(x, y)$;
(2) for all $(x, y) \in \operatorname{dom} h \setminus \operatorname{dom} g$, if $x \in I$, then $h(x, y) = A(x)$.[18]

18: Even if A is not regular, the set I being of standard cardinality, the restriction $A{\restriction}I$ belongs to M. Therefore, the extension relation is Δ^0_1-definable with parameters in $\mathcal{M}$.

A condition (h, J) *extends* (g, I) (denoted $(h, J) \leq (g, I)$) if $J \supseteq I$ and $h \in [g, I]$.

For every condition (g, I) and every $x \in M$, $(g, I \cup \{x\})$ is a valid extension. Moreover, for every condition (g, I) and every $(x, y) \in M^2$, there is an extension $(h, I) \leq (g, I)$ such that $(x, y) \in \operatorname{dom} h$. Therefore, if $\mathcal{F}$ is a sufficiently generic filter, then, letting $f_{\mathcal{F}} = \bigcup\{g : (g, I) \in \mathcal{F}\}$, $\operatorname{dom} f_{\mathcal{F}} = M^2$ and every column will eventually be locked, so $f_{\mathcal{F}}$ is stable with limit A.

Given a condition (g, I) and a Σ^0_1-formula (with parameters in $\mathcal{M}$) $\varphi(G) \equiv \exists y \psi(y, G{\restriction}y)$, let $(g, I) \mathrel{?{\vdash}} \varphi(G)$ hold if there is a finite $h \in [g, I]$ and some $y \in M$ such that $\psi(y, h{\restriction}y)$ holds. The formula is Σ^0_1-preserving. We show that it is a forcing question in a strong sense, that is, if it does not hold, then the condition already forces the Π^0_1 formula.

Lemma 7.3.9. Let (g, I) be a condition and $\varphi(G)$ be a Σ^0_1 formula.

- If $(g, I) \mathrel{?{\vdash}} \varphi(G)$ then there is an extension (h, I) forcing $\varphi(G)$;
- If $(g, I) \mathrel{?{\nvdash}} \varphi(G)$, then (g, I) forces $\neg\varphi(G)$. ★

Proof. Say $\varphi(G) \equiv \exists y \psi(y, G{\restriction}_y)$.

1. Suppose $(g, I) \mathrel{?{\vdash}} \varphi(G)$. Then, letting $h \in [g, I]$ and $y \in M$ witness it, the condition (h, I) is an extension forcing $\varphi(G)$.
2. Suppose $(g, I) \mathrel{?{\nvdash}} \varphi(G)$. Suppose for the contradiction that there is some $h \in [g, I]$ such that $\varphi(h)$ holds. Unfolding the definition, there is some $y \in M$ such that $\psi(y, h{\restriction}y)$ holds. Let $h_1 \subseteq h$ be a finite function such that $\operatorname{dom} g \subseteq \operatorname{dom} h_1$ and $h{\restriction}y = h_1{\restriction}y$, then y and h_1 witness the fact that $(g, I) \mathrel{?{\vdash}} \varphi(G)$. Contradiction. So (g, I) forces $\neg\varphi(G)$. ■

It follows from Lemma 7.3.9 that if $\varphi(G)$ and $\psi(G)$ are two Σ^0_1-formulas such that $(g, I) \mathrel{?{\vdash}} \varphi(G)$ and $(g, I) \mathrel{?{\nvdash}} \psi(G)$, then there is an extension $(h, I) \leq (g, I)$ forcing $\varphi(G) \wedge \neg\psi(G)$. The following lemma shows that if $\mathcal{F}$ is sufficiently generic, then $\mathcal{M} \cup \{f_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$.

Lemma 7.3.10. Let (g, I) be a condition and $\varphi(x, X)$ be a Σ^0_1 formula such that (g, I) forces $\neg\varphi(b, G)$ for some $b \in M$. Then there is an extension $(h, I) \leq (g, I)$ and some $a \in M$ such that (h, I) forces $\neg\varphi(a, G)$, and if $a > 0$, (h, I)

forces $\varphi(a-1, G)$.[19] ★

19: Note the similarity of the proof of Lemma 7.3.10 with the proof of Lemma 7.3.6. We again only exploit some abstract properties of the forcing question.

Proof. Let $A = \{x \in M : (g, I) \mathrel{?\vdash} \varphi(x, G)\}$. Since the forcing question is Σ^0_1-preserving, the set A is $\Sigma^0_1(\mathcal{M})$. Moreover, (g, I) forces $\neg\varphi(b, G)$, so by Lemma 7.3.9, $(g, I) \mathrel{?\nvdash} \varphi(b, G)$, hence $b \notin A$. Since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$, and $A \neq M$, there is some $a \in M$ such that $a \notin A$, and if $a > 0$, then $a - 1 \in A$. By Lemma 7.3.9, (g, I) forces $\neg\varphi(a, G)$. Moreover, if $a > 0$, then since $a-1 \in A$, by Lemma 7.3.9, there is an extension (h, I) forcing $\varphi(a-1, G)$. Note that (h, I) forces $\neg\varphi(a, G)$. This completes the proof of Lemma 7.3.10. ■

We are now ready to prove Theorem 7.3.8. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. As mentioned, it induces a stable function $f_{\mathcal{F}} = \bigcup\{g : (g, I) \in \mathcal{F}\}$ whose limit is A. By Lemma 7.3.10, $\mathcal{M} \cup \{f_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$, so by Exercise 7.3.1, $\mathcal{M}[f_{\mathcal{F}}] \models \mathsf{RCA}_0$. This completes the proof of Theorem 7.3.8.■

The careful reader will have recognized some common pattern in the proofs of Theorem 7.3.3 and Theorem 7.3.8. Indeed, in both theorems, the lemma stating the preservation of Σ^0_1-induction used the existence of a Σ^0_1-preserving function which was able to give simultaneously a positive and a negative answer to two independent Σ^0_1 questions. This motivates the following definition.

Definition 7.3.11. Given a notion of forcing $(\mathbb{P}, \leq)$ and some $n \in \mathbb{N}$, a forcing question is (Σ^0_n, Π^0_n)-*merging* if for every $p \in \mathbb{P}$ and every pair of Σ^0_n formulas $\varphi(G), \psi(G)$ such that $p \mathrel{?\vdash} \varphi(G)$ but $p \mathrel{?\nvdash} \psi(G)$, then there is an extension $q \leq p$ forcing $\varphi(G) \wedge \neg\psi(G)$. ◊

Recall that a forcing question can be seen as a dividing line within the slice of conditions which do not already decide a formula (see Figure 7.2).

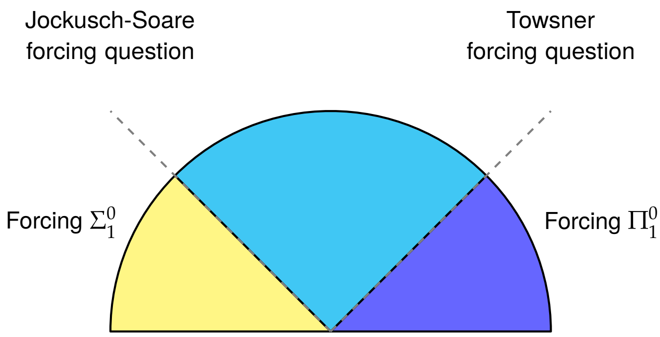

Figure 7.2: The yellow part and the dark blue part represent the conditions forcing a fixed Σ^0_1 and its negation, respectively. The light blue part represent the conditions of the third category. With Jockusch-Soare forcing (Theorem 7.3.3), the dividing line is at the left-most position, while for Towsner forcing (Theorem 7.3.8), the dividing line is at the opposite position.

As shown in the picture, Jockush-Soare forcing and Towsner forcing have extremal values. Any forcing question at one of these extremes is (Σ^0_1, Π^0_1)-merging, as if $p \mathrel{?\vdash} \varphi(G)$ and $p \mathrel{?\nvdash} \psi(G)$ for two Σ^0_1 formulas φ and ψ, then either p forces $\varphi(G)$ or p forces $\neg\psi(G)$, and one simply has to take the extension witnessing the answer to the other question. We now prove the abstract theorem associated to preservation of Σ^0_1-induction.

Theorem 7.3.12
Let $\mathcal{M} = (M, S) \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$ be a countable model and let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving (Σ^0_1, Π^0_1)-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $\mathcal{M} \cup \{G_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$.

Proof. It suffices to prove the following lemma:

Lemma 7.3.13. For every condition $p \in \mathbb{P}$ and every Σ^0_1-formula such that p forces $\neg\varphi(b, G)$ for some $b \in M$, there is an extension $q \leq p$ and some $a \in M$ such that q forces $\neg\varphi(a, G)$, and if $a > 0$, then q forces $\varphi(a-1, G)$. ⋆

Proof. Let $A = \{x \in M : p \,?\!\vdash \varphi(x, G)\}$. Since the forcing question is Σ^0_1-preserving, the set A is $\Sigma^0_1(\mathcal{M})$. Moreover, p forces $\neg\varphi(b, G)$, so by definition of the forcing question, $p \,?\!\nvdash \varphi(b, G)$, hence $b \notin A$. Since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$, and $A \neq M$, there is some $a \in M$ such that $a \notin A$, and if $a > 0$, then $a - 1 \in A$. If $a = 0$, then by definition of the forcing question, there is an extension $q \leq p$ forcing $\neg\varphi(0, G)$. If $a > 0$, then since the forcing question is (Σ^0_1, Π^0_1)-merging, there is an extension $q \leq p$ forcing $\neg\varphi(a, G)$ and $\varphi(a-1, G)$. ■

We are now ready to prove Theorem 7.3.12. Given a Σ^0_1 formula φ, let $\mathcal{D}_\varphi$ be the set of all conditions $q \in \mathbb{P}$ forcing either $\forall b\varphi(b, G)$, or $\neg\varphi(0, G)$, or $\varphi(a-1, G) \wedge \neg\varphi(a, G)$ for some $a > 0$. It follows from Lemma 7.3.13 that every $\mathcal{D}_\varphi$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_\varphi : \varphi \in \Sigma^0_1\}$-generic, so $\mathcal{M} \cup \{G_\mathcal{F}\} \models \mathsf{I}\Sigma^0_1$. This completes the proof of Theorem 7.3.12.■

Exercise 7.3.14 (Cholak, Jockusch and Slaman [25]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ be a countable model and $\vec{R} = R_0, R_1, \dots$ be a sequence of sets in $\mathcal{M}$. Use a formalized notion of computable Mathias forcing (see Exercise 3.2.8) to prove the existence of an infinite $\vec{R}$-cohesive set $G \subseteq M$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0$. Deduce that $\mathsf{RCA}_0 + \mathsf{COH}$ is Π^1_1-conservative over RCA_0. ⋆

7.4 Isomorphism theorem

The choice of RCA_0 as a base theory capturing computable mathematics can be questioned because of Σ^0_1-induction. Indeed, by Proposition 7.2.5, Σ^0_n-induction corresponds to Σ^0_n-regularity, so Σ^0_1-induction will add every bounded c.e. set in the model. By Post's theorem, one would arguably restrict the base theory to Δ^0_1-induction to have Δ^0_1-regularity.[20] Simpson and Smith [50] introduced RCA^*_0, the theory based on Robinson arithmetic (Q), together with the Δ^0_1-comprehension scheme, the Δ^0_0-induction scheme ($\mathsf{I}\Delta^0_0$) and the statement of the totality of the exponential (exp).

20: There are mostly two reasons why RCA_0 was chosen as the base theory rather than RCA^*_0: a historical and a pragmatical one.
Historically, Friedman used a language of functions rather than sets, with a Δ^0_0-recursion principle which turned out to be equivalent to Σ^0_1-induction. See Hirschfeldt [6, Chapter 4] for a more thorough discussion on the subject.
Pragmatically, basic features such as the equivalence of the various notions of infinity, are equivalent to Σ^0_1-induction. One expects from a base theory to be able to prove the robustness of the core concepts. In particular, the provably total functions over RCA_0 are the primitive recursive functions, while RCA^*_0 only proves the totality of the elementary recursive functions.

Exercise 7.4.1. Show that RCA^*_0 proves $\mathsf{I}\Delta^0_1$ and $\mathsf{B}\Sigma^0_1$. ⋆

Although RCA_0 remains the mainstream base theory to found reverse mathematics, RCA^*_0 is useful to compare very weak statements of arithmetic [50]. In particular, the notion of infinity is not robust in RCA^*_0, as some unbounded sets may not be in bijection with $\mathbb{N}$. As it turns out, RCA^*_0 became an essential tool in the study of models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, through the notion of jump model.

Definition 7.4.2. Given a model $\mathcal{M} = (M, S)$, its *jump model* is the structure $\mathcal{N} = (M, \Delta^0_2\text{-Def}(\mathcal{M}))$, where Δ^0_2-Def($\mathcal{M}$) denotes the Δ^0_2 definable sets with parameters in $\mathcal{M}$. We then call $\mathcal{M}$ a *ground model* of $\mathcal{N}$. ◇

The following exercise puts a bridge between models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ and models of RCA^*_0.

Exercise 7.4.3 (Belanger [51]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$. Show that $\mathcal{M} \models \mathsf{B}\Sigma^0_2$ iff $(M, \Delta^0_2\text{-Def}(\mathcal{M})) \models \mathsf{RCA}^*_0$. ★

Models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ play an important role in the study of Ramsey's theorem for pairs. Let RT^1 be the statement $\forall a \mathsf{RT}^1_a$. This statement easily follows from $\mathsf{RCA}_0 + \mathsf{RT}^2_2$. Indeed, given a coloring $f : \mathbb{N} \to a$ for some $a \in \mathbb{N}$, one can define the coloring $g : [\mathbb{N}]^2 \to 2$ by $g(x, y) = 1$ iff $f(x) = f(y)$. Any infinite g-homogeneous set is f-homogeneous. The following proposition therefore shows that any model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ satisfies $\mathsf{B}\Sigma^0_2$.

Proposition 7.4.4 (Hirst [52]). $\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_2 \leftrightarrow \mathsf{RT}^1$. ★

Proof.

- Assume $\mathsf{B}\Sigma^0_2$. Let $f : \mathbb{N} \to a$ be an instance of RT^1 for some $a \in \mathbb{N}$. Suppose that there is no infinite f-homogeneous set. Then $(\forall x < a)(\exists y)(\forall w)[w > y \to f(w) \neq x]$. Then by $\mathsf{B}\Sigma^0_2$, there is some $b \in \mathbb{N}$ such that $(\forall x < a)(\exists y < b)(\forall w)[w > y \to f(w) \neq x]$. Then $(\forall x < a)[f(b) \neq x]$, contradiction.
- Assume RT^1. Let $\theta(x, y, w)$ be a Δ^0_0-formula. Fix $a \in \mathbb{N}$ and suppose that $(\forall x < a)(\exists y)(\forall z)\theta(x, y, w)$. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(t)$ is the least $b < t$ such that $(\forall x < a)(\exists y < b)(\forall w < t)\theta(x, y, w)$, if such a b exists. Otherwise, let $f(t) = t$. Suppose first that there exists an infinite f-homogeneous set H, for some color b. Then $(\forall x < a)(\exists y < b)\forall w \theta(x, y, w)$ holds by RT^1. Suppose now that there is no infinite f-homogeneous set. Then by RT^1, the range of f is unbounded. Construct a strictly increasing sequence $(t_s)_{s \in \mathbb{N}}$ such that $f(t_s) < f(t_{s+1})$ for every $s \in \mathbb{N}$. Let $g : \mathbb{N} \to a$ be such that $g(s)$ is the least $x < a$ such that $(\forall y < f(t_s) - 1)(\exists w < t_s)\neg\theta(x, y, w)$. By RT^1, there is an infinite g-homogeneous set S for some color x. Fix some $y \in \mathbb{N}$. Since S is infinite, there is some $s \in S$ such that $f(t_s) - 1 > y$. So $(\exists w < t_s)\neg\theta(x, y, w)$ holds. Hence $(\forall y)(\exists w)\neg\theta(x, y, w)$, contradiction. ■

Π^1_1-conservation theorems over RCA^*_0 follow the same structure as over RCA_0, mutatis mutandis.

Exercise 7.4.5 (Simpon and Smith [50]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}^*_0$ and fix a set $G \subseteq M$. Show that

1. If G is M-regular, then $\mathcal{M}[G] \models \mathsf{I}\Delta^0_0$.
2. If moreover $\mathcal{M} \cup \{G\} \models \mathsf{B}\Sigma^0_1$, then $\mathcal{M}[G] \models \mathsf{RCA}^*_0$. ★

Exercise 7.4.6 (Simpon and Smith [50]). Let P be a Π^1_2 problem. Suppose that for every countable topped model $\mathcal{M} = (M, S) \models \mathsf{RCA}^*_0$, and every $X \in S$ such that $\mathcal{M} \models X \in \operatorname{dom} \mathsf{P}$, there is set $Y \subseteq M$ such that $\mathcal{M}[Y] \models \mathsf{RCA}^*_0 + (Y \in \mathsf{P}(X))$. Adapt the proof of Proposition 7.3.2 to show that $\mathsf{RCA}^*_0 + \mathsf{P}$ is Π^1_1-conservative over RCA^*_0. ★

Let WKL^*_0 be the theory RCA^*_0 augmented with the statement "Every infinite binary tree admits an infinite path". Simpson and Smith proved that WKL^*_0 is Π^1_1-conservative over RCA^*_0, and we shall see that this is the best result possible, in the sense that weak König's lemma is the strongest Π^1_2 statement that is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$.

Theorem 7.4.7 (Simpson and Smith [50])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0^*$ be a countable model and $T \subseteq 2^{<M}$ be an infinite tree in S. There is an M-regular path $G \in [T]$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0^*$.[21]

Proof. The proof of Theorem 7.4.7 is very similar to that of Theorem 7.3.3. It also uses Jockusch-Soare forcing whose conditions are infinite trees $T_1 \subseteq T$ in S, partially ordered by inclusion. Lemma 7.3.4 and Lemma 7.3.5 both hold in models of RCA_0^*, so for every sufficiently generic filter $\mathcal{F}$, $\bigcap_{T_1 \in \mathcal{F}}[T_1]$ is a singleton $G_{\mathcal{F}}$, which is M-regular. The main difference lies in the following lemma:

Lemma 7.4.8. Let T_1 be a condition, $a \in M$, and $\varphi(x, y, X)$ be a Σ^0_1 formula forcing $(\forall x < a)(\exists y)\varphi(x, y, G)$. Then there is some $b \in M$ such that T_1 forces $(\forall x < a)(\exists y < b)\varphi(x, y, G)$. ★

Proof. Let $\theta(x, z) \equiv T_1 \mathrel{?\vdash} (\exists y < z)\varphi(x, y, G)$. Since the forcing question is Σ^0_1-preserving, the formula θ is $\Sigma^0_1(\mathcal{M})$. Moreover, T_1 forces $(\forall x < a)(\exists y)\varphi(x, y, G)$, so by Lemma 7.3.5, for every $x < a$, $T_1 \mathrel{?\vdash} \exists y \varphi(x, y, G)$. By Σ^0_1-compactness[22] of the forcing question, for every $x < a$, there is some $z \in M$ such that $T_1 \mathrel{?\vdash} (\exists y < z)\varphi(x, y, G)$. Thus, for every $x < a$, there is some $z \in M$ such that $\theta(x, z)$ holds. By $\mathsf{B}\Sigma^0_1$, there is some $b \in M$ such that $(\forall x < a)(\exists z < b)\theta(x, z)$. Unfolding the definition of θ, $(\forall x < a)(\exists z < b)T_1 \mathrel{?\vdash} (\exists y < z)\varphi(x, y, G)$. By Lemma 7.3.5, for every $x < a$, there is some $z < b$ such that T_1 forces $(\exists y < z)\varphi(x, y, G)$, so T_1 forces $(\exists y < b)\varphi(x, y, G)$. ■

We are now ready to prove Theorem 7.4.7. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. By Lemma 7.3.4, there is a unique M-regular set $G \in \bigcap_{T_1 \in \mathcal{F}}[T_1]$. In particular, $G \in [T]$. By Lemma 7.3.6, $\mathcal{M} \cup \{G\} \models \mathsf{B}\Sigma^0_1$, so by Exercise 7.4.5, $\mathcal{M}[G] \models \mathsf{RCA}_0^*$. This completes the proof of Theorem 7.4.7.■

21: The proof of preservation of $\mathsf{B}\Sigma^0_1$ (Lemma 7.4.8) uses the existence of a Σ^0_1-preserving, Σ^0_1-compact forcing question such that if $p \mathrel{?\vdash} \varphi(G)$ holds for some Σ^0_1 formula φ, then p already forces $\varphi(G)$. Since weak König's lemma is the strongest Π^1_2 theory which is Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$, the Jockusch-Soare forcing is in some sense the strongest notion of forcing with the existence of a forcing question with the above mentioned properties.

22: Recall that a forcing question is Σ^0_n-*compact* if for every $p \in \mathbb{P}$ and every Σ^0_n formula $\varphi(G, x)$, if $p \mathrel{?\vdash} \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ such that $p \mathrel{?\vdash} \exists x \in F\ \varphi(G, x)$.

Corollary 7.4.9 (Simpson and Smith [50])
WKL_0^* *is a Π^1_1-conservative extension of* RCA_0^*.

Proof. Immediate by Theorem 7.4.7 and Exercise 7.4.6. ■

Fiori-Carones, Kołodziejczyk, Wong and Yokoyama [53] proved a beautiful isomorphism theorem for countable models of $\mathsf{WKL}_0^* + \neg\mathsf{I}\Sigma^0_1$ with many consequences, not only for provability over RCA_0^*, but also for conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Theorem 7.4.10 (Fiori-Carones et al [53])
Let (M, S_0) and (M, S_1) be countable models of WKL_0^* *such that $(M, S_0 \cap S_1) \models \neg\mathsf{I}\Sigma^0_1$. Let $\vec{c}$ be a tuple of elements of M and $\vec{C}$ be a tuple of elements of $S_0 \cap S_1$. Then there is an isomorphism h between (M, S_0) and (M, S_1) such that $h(\vec{c}) = \vec{c}$ and $h(\vec{C}) = \vec{C}$.*

Proof. Let $\mathcal{M} = (M, S_0 \cap S_1)$ and $\mathcal{M}_i = (M, S_i)$ for each $i < 2$. A *cut* is an initial segment of M which is closed under successor. Any model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$ contains a proper Σ^0_1-definable cut. Indeed, since $\varphi(x)$ be a Σ^0_1 formula such that $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))$ holds, but $\neg\varphi(a)$ for some $a \in \mathbb{N}$.

Let $I = \{x \in \mathbb{N} : (\forall x' < x)\varphi(x')\}$. By $\mathsf{B}\Sigma^0_1$, I is Σ^0_1-definable, and by construction, I is a proper cut. Such a cut I is not necessarily closed under other operations such as addition, multiplication or exponentiation. With some extra work, one can prove that every model of $\mathsf{I}\Delta^0_0 + \exp + \neg\mathsf{I}\Sigma^0_1$ contains a proper Σ^0_1-definable cut which is closed under $\exp$ (see [54, Lemma 9]). Therefore, fix a $\Sigma^0_1(\mathcal{M})$ proper cut I which is closed under $\exp$.

Let $\psi(x, y)$ be a $\Delta^0_0(\mathcal{M})$ formula such that $I = \{x \in M : \mathcal{M} \models \exists y \psi(x, y)\}$. Let $a_0 \in M \setminus I$ and let B be the set of all pairs $\langle i, a_i\rangle \in \mathbb{N}$ such that a_{i+1} is the least element greater than a_i satisfying $(\forall x \leq i)(\exists y \leq a_{i+1})\psi(x, y)$. The set B is $\Delta^0_0(\mathcal{M})$-definable, of cardinality I and the sequence $(a_i)_{i\in I}$ is enumerated in increasing order and cofinal in M. Note that B belongs $S_0 \cap S_1$ by Δ^0_0-comprehension. By adding the set B to the tuple $\vec{C}$, we ensure that the relation $\theta(x, i) \equiv x = a_i$ is $\Delta_0(\vec{C})$.

We build the isomorphism h by a back-and-forth construction. Let $\mathbb{P}$ be the notion of forcing[23] whose conditions are tuples $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ such that

1. $\vec{r}$ and $\vec{s}$ are finite vectors of same standard length, of elements of M ;
2. $\vec{R}$ and $\vec{S}$ are finite vectors of same standard length, of elements of S_0 and S_1, respectively ;
3. $b \in M$ is such that $b > I$;
4. for each $i \in I$ and each Δ^0_0-formula δ with $\ulcorner\delta\urcorner < b$, $\mathcal{M}_0 \models \delta(a_i, \vec{r}, \vec{R})$ iff $\mathcal{M}_1 \models \delta(a_i, \vec{s}, \vec{S})$.[24] [25]

23: The construction uses the language of forcing for convenience, but it will not use its whole machinery, such as the forcing relation.

24: We write $\ulcorner\delta\urcorner$ for the Gödel number of a formula. One can think of it as the integer whose binary representation is the string of the formula. In particular, the Gödel number of a standard formula is a standard integer. Note that we work with Δ^0_0-formulas with first-order parameters, that is, in a language enriched with symbol constants for each first-order element. The constraint $\ulcorner\delta\urcorner < b$ prevents from using first-order parameters larger than $\log b$.

25: Since we also consider non-standard Δ^0_0-formulas, the satisfaction relation $\models$ is replaced by a Σ^0_1-formula Sat_0 expressing the truth definition for Δ^0_0-formulas (see Hájek and Pudlák [41]).

Intuitively, a condition $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ is a partial assignment of h over the domain $\vec{r} \cup \vec{R}$ and with range $\vec{s} \cup \vec{S}$. The initial condition is $(\vec{c}, \vec{c}, \vec{C}, \vec{C}, b)$ for a fixed $b > I$. A condition $(\vec{r}', \vec{s}', \vec{R}', \vec{S}', b')$ *extends* $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ if $b' \leq b$, $\vec{r} \preceq \vec{r}'$, $\vec{s} \preceq \vec{s}'$, $\vec{R} \preceq \vec{R}'$ and $\vec{S} \preceq \vec{S}'$.

Before proving our main density lemmas, we need to state a technical coding lemma which generalizes Proposition 7.2.5.

Lemma 7.4.11 (Chong and Mourad [55]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}^*_0$. Then for every pair of bounded disjoint Σ^0_1 sets $X, Y \subseteq M$, there exists some $s \in M$ such that $\mathrm{Ack}(s) \cap (X \cup Y) = X$.[26] ⋆

26: Recall that given $s \in M$, we write $\mathrm{Ack}(s)$ for the set $F \subseteq M$ coded by s, that is, such that $s = \sum_{x\in F} 2^x$.

Proof. Let φ and ψ be two Δ^0_0 formulas such that $X = \{x \in M : \mathcal{M} \models (\exists z)\varphi(x, z)\}$ and $Y = \{x \in M : \mathcal{M} \models (\exists z)\psi(x, z)\}$. Let $a \in M$ be a common bound for X and Y and let $b \in M$ be such that $\mathrm{Ack}(b) = \{0, \dots, a-1\}$. Suppose for the contradiction that for all $s \leq b$, $\mathrm{Ack}(s) \cap (X \cup Y) \neq X$. Then

$$(\forall s < b)(\exists x < a)[(x \in \mathrm{Ack}(s) \wedge x \in Y) \vee (x \notin \mathrm{Ack}(s) \wedge x \in X)]$$

By $\mathsf{B}\Sigma^0_1$, there is a uniform bound $\hat{z} \in M$ such that

$$(\forall s < b)(\exists x < a)\left[\begin{array}{cc} & (x \in \mathrm{Ack}(s) \wedge (\exists z < \hat{z})\psi(x, z)) \\ \vee & (x \notin \mathrm{Ack}(s) \wedge (\exists z < \hat{z})\varphi(x, z)) \end{array}\right]$$

Let $S = \{x < a : (\forall z < \hat{z})\neg\psi(x, z)\}$. The set S is Δ^0_0, hence is M-coded by some $s \leq b$. Moreover, $S \cap (X \cup Y) = X$, contradiction. ■

The following lemma shows that one can add any first-order element to the domain of h while preserving the invariant. Since the models (M, S_0) and (M, S_1) play a symmetric role, it is also dense to add any first-order element to the range of h.

Lemma 7.4.12. Let $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ be a condition and $d \in M$. There is an extension $(\vec{r}d, \vec{s}e, \vec{R}, \vec{S}, b')$ for some $e, b' \in M$. ⋆

Proof. Let $b' > I$ be sufficiently small with respect to b. Let $D \subseteq I \times b'$ be the following set

$$\{(i, \ulcorner\delta\urcorner) \in I \times b' : \delta \text{ is } \Delta_0^0 \text{ and } \mathcal{M}_0 \models \delta(a_i, \vec{r}d, \vec{R})\}$$

Both D and $(I \times b') \setminus D$ are bounded and Σ_1^0-definable, so by Lemma 7.4.11, there is some $t \in M$ such that $\mathrm{Ack}(t) \cap (I \times b') = D$. Moreover, since $D \subseteq I \times b'$ and $I < b'$, we can assume $t < 2^{b' \times b'}$. Let $i' \in I$ be such that $d \leq a_{i'}$. By choice of t, for every $i \in I$, the structure $\mathcal{M}_0$ satisfies

$$(\exists y \leq a_{i'})(\forall j \leq i) \bigwedge_{\ulcorner\delta\urcorner < b'} [\delta(a_j, \vec{r}y, \vec{R}) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(t)]$$

as witnessed by taking $y = d$. For every $i \in I$ such that $i \geq i'$, $\mathcal{M}_0$ therefore satisfies the Δ_0^0-formula $\gamma(a_i, \vec{r}, \vec{R})$ defined by

$$(\exists x, z \leq a_i)(\exists y \leq x)(x = a_{\mathbf{i'}} \wedge z = \mathbf{t} \wedge (\forall j \leq i)(\forall v \leq a_i)$$
$$(v = a_j \rightarrow \wedge_{\ulcorner\delta\urcorner < \mathbf{b'}}[\delta(v, \vec{r}y, \vec{R}) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(z)]))$$

For each $i \in I$, the formula γ is written in a language enriched with symbol constants for i', b', t.[27] The formula γ written in binary starts with a part of length $\mathcal{O}(\log(i') + \log(b') + \log(t))$. It is then followed by a conjunction composed of b' conjuncts, each of length $\mathcal{O}(b')$. Since $i' < b'$ and $\log(t) < b' \cdot b'$, the formula γ has length $\mathcal{O}(b' \times b')$. Since I is an exponential cut, we can take b' sufficiently small so that $\ulcorner\gamma\urcorner < b$.

27: The relation $\theta(x, i) \equiv x = a_i$ being $\Delta_0(\vec{C})$, the parameter i can be obtained from a_i, and conversely, $a_{i'}$ can be obtained from i'. Thus, i and $a_{i'}$ are not considered as parameters.

The big conjunction is not part of the language, hence is a shorthand for a non-standard conjunction with b' many conjuncts. Because of this and because of the non-standard parameters i', b' and t, the formula has a non-standard length.

The variable z is introduced to move the parameter t outside of the big conjunction. Therefore, t is coded only once, instead of b' many times.

By definition of a condition, $\mathcal{M}_1 \models \gamma(a_i, \vec{s}, \vec{S})$ for each $i \in I$ such that $i \geq i'$. Therefore $\mathcal{M}_1$ satisfies

$$(\exists y \leq a_{i'})(\forall j \leq i) \bigwedge_{\ulcorner\delta\urcorner < b'} [\delta(a_j, \vec{s}y, \vec{S}) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(t)]$$

Since $\mathcal{M}_1 \models \mathsf{B}\Sigma_1^0$, there is some fixed $e \in M$ that witnesses the first existential above for every $i \in I$ such that $i \geq i'$. Then $(\vec{r}d, \vec{s}e, \vec{R}, \vec{S}, b')$ is our desired extension. ∎

The following lemma shows that one can add any second-order element to the domain of h. Here again, by symmetry, any second-order element can also be added to the range of h.

Lemma 7.4.13. Let $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ be a condition and $X \in S_0$. There is an extension $(\vec{r}, \vec{s}, \vec{R}X, \vec{S}Y, b')$ for some $b' \in M$ and $Y \in S_1$. ⋆

Proof. Let $b' > I$ be sufficiently small with respect to b and $D \subseteq I \times b'$ be the following set

$$\{(i, \ulcorner\delta\urcorner) \in I \times b' : \delta \text{ is } \Delta_0^0 \text{ and } \mathcal{M}_0 \models \delta(a_i, \vec{r}, \vec{R}X)\}$$

Again, D and $(I \times b') \setminus D$ are bounded and Σ_1^0-definable, so by Lemma 7.4.11, there is some $t < 2^{b' \times b'}$ such that $\mathrm{Ack}(t) \cap (I \times b') = D$. By choice of t, there is some $i' \in I$ such that for every $i \in I$ with $i \geq i'$, the structure $\mathcal{M}_0$ satisfies the formula

$$(\exists F \subseteq [0, \log a_i))(\forall j \leq i)(\forall v \leq \log\log a_i)$$
$$(v = a_j \to \bigwedge_{\ulcorner\delta\urcorner < b'} [\delta(a_j, \vec{r}, \vec{R}F) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(t)]$$

as witnessed by taking $F = X \cap [0, \log a_i)$.[28] For every $i \in I$ such that $i \geq i'$, $\mathcal{M}_0$ therefore satisfies the Δ^0_0-formula $\gamma(a_i, \vec{r}, \vec{R})$ defined by

28: It is not clear at first sight that $\mathcal{M}_0$ satisfies this formula, since δ is witnessed by $F = X \cap [0, \log a_i)$ instead of X. However, since the first-order parameters of δ are smaller than $\max(\log\log a_i, \vec{r})$, then the gödel number the formula δ evaluated on its parameters is smaller than $\log a_i$, hence its evaluation is left unchanged by replacing X with $X \cap [0, \log a_i)$.

$$(\exists F \subseteq [0, \log a_i))(\exists z \leq a_i)(\forall j \leq i)(\forall v \leq \log\log a_i)$$
$$(z = \mathbf{t} \wedge v = a_j \to \bigwedge_{\ulcorner\delta\urcorner < \mathbf{b'}} [\delta(a_j, \vec{r}, \vec{R}F) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(z)]$$

For each $i \in I$, the formula γ is written in a language enriched with symbol constants for b' and t. By a similar analysis to Lemma 7.4.12, if b' is sufficiently small with respect to b, then $\ulcorner\gamma\urcorner < b$. Thus by definition of a condition, for every $i \in I$ such that $i \geq i'$, $\mathcal{M}_1$ satisfies

$$(\exists F \subseteq [0, \log a_i))(\forall j \leq i)(\forall v \leq \log\log a_i)$$
$$(v = a_j \to \bigwedge_{\ulcorner\delta\urcorner < b'} [\delta(a_j, \vec{s}, \vec{S}F) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(t)]$$

Let $T \subseteq 2^{<M}$ be the Π^0_1 tree of all σ such that for every $i \in I$ with $i' \leq i \leq |\sigma|$, the set $F = \{s < \log a_i : \sigma(s) = 1\}$ witnesses the first existential of the previous formula. Since $\mathcal{M}_1 \models \mathsf{WKL}^*_0$, there is an infinite path Y through T in $\mathcal{M}_1$. Then $(\vec{r}, \vec{s}, \vec{R}X, \vec{S}Y, b')$ is our desired extension. ∎

We are now ready to prove Theorem 7.4.10. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. Let h be the function induced by $\mathcal{F}$. By Lemma 7.4.12 and Lemma 7.4.13, h is a bijection from $M \cup S_0$ to $M \cup S_1$.

We claim that h is an isomorphism. We only prove the case of addition. Let $+_0$ and $+_1$ be the interpretation of the addition symbol in (M, S_0) and (M, S_1), respectively. Given $u, v \in M$, consider the Δ^0_0-formula

$$\delta(a, x, y, z) \equiv x + y = z$$

Let $w = u +_0 v$, and let $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b) \in \mathcal{F}$ be a condition such that $u, v, w \in \vec{r}$. Since the formula δ is standard, then $\ulcorner\delta\urcorner \in \omega < b$, so by definition of a condition, for each $i \in I$,

$$\mathcal{M}_0 \models \delta(a_i, u, v, w) \text{ iff } \mathcal{M}_1 \models \delta(a_i, h(u), h(v), h(w))$$

Since $u +_0 v = w$, then $\mathcal{M}_0 \models \delta(a_i, u, v, w)$, so $\mathcal{M}_1 \models \delta(a_i, h(u), h(v), h(w))$, and therefore $h(u) +_1 h(v) = h(w) = h(u +_0 v)$. This completes the proof of Theorem 7.4.10. ∎

As an immediate consequence of Theorem 7.4.10, weak König's lemma is the maximal Π^1_2-problem which is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$.

Theorem 7.4.14 (Fiori-Carones et al [53])
Let P *be a* Π^1_2*-problem. Then* $\mathsf{RCA}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ *is* Π^1_1*-conservative over* $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ *iff* $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$.

Proof. First, by Theorem 7.4.7, $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$, so if $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$, $\mathsf{RCA}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$. We prove the other direction.

If $\mathsf{RCA}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$, then by Theorem 7.4.7 and a standard amalgamation argument (see Yokoyama [56]), $\mathsf{WKL}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$. Let $\mathcal{M} \models \mathsf{WKL}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ be a countable model. By Theorem 7.4.10, every coded ω-model of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ in $\mathcal{M}$ is elementarily equivalent to $\mathcal{M}$, hence satisfies P, so by Gödel's completeness theorem, $\mathsf{WKL}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ proves that every coded ω-model of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ satisfies P. By Π^1_1-conservation, $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ proves the same statement.

Let $\mathcal{M}$ be a countable model of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ and $A \in \mathcal{M}$ witness $\neg\mathsf{I}\Sigma^0_1$. By Theorem 4.3.2, $\mathcal{M}$ contains a coded ω-model $\mathcal{N}$ of WKL^*_0 with $A \in \mathcal{N}$. In particular, $\mathcal{N} \models \mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$, so $\mathcal{N} \models \mathsf{P}$. Again by Theorem 7.4.10, $\mathcal{N}$ is an elementary submodel of $\mathcal{M}$, so $\mathcal{M} \models \mathsf{P}$. By Gödel's completeness theorem, $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$. ■

7.5 Conservation over $\mathsf{B}\Sigma^0_2$

The system $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ plays an important role in reverse mathematics for two reasons. First, it characterizes the first-order part of some statements related to Ramsey's theorem for pairs [57]. Second, it is the highest level in the hierarchy of induction which satisfies Hilbert's program. Indeed, $\mathsf{I}\Sigma^0_2$ is not finitistically reducible, as it proves the consistency of $\mathsf{I}\Sigma^0_1$, which is a Π_1 statement not provable over $\mathsf{I}\Sigma^0_1$ (see Hájek and Pudlák [41, Theorem 4.33]). On the other hand, by Parsons, Paris and Friedman (see [58]), $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is $\forall\Pi^0_3$-conservative over RCA_0.[29] In particular, $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is a Π_2-conservative extension of PRA.

29: $\forall\Pi^0_n$ is the class of formulas starting with a universal set quantifier, followed by a Π^0_n formula. Every Π^1_1-formula is $\forall\Pi^0_n$ for some $n \in \mathbb{N}$.

Exercise 7.5.1. Let P be a Π^1_2 problem. Suppose that for every countable topped model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, and every $X \in S$ such that $\mathcal{M} \models X \in \operatorname{dom} \mathsf{P}$, there is a set $Y \subseteq M$ such that $\mathcal{M}[Y] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + (Y \in \mathsf{P}(X))$. Adapt the proof of Proposition 7.3.2 to show that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{P}$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ★

Conservation over RCA_0 involved first-jump control to build sets while preserving $\mathsf{I}\Sigma^0_1$. One would therefore expect conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ to involve second-jump control to preserve $\mathsf{B}\Sigma^0_2$. However, as mentioned in Section 4.1, effectivization of first-jump control can often be used to obtain simple proofs of jump preservations. First-jump control being usually significantly simpler than second-jump control, one usually prefers to use the former technique. Actually, as a consequence of the isomorphism theorem for $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$, in the context of Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$, effective first-jump control can be used without loss of generality (see Fiori-Carones et al. [53]).

Exercise 7.5.2. Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set $Y \subseteq M$. Let $G \subseteq M$ be such that $(G \oplus Y)' \leq_T Y'$.[30] Use Exercise 7.4.3 and Exercise 7.4.5 to show that $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ★

30: $\mathsf{Q} + \mathsf{I}\Sigma^0_1$ is enough to prove the existence of a universal Σ^0_1-formula. From it, we can define a robust notion of Turing jump X' as the set of all codes of true $\Sigma^0_1(X)$ formulas.

Recall that the Turing reduction is robust in models of RCA^*_0 (see Groszek and Slaman [40]). If $\mathcal{M} = (M, S) \models RCA_0 + \mathsf{B}\Sigma^0_2$ then its jump model $\mathcal{N} = (M, \Delta^0_2\text{-Def}(\mathcal{M}))$ satisfies RCA^*_0, so the Turing reduction is robust between Δ^0_2 sets in models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Effective constructions in the context of weak arithmetic raise an issue that already occurs in higher computability theory. Many effectiveness constructions are done inductively along the integers, satisfying a requirement at each step.

In the case of a non-standard model of weak arithmetic, some steps are non-standard, hence are preceded by infinitely many other steps.[31] If induction fails, it might be the case that the set of steps of the construction forms a proper cut, and thus that some requirement at a non-standard step is never satisfied. Even if the model is countable, since the construction is internal, one cannot fix a countable enumeration of the integers.

31: Models of weak arithmetic have common similarities with ordinals. Indeed, one can reason inductively among both, and a non-standard integer, like an infinite ordinal, is infinite from an external point of view, but there is no infinite decreasing sequence starting from it.

Consider for example Cohen forcing over a non-standard model $\mathcal{M} = (M, S)$. Let $(D_a)_{a \in M}$ be a collection of dense sets. The naive approach to the construction of a $\vec{D}$-generic set G would consist in letting $\sigma_0 = \epsilon$, and σ_{a+1} be the lexicographically least extension of σ_a belonging to D_a. If the dense sets are to complex with respect to the level of induction in $\mathcal{M}$, the set $I = \{a \in M : \sigma_a \text{ is defined }\}$ might be a proper cut, while the set $\{|\sigma_a| : a \in I\}$ will be cofinal in M.

To circumvent this problem, one resorts to a technique from higher computability theory called *Shore blocking*.[32] Suppose one proves that the collection $(D_a)_{a \in M}$ is dense in a strong sense: for every $b \in M$ and every $\sigma \in 2^{<M}$, there exists an extension $\tau \geq \sigma$ intersecting every $(D_a)_{a<b}$ simultaneously. One can then build a $\vec{D}$-generic set G by letting $\sigma_0 = \epsilon$, and σ_{a+1} be the lexicographically least extension of σ_a intersecting $(D_c)_{c<|\sigma_a|}$ simultaneously. Then, even if the set $I = \{a \in M : \sigma_a \text{ is defined }\}$ is a proper cut, the resulting set G will be $\vec{D}$-generic, as for every $c \in M$, there is a stage $a \in I$ such that $|\sigma_a| > c$, hence σ_{a+1} intersects D_c. The main difficulty of conservation theorems over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ consists of proving the blocking lemma.

32: The "blocking" terminology might be confusing. It should be understood as satisfying blocks of requirements simultaneously instead of one by one.

Our first proof of Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is based on a formalization in weak arithmetic by Hájek [59] of the low basis theorem from Jockusch and Soare [24].

Theorem 7.5.3 (Hájek [59])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $P \in [T]$ such that $(P \oplus Y)' \leq_T Y'$ and $\mathcal{M}[P] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.[33]

33: The proof of Theorem 7.5.3 is slightly more verbose than necessary, but it is more modular, in that it is easy to interleave other blocking lemmas to satisfy more requirements. This will be useful for Theorem 7.6.16.

Proof. Consider the notion of forcing whose *conditions* are pairs (σ, T_1) where

- T_1 is a primitive Y-recursive infinite subtree of T ;
- $\sigma \in 2^{<M}$ is a *stem* of T_1, that is, every element in T_1 is comparable with σ.

The *interpretation* of a condition (σ, T_1) is $[\sigma, T_1] = [T_1]$. A condition (τ, T_2) *extends* (σ, T_1) (written $(\tau, T_2) \leq (\sigma, T_1)$) if $\sigma \preceq \tau$ and $T_2 \subseteq T_1$. A *code* of a condition (σ, T_1) is a pair $\langle \sigma, a \rangle$ such that a is a primitive Y-recursive code for T_1.

We need to satisfy the following requirements for every $b \in M$:

- $\mathcal{T}_b$: $G \upharpoonright_b$ is decided[34]
- $\mathcal{R}_b$: $(G \oplus Y)' \upharpoonright_b$ is decided

34: Technically, this requirement is not necessary, as deciding $(G \oplus Y)'$ implies deciding G. However, explicitly satisfying this requirement will be convenient for the construction.

For this, we prove a blocking lemma to decide the jump, Lemma 7.5.4. Given a condition (σ, T_1) and $e \in M$, let

- $(\sigma, T_1) \Vdash \Phi^{G \oplus Y}_e(e)\downarrow$ if $\Phi^{\sigma \oplus Y}_e(e)\downarrow$;
- $(\sigma, T_1) \Vdash \Phi^{G \oplus Y}_e(e)\uparrow$ if for every $\tau \in T_1$, $\Phi^{\tau \oplus Y}_e(e)\uparrow$;
- $(\sigma, T_1) \Vdash \rho \prec (G \oplus Y)'$ for some $\rho \in 2^{<M}$ if for every $e < |\rho|$, if $\rho(e) = 1$ then $(\sigma, T_1) \Vdash \Phi^{G \oplus Y}_e(e)\downarrow$, and if $\rho(e) = 0$ then $(\sigma, T_1) \Vdash \Phi^{G \oplus Y}_e(e)\uparrow$.

Note that the predicate $(\sigma, T_1) \Vdash \rho \prec (G \oplus Y)'$ is $\Pi^0_1(Y)$ uniformly in σ, T_1 and ρ.

Lemma 7.5.4. For every condition (σ, T_1) and $b \in M$, there is an extension (τ, T_2) and some M-coded $\rho \in 2^b$ such that $(\tau, T_2) \Vdash \rho \prec (G \oplus Y)'$. ★

Proof. Let U be the set of all $\rho \in 2^b$ such that the tree

$$T_\rho = \{\tau \in T_1 : (\forall e < b)(\rho(e) = 0 \rightarrow \Phi_e^{\rho \oplus Y}(e)\uparrow)\}$$

is infinite. U is $\Pi^1_0(Y)$ and hence M-finite, and it is non-empty as it contains the string $1111\dots$.

Let $\rho \in U$ be its lexicographically smallest element. For every $e < b$ such that $\rho(e) = 1$, the minimality of ρ implies that the set of $\tau \in T_\rho$ such that $\Phi_e^{\tau \oplus Y}(e)\uparrow$ is M-finite, so there is a level ℓ_e such that for every $\tau \in T_\rho \cap 2^{\ell_e}$, $\Phi_e^{\tau \oplus Y}(e)\downarrow$. The set $\{e < b : \rho(e) = 1\}$ is M-finite, so by $\mathsf{B}\Sigma^0_1$, there is an upper-bound ℓ of all the ℓ_e's. Finally, by Lemma 7.3.4, there is a node $\tau \in T_\rho \cap 2^\ell$ such that $T_2 = \{\mu \in T_\rho : \mu \textit{ is comparable with } \tau\}$ is M-infinite.

We claim that $(\tau, T_2) \Vdash \rho \prec (G \oplus Y)'$. Fix some $e < b$. Suppose $\rho(e) = 0$. Then $\Phi_e^{\mu \oplus Y}(e)\uparrow$ for every $\mu \in T_2$ since $T_2 \subseteq T_\rho$. Hence, $(\tau, T_2) \Vdash \Phi_e^{G \oplus Y}(e)\uparrow$. Suppose $\rho(e) = 1$. The definition of τ ensure that $\Phi_e^{\tau \oplus Y}(e)\downarrow$, so $(\tau, T_2) \Vdash \Phi_e^{G \oplus Y}(e)\downarrow$. ■

We are now ready to prove Theorem 7.5.3.

Construction. We build a decreasing sequence (σ_s, T_s) of conditions and then take G for the union of the σ_s. We also build an increasing sequence (ρ_s) such that $(G \oplus Y)'$ will be the union of the ρ_s. Initially, let $\sigma_0 = \sigma'_0 = \epsilon$ and $T_0 = T$. During the construction, we will ensure that $\langle \sigma_s, T_s \rangle, |\rho_s| \leq s$. Each stage will be either of type $\mathcal{T}$, or of type $\mathcal{R}$. The stage 0 is of type $\mathcal{T}$.

Assume that (σ_s, T_s) and ρ_s are already defined. Let $s_0 < s$ be the latest stage at which we switched the stage type. We have two cases.

Case 1: s is of type $\mathcal{T}$. If there a code $\langle \tau, \hat{T} \rangle \leq s$ such that $(\tau, \hat{T}) \leq (\sigma_s, T_s)$ and $|\tau| \geq s_0$, then let $\sigma_{s+1} = \tau$, $T_{s+1} = \hat{T}$, $\rho_{s+1} = \rho_s$ and let $s + 1$ be of type $\mathcal{R}$. Otherwise, the elements are left unchanged and we go to the next stage.

Case 2: s is of type $\mathcal{R}$. If there a code $\langle \tau, \hat{T} \rangle \leq s$ such that $(\tau, \hat{T}) \leq (\sigma_s, T_s)$ and $(\sigma_s, \hat{T}) \Vdash \rho \prec (G \oplus Y)'$ for some $\rho \in 2^{s_0}$, then let $\sigma_{s+1} = \tau$, $T_{s+1} = \hat{T}$, $\rho_{s+1} = \rho$ and let $s+1$ be of type $\mathcal{T}$. Otherwise, the elements are left unchanged and we go to the next stage.

This completes the construction.

Verification. Since the size of σ_s, ρ_s and the index of T_s are bounded by s, there is a $\Delta^0_1(Y')$-formula $\phi(s)$ stating that the construction can be pursued up to stage s. Our construction implies that the set $\{s | \phi(s)\}$ is $\Delta^0_1(Y')$ and forms a cut, so by $\mathsf{I}\Delta^0_1(Y')$, the construction can be pursued at every stage.

Let $G = \bigcup_{s \in M} \sigma_s$. By Lemma 7.3.4 and Lemma 7.5.4, each type of stage changes M-infinitely often. Thus, $\{|\sigma_s| : s \in M\}$ and $\{|\rho_s| : s \in M\}$ are M-infinite. In particular, G is an M-regular path in T and $Y' \geq_T (G \oplus Y)'$. By Exercise 7.5.2, $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

This completes the proof of Theorem 7.5.3.

35: Exercise 7.5.1 and Corollary 7.5.5 easily adapt to prove that for every $n \geq 2$ that $\mathsf{WKL}_0 + \mathsf{I}\Sigma^0_n$ and $\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_n$ are Π^1_1-conservative extensions of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n$, respectively.

Corollary 7.5.5 (Hájek [59])
$\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.[35]

Proof. Immediate by Theorem 7.5.3 and Exercise 7.5.1. ■

We have seen in Theorem 7.3.8 that Δ^0_2 sets are indistinguishable from arbitrary sets from the viewpoint of models of RCA_0, in that every countable model of RCA_0 can be ω-extended into another model of RCA_0 relative to which a fixed set becomes Δ^0_2. This is not true anymore when considering models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Indeed, by Theorem 7.2.11and Exercise 7.2.12, given a countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ and a non-M-regular set $A \subseteq M$, there is no ω-extension $\mathcal{N} \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ of $\mathcal{M}$ relative to which A is Δ^0_2, since it would imply M-regularity of A. On the other hand, Belanger [51] proved a formalized Friedberg jump inversion theorem with some extra assumptions on the set A.

Theorem 7.5.6 (Belanger [51])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y, and $A \subseteq M$ be a set such that $\mathcal{M}[A \oplus Y'] \models \mathsf{RCA}^*_0$. Then there is a set $G \subseteq M$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ and $A \oplus Y' \equiv_T (G \oplus Y)'$

Proof. Based on Shoenfield's limit lemma [7], we will construct a function $f : \mathbb{N}^2 \to 2$ such that for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists and equals $A(x)$. We are therefore going to build directly the function f by forcing, and let G be the graph of f.

36: Contrary to Theorem 7.3.8, the set $A \oplus Y'$ is M-regular, so we can work with pairs (g, a) and lock a non-standard number of columns simultaneously.

Consider the notion of forcing whose *conditions* is a pairs (g, a)[36], such that

- $g \subseteq M^2 \to \{0, 1\}$ is a partial function with two parameters whose domain is M-finite, representing an initial segment of the function f that we are building.
- $a \in M$ is the number of "locked" columns, meaning that from now on, when we extend the domain of g with a new pair (x, y), if $x < a$ then $g(x, y) = (A \oplus Y')(x)$.

The *interpretation* $[g, a]$ of a condition (g, a) is the class of all partial or total functions $h \subseteq M^2 \to 2$ such that

(1) $g \subseteq h$, i.e. $\operatorname{dom} g \subseteq \operatorname{dom} h$ and for all $(x, y) \in \operatorname{dom} g$, $g(x, y) = h(x, y)$;
(2) for all $(x, y) \in \operatorname{dom} h \setminus \operatorname{dom} g$, if $x < a$, then $h(x, y) = (A \oplus Y')(x)$.

A condition (h, b) *extends* (g, a) (denoted $(h, b) \leq (g, a)$) if $b \geq a$ and $h \in [g, a]$.

We will need to satisfy three kind of requirements for every $b \in M$:

- $\mathcal{T}_b$: $b^2 \subseteq \operatorname{dom} f$
- $\mathcal{R}_b$: $(f \oplus Y)' \upharpoonright_b$ is decided
- $\mathcal{S}_b$: $(\forall a < b) \lim_y f(a, y)$ exists

For this, we prove two lemmas, Lemma 7.5.7 and Lemma 7.5.8, stating that the set of conditions forcing $\mathcal{T}_b$ and $\mathcal{R}_b$ is dense for every $b \in M$. Density of the requirement $\mathcal{S}_b$ simply consists, given a condition (g, a), of taking the extension $(g, \max(a, b))$.

Lemma 7.5.7. For every condition (g, a) and $b \in M$, there is an extension $(h, a) \le (g, a)$ such that $b^2 \subseteq \operatorname{dom} h$. ★

Proof. Since $A \oplus Y'$ is M-regular, the string $\sigma = (A \oplus Y')\upharpoonright_a$ is M-coded. By Δ^0_0-comprehension, the set $h = g \cup \{(x, y, \sigma(x)) \in b^2 \times 2 : (x, y) \notin \operatorname{dom} g\}$ is M-coded. By construction, $h \in [g, a]$ and $b^2 \subseteq \operatorname{dom} h$, so (h, a) is the desired extension. ■

Given a condition (g, a) and $e \in M$, let

- $(g, a) \Vdash \Phi^{f\oplus Y}_e(e)\downarrow$ if $\Phi^{g\oplus Y}_e(e)\downarrow$;
- $(g, a) \Vdash \Phi^{f\oplus Y}_e(e)\uparrow$ if for every finite $h \in [g, a]$, $\Phi^{h\oplus Y}_e(e)\uparrow$;
- $(g, a) \Vdash \rho \prec (f \oplus Y)'$ for some $\rho \in 2^{<M}$ if for every $e < |\rho|$, if $\rho(e) = 1$ then $(g, a) \Vdash \Phi^{f\oplus Y}_e(e)\downarrow$, and if $\rho(e) = 0$ then $(g, a) \Vdash \Phi^{f\oplus Y}_e(e)\uparrow$.

Note that the predicate $(g, a) \Vdash \rho \prec (f \oplus Y)'$ is $\Delta^0_2(Y)$ uniformly in g, a and ρ.

Lemma 7.5.8. For every condition (g, a) and $b \in M$, there is an extension $(h, a) \le (g, a)$ and some M-coded $\rho \in 2^b$ such that $(h, a) \Vdash \rho \prec (f \oplus Y)'$.★

Proof. Let U be the set of all $\rho \in 2^b$ such that

$$(\exists h \in [g, a])(\exists t)(\forall e < b)(\rho(e) = 1 \rightarrow \Phi^{h\oplus Y}_e(e)[t]\downarrow)$$

Note that U is $\Sigma^0_1(Y)$, hence is M-finite. Moreover, U is non-empty, as it contains the string $000\dots$. Let $\rho \in U$ be the lexicographically maximal element, and let $h \in [g, a]$ witness that $\rho \in U$.

We claim that (h, a) forces $\rho \prec (G \oplus Y)'$. Fix some $e < b$. Suppose $\rho(e) = 1$. Then $\Phi^{h\oplus Y}_e(e)\downarrow$, hence $(h, a) \Vdash \Phi^{f\oplus Y}_e(e)\downarrow$. Suppose $\rho(e) = 0$. The maximality of ρ ensures that for every $\hat{h} \in [h, a]$, $\Phi^{\hat{h}\oplus Y}_e(e)\uparrow$. It follows that $(h, a) \Vdash \Phi^{f\oplus Y}_e(e)\uparrow$. ■

We are now ready to prove Theorem 7.5.6.

Construction. We will build a decreasing sequence (g_s, a_s) of conditions and then take for f the union of the g_s. We will also build an increasing sequence (ρ_s) such that $(f \oplus Y)'$ will be the union of the ρ_s. Initially, let $g_0 = \rho_0 = \epsilon$ and $a_0 = 0$. Each stage will be either of type $\mathcal{T}$, of type $\mathcal{R}$ or of type $\mathcal{S}$. The stage 0 is of type $\mathcal{T}$.

Assume that (g_s, a_s) and ρ_s are already defined. Let $s_0 < s$ be the latest stage at which we switched the stage type. We have three cases.

Case 1: s is of type $\mathcal{T}$. If there exists some $h \in 2^{\le s \times \le s}$ such that $(h, a_s) \le (g_s, a_s)$ and $s_0 \times s_0 \subseteq \operatorname{dom} h$, then let $g_{s+1} = h$, $a_{s+1} = a_s$, $\rho_{s+1} = \rho_s$, and let $s + 1$ be of type $\mathcal{R}$. Otherwise, the elements are left unchanged and we go to the next stage.

Case 2: s is of type $\mathcal{R}$. If there exists some $h \in 2^{\le s \times \le s}$ and some $\mu \in 2^{s_0}$ such that $(h, a_s) \le (g_s, a_s)$, and $(h, a_s) \Vdash \mu \prec (f \oplus Y)'$, then let $g_{s+1} = h$, $a_{s+1} = a_s$, $\rho_{s+1} = \mu$, and let $s + 1$ be of type $\mathcal{S}$. Otherwise, the elements are left unchanged and we go to the next stage.

Case 3: s is of type $\mathcal{S}$. Let $g_{s+1} = g_s$, $a_{s+1} = s$, $\rho_{s+1} = \rho_s$, and let $s + 1$ be of type $\mathcal{T}$. This completes the construction.

Verification. Since the size of g_s, a_s and ρ_s are bounded by s, there is a $\Delta^0_1(A \oplus Y')$-formula $\phi(s)$ stating that the construction can be pursued up to stage s. Our construction implies that the set $\{s|\phi(s)\}$ is a cut, so since $\mathcal{M}[A \oplus Y'] \models \mathsf{I}\Delta^0_1$, the construction can be pursued at every stage.

Let $f = \bigcup_{s \in M} g_s$. By Lemma 7.5.7 and Lemma 7.5.8, each type of stage changes M-infinitely often. Thus, $\operatorname{dom} f = M^2$, and $\{a_s : s \in M\}$ and $\{|\rho_s| : s \in M\}$ are both cofinal in M. It follows that f is stable and $A \oplus Y' \geq_T (f \oplus Y)'$. Since $\mathcal{M}[A \oplus Y'] \models \mathsf{RCA}^*_0$, then $\mathcal{M}[(f \oplus Y)'] \models \mathsf{RCA}_0$, so by Exercise 7.4.3, $\mathcal{M}[f] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Conversely, since $\lim_y f(\cdot, y) = A \oplus Y'$, then $A \oplus Y' \equiv_T (f \oplus Y)'$. This completes the proof of Theorem 7.5.6. ∎

We now prove that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Recall that thanks to the characterization of COH in terms of Δ^0_2 approximations of paths through infinite Δ^0_2 binary trees (Exercise 3.4.3), there exist two main ways to build solutions to instances of COH: either picking a path, and constructing a Δ^0_2 approximation of it, or directly building a cohesive set through computable Mathias forcing. We shall start with the former approach. Belanger [51] proved that the above characterization holds over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Exercise 7.5.9 (Belanger [51]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$. Show that $\mathcal{M} \models \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ iff $(M, \Delta^0_2\text{-Def}(\mathcal{M})) \models \mathsf{WKL}^*_0$. ★

Theorem 7.5.10 (Chong, Slaman and Yang [57])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable topped model and $\vec{R} = R_0, R_1, \dots$ be a uniform sequence in S. Then there is an infinite $\vec{R}$-cohesive set $C \subseteq M$ such that $\mathcal{M}[C] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Proof. Say $\mathcal{M}$ is topped by a set Y. Given $\sigma \in 2^{<M}$, let

$$R_\sigma = \bigcap_{\sigma(n)=0} \overline{R}_n \bigcap_{\sigma(n)=1} R_n$$

Let $T = \{\sigma \in 2^{<M} : (\exists x > |\sigma|) x \in R_\sigma\}$. The tree T is infinite and $\Sigma^0_1(\mathcal{M})$. Since $(M, \Delta^0_2\text{-Def}(\mathcal{M})) \models \mathsf{RCA}^*_0$, by Theorem 7.4.7, there is a path $P \in [T]$ such that $\mathcal{M}[P \oplus Y'] \models \mathsf{RCA}^*_0$. By Theorem 7.5.6, there is a set $G \subseteq M$ such that $P \oplus Y' \leq_T (G \oplus Y)'$ and $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Let $(P_s)_{s \in M}$ be a Δ^0_2 approximation of P in $\mathcal{M}[G]$. Let $(x_a)_{a \in M}$ be inductively defined as follows: First, $x_0 = 0$. Given x_a, let $\langle s, x \rangle$ be the least tuple such that $s, x > x_a$ and $x \in R_{P_s \upharpoonright x_a}$. Such a tuple exists, since by $\mathsf{B}\Sigma^0_2$, there is some $s > x_a$ such that $P_s \upharpoonright x_a = P \upharpoonright x_a$, and that $R_{P \upharpoonright x_a}$ is infinite. Then let $x_{a+1} = x$. This completes the construction.

By Σ^0_1-induction, x_a is defined for every $a \in M$. Let $D = \{x_a : a \in M\}$. We claim that D is $\vec{R}$-cohesive. Indeed, given $a \in M$, by $\mathsf{B}\Sigma^0_2$, there is some $k > a$ such that for every $t > k$, $P_t \upharpoonright a = P \upharpoonright_a$. For every $t > k$, $x_{t+1} \in R_{P_s \upharpoonright x_t}$ for some $s > x_t$. Since $s > x_t > t > k > a$, $R_{P_s \upharpoonright x_t} \subseteq R_{P_s \upharpoonright a} = R_{P \upharpoonright a}$, so for all but finitely many $t \in M$, $x_t \in R_{P \upharpoonright a}$.

Since D is Σ^0_1, it contains an infinite Δ^0_1 subset $C \subseteq D$. In particular, $C \in \mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, so $\mathcal{M}[C] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ∎

Corollary 7.5.11 (Chong, Slaman and Yang [57])
$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ *is a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Proof. Immediate by Theorem 7.5.10 and Exercise 7.5.1. ■

There exists another more direct construction of an $\vec{R}$-cohesive set by Mathias forcing, which does not involve the formalized Friedberg jump inversion theorem.

Exercise 7.5.12 (Le Houérou, Levy Patey and Yokoyama [60]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y, and let $\vec{R} = R_0, R_1, \dots$ be a uniform sequence in S. Let P be as in the proof of Theorem 7.5.10. A *condition* is a pair (σ, a) where $\sigma \in 2^{<M}$ and $a \in M$. The *interpretation* $[\sigma, a]$ of a condition (σ, a) is the class of all G such that $\sigma \prec G$ and $G \subseteq \sigma \cup R_{P \upharpoonright a}$. In other words, the interpretation of (σ, a) is the interpretation of the Mathias condition $(\sigma, R_{P\upharpoonright a} \setminus \{0, \dots, |\sigma|\})$. Build a $\Delta^0_1(P \oplus Y')$ infinite decreasing sequence of conditions while deciding the jump as in the proof of Theorem 7.5.6. ★

Recall that by Theorem 4.5.2, if a Σ^0_2 set A is co-hyperimmune, then it admits an infinite low subset. This theorem was then used by Hirschfeldt and Shore [20] to prove that every infinite computable stable linear order admits an infinite ascending or descending sequence of low degree (see Exercise 4.5.4). The proof of Theorem 4.5.2 does not seem to be formalizable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ because of Shore blocking. However, Chong, Slaman and Yang [57] used the transitive features of linear orders to prove that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{SADS}$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, where SADS is the Π^1_2-problem whose instances are stable linear orders, and solutions are infinite ascending or descending sequences.[37]

37: Actually, SADS implies $\mathsf{B}\Sigma^0_2$ over RCA_0, but the proof is non-trivial and involved a model-theoretic argument. See Hirschfeldt and Shore [20] and Chong, Lempp and Yang [61].

Exercise 7.5.13 (Chong, Slaman and Yang [57]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y. Let $\mathcal{L} = (M, <_{\mathcal{L}})$ be a computable stable linear order in $\mathcal{M}$.

1. Show that $\mathcal{M}$ does not contain any infinite descending sequence, then there is an M-regular infinite ascending sequence $G \subseteq M$ such that $(G \oplus Y)' \leq_T Y'$.
2. Deduce that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{SADS}$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ★

7.6 Shore blocking and BME

The most naive way to prove a blocking lemma given a family $(D_a)_{a<b}$ of dense sets would be to start from a condition p_0, and then inductively letting p_{a+1} be an extension of p_a in D_a for every $a < b$. Then, p_b would be an extension simultaneously intersecting all the dense sets simultaneously. However, as explained above, in models of weak arithmetic, the set $I = \{a : p_a \text{ is defined }\}$ might be a proper cut bounded by b. We therefore used some combinatorial features of each construction to prove conservation theorems over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. As usual, these can often be formulated as properties of the forcing questions.

The main concern for Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is to prove a blocking lemma to decide an initial segment of the jump. If an extension witnessing a positive answer to the forcing question can be found uniformly in the condition, then the naive sequential approach holds.

Definition 7.6.1. Let $(\mathbb{P}, \leq)$ be a notion of forcing and $n \geq 1$. A forcing question is *uniformly Σ^0_n-preserving* if for every Σ^0_n formula $\varphi(G, x, y)$, there is a Σ^0_n set $W \subseteq \mathbb{P} \times \mathbb{N} \times \mathbb{P} \times \mathbb{N}$ such that

- For every $(p, n, q, m) \in W$, $q \leq p$ and q forces $\varphi(G, m, n)$;
- For every condition $p \in \mathbb{P}$ and $n \in \mathbb{N}$, $p \mathrel{?\!\vdash} \exists x \varphi(G, x, n)$ if and only if $(p, n, q, m) \in W$ for some $q \leq p$ and $m \in \mathbb{N}$. ◇

Note that any uniformly Σ^0_n-preserving forcing question is Σ^0_n-preserving.[38]

38: Uniform Σ^0_n-preservation has two levels of uniformity: deciding a Σ^0_n-formula is Σ^0_n uniformly in the conditions, and if the forcing question holds, then one can find an extension witnessing the positive answer uniformly.

This assumes of course that there is a notion of computability over forcing conditions, which can be obtained by manipulating conditions through their codes.

Theorem 7.6.2
Let $\mathcal{M} = (M, S) \models \mathsf{Q}+\mathsf{I}\Sigma^0_1$ be a countable model topped by Y and let $(\mathbb{P}, \leq)$ be a notion of forcing with a uniformly Σ^0_1-preserving forcing question. For every condition $p \in \mathbb{P}$ and $b \in M$, there is an extension $q \leq p$ and some $\rho \in 2^{<M}$ of length b such that q forces $\rho \prec (G \oplus Y)'$.

Proof. Let $\varphi(G, F, y)$ be the following $\Sigma^0_1(\mathcal{M})$-formula, where F is a first-order variable coding a set

$$(\exists t)(F \subseteq \{0, \dots, b-1\} \wedge \operatorname{card} F = y \wedge (\forall e \in F)\Phi^{G \oplus Y}_e(e)[t]\downarrow)$$

Let W be the $\Sigma^0_1(\mathcal{M})$ set witnessing that the function is uniformly Σ^0_1-preserving. Let U be the $\Sigma^0_1(\mathcal{M})$ set of all $F \subseteq \{0, \dots, b-1\}$ such that there is some $k \in M$ and a sequence $\langle p_0, F_0, \dots, p_{k-1}, F_{k-1}, p_k \rangle$ satisfying

- $p_0 = p$; $F = F_{k-1}$;
- $(p_s, s, p_{s+1}, F_s) \in W$ for every $s < k$.

We claim that $\emptyset \in U$. Indeed, $p \mathrel{?\!\vdash} (\exists F)\varphi(G, F, 0)$, so there is some F such that $\operatorname{card} F = 0$ and some $q \leq p$ such that $(p, 0, q, F) \in W$. In particular, $F = \emptyset$, and the sequence $(p, \emptyset, q)$ witnesses that $\emptyset \in F$.

By Exercise 7.2.3, there is a maximal element $F \in U$ for inclusion. Let $\rho \in 2^b$ be such that $\{e < b : \rho(e) = 1\} = F$ and let $\langle p_0, F_0, \dots, p_{k-1}, F_{k-1}, p_k \rangle$ witness that $F \in U$. By definition of W, p_k forces $\varphi(G, F, k-1)$, and by maximality of F, $p_k \mathrel{?\!\nvdash} (\exists F)\varphi(G, F, k)$. By definition of the forcing question, there is an extension $q \leq p_k$ forcing $(\forall F)\neg\varphi(G, F, k)$.

We claim that q forces $\rho \prec (G \oplus Y)'$. By definition of φ, for every $e \in F$, p_k forces $\Phi^{G \oplus Y}_e(e)\downarrow$. Let $e < b$ be such that $e \notin F$. There is no extension of q forcing $\Phi^{G \oplus Y}_e(e)\downarrow$, otherwise $F \cup \{e\}$ would contradict the fact that q forces $\neg\varphi(G, F, k)$. Thus, q forces $\Phi^{G \oplus Y}_e(e)\uparrow$. This completes the proof of Theorem 7.6.2. ■

Exercise 7.6.3. Show that Cohen forcing admits a uniformly Σ^0_1-preserving forcing question. ★

Exercise 7.6.4. Let $(\mathbb{P}, \leq)$ be the notion of forcing of Theorem 7.5.6, and given $a \in M$, let $\mathbb{P}_a$ be the set of conditions of the form (g, a).

1. Show that for every $a \in M$, $(\mathbb{P}_a, \leq)$ admits a uniformly Σ^0_1-preserving forcing question.

2. Show that if a condition (g, a) forces a Σ^0_1 or a Π^0_1 property over $(\mathbb{P}_a, \leq)$, then so does it over $(\mathbb{P}, \leq)$.
3. Deduce the existence of a blocking lemma to decide the jump for $(\mathbb{P}, \leq)$.
★

Many forcing questions appearing in practice are not Σ^0_1-uniform. Thankfully, it often represents a dividing line at one of the extremes of Figure 7.2. In this case again, one can prove a blocking lemma to decide an initial segment of a the jump.

Definition 7.6.5. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*extremal* if for every formula $\varphi \in \Gamma$ and every condition $p \in \mathbb{P}$, if $p \mathrel{?\vdash} \varphi(G)$ then p forces $\varphi(G)$. ◊

By extension, we say that a forcing question for Σ^0_n-formulas is Π^0_n-*extremal* if for every Σ^0_n-formula φ and every condition $p \in \mathbb{P}$, if $p \mathrel{?\nvdash} \varphi(G)$, then p forces $\neg\varphi(G)$. Many notions of forcing considered in practice admit a Σ^0_1-preserving forcing question which is Π^0_1-extremal. In this case, one can obtain an abstract blocking lemma to decide the jump.

Theorem 7.6.6
Let $\mathcal{M} = (M, S) \models \mathsf{Q}+\mathsf{I}\Sigma^0_1$ be a countable model topped by Y and let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving Π^0_1-extremal forcing question. For every condition $p \in \mathbb{P}$ and $b \in M$, there is an extension $q \leq p$ and some $\rho \in 2^{<M}$ of length b such that q forces $\rho \prec (G \oplus Y)'$.

Proof. Consider the following set

$$U = \{\rho \in 2^b : q \mathrel{?\vdash} (\exists t)(\forall e < b)(\rho(e) = 1 \to \Phi^{G\oplus Y}_e(e)[t]\downarrow)\}$$

The set U is $\Sigma^0_1(\mathcal{M})$ since the forcing question is Σ^0_1-preserving. Moreover, U is non-empty, as it contains the string $000\dots$. By Exercise 7.2.3, there is a lexicographically maximal element $\rho \in U$. By maximality, for every $e' < |\sigma|$ such that $\sigma(e') = 0$,

$$p \mathrel{?\nvdash} (\exists t)(\forall e < b)((\rho(e) = 1 \vee e = e') \to \Phi^{G\oplus Y}_e(e)[t]\downarrow)$$

so since the forcing question is Π^0_1-extremal, p forces

$$(\forall t)(\exists e < b)((\rho(e) = 1 \vee e = e') \wedge \Phi^{G\oplus Y}_e(e)[t]\uparrow)$$

Since $\rho \in U$, there is an extension $q \leq p$ and some $t \in \mathbb{N}$ such that q forces $(\forall e < b)(\rho(e) = 1 \to \Phi^{G\oplus Y}_e(e)[t]\downarrow)$. In particular, for every $e' < |\sigma|$ such that $\sigma(e') = 0$, q forces $\Phi^{G\oplus Y}_e(e)\uparrow$. It follows that q forces $\rho \prec (G \oplus Y)'$. This completes the proof of Theorem 7.6.6. ■

Exercise 7.6.7. Show that Theorem 7.6.6 also holds with a Σ^0_1-preserving Σ^0_1-extremal forcing question. ★

Recall that Ramsey's theorem for pairs can be decomposed into the cohesiveness principle (COH) and the pigeonhole principle for Δ^0_2 instances ($\mathsf{RT}^{1\prime}_2$). By Corollary 7.5.11 and an amalgamation theorem of Yokoyama [56], $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ iff so is $\mathsf{RCA}_0 + \mathsf{RT}^{1\prime}_2$. One would naturally want to adapt the proof that $\mathsf{RT}^{1\prime}_2$ admits a weakly low basis

(Theorem 4.7.5). However, the natural forcing question for the pigeonhole principle is neither uniformly Σ^0_1-preserving, nor extremal. It is therefore not clear how to prove a blocking lemma deciding the jump.

Question 7.6.8. Is $\mathsf{RCA}_0+\mathsf{RT}^2_2$ a Π^1_1-conservative extension of $\mathsf{RCA}_0+\mathsf{B}\Sigma^0_2$?★

As mentioned, the forcing question for the pigeonhole principle is not uniformly Σ^0_1-preserving, but enjoys a weaker uniformity property: if the answer to a Σ^0_1 question is positive, then one can effectively find a finite set of *pre-conditions*[39], one of each being a valid condition forcing the Σ^0_1 property. Successive applications of the forcing question to prove a blocking lemma then yields a c.e. tree of bounded depth, motivating the following definition.

39: A *Mathias pre-condition* is a pair (σ, X), where X is not longer required to be infinite. Given a Turing ideal $\mathcal{M}$ coded by a set M, the set of all Mathias pre-conditions over $\mathcal{M}$ is M-computable, while the set of Mathias conditions over $\mathcal{M}$ is not.

Definition 7.6.9. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a c.e. tree.

- A *monotone enumeration* of T is a uniformly computable sequence of finite coded[40] trees $T_0, T_1, \dots$ such that $T_0 = \{\epsilon\}$, $\bigcup_s T_s = T$ and for every stage s such that $T_{s+1} \neq T_s$, every node in $T_{s+1} \setminus T_s$ is an immediate extension of a leaf in T_s.
- The tree T is *k-bounded* if every node in T has length at most k. A tree is *bounded* if it is k-bounded for some $k \in \mathbb{N}$.[41] ◊

40: A monotone enumeration can be represented as a sequence of integers, each of them being the canonical code of a finite tree. Thus, the complete information about each tree is known.

41: Technically, the tree being Σ^0_1, it may not belong to the model. However, a Σ^0_1 tree is k-bounded if at any stage, it contains nodes of length at most k.

A monotone enumeration of a tree is such that all the immediate successors of a node are enumerated in one block at the same stage. Therefore, it is not possible to add immediate children at a later stage. On the other hand, it is not possible to decide ahead of time whether a node is a leaf or not. An easy induction over k shows that every k-bounded Σ^0_1 tree with a monotone enumeration is finite. Let BME_* be the Π^1_2-problem whose instances are enumerations of k-bounded Σ^0_1 trees for some $k \in \mathbb{N}$, and whose solutions are canonical codes for the tree.[42]

42: Given a monotone enumeration $(T_s)_{s\in\mathbb{N}}$, a stage s is *expansionary* if $T_{s+1} \neq T_s$. Over RCA^*_0, BME_* is equivalent to stating that the expansionary stages of a bounded monotone enumeration are bounded. Indeed, letting $s \in \mathbb{N}$ be such a bound, then $T_s = T$, but T_s is finite, hence so is T. On the other direction, if T is finite, then for every $\sigma \in T$, there is a stage s such that $\sigma \in T_s$. By $\mathsf{B}\Sigma^0_1$, there is a uniform bound on such stages.

Exercise 7.6.10 (Chong, Slaman and Yang [27]). Show that $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_2 \to \mathsf{BME}_*$. ★

Over RCA_0, the Bounded Monotone Enumeration principle and $\mathsf{B}\Sigma^0_2$ are incomparable, and their conjunction is strictly weaker than $\mathsf{I}\Sigma^0_2$. In fact, BME_* happens to be equivalent to multiple existing principles, and therefore has an arguably natural proof-theoretic strength.

Exercise 7.6.11 (Kreuzer and Yokoyama [62]). A formula $\phi(x, y)$ represents a partial function if $(\forall x, y, z)(\phi(x, y) \wedge \phi(x, z) \to y = z)$. A string $\sigma \in \mathbb{N}^{<\mathbb{N}}$ is an *approximation*[43] of a partial function $\phi(x, y)$ if

$$(\forall i < |\sigma| - 1)(\forall x, y)[(x < \sigma(i) \wedge \phi(x, y)) \to y < \sigma(i+1)]$$

43: The notion was introduced by Paris and Hájek [63], who proved that $\mathsf{B}\Sigma^0_2$ and $\mathsf{P}\Sigma^0_1$ are incomparable over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

Given a collection of formulas Γ, let $\mathsf{P}\Gamma$ be the scheme "For every partial function $\phi \in \Gamma$ and every length $k \in \mathbb{N}$, there is an approximation of length k." Show that $\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \mathsf{BME}_* \leftrightarrow \mathsf{P}\Sigma^0_1$. ★

The Bounded Monotone Enumeration principle can also be understood in terms of well-foundedness of ordinals. It requires first to fix a representation of ordinals. By Cantor normal form, every ordinal α can be uniquely written as $\omega^{\beta_0}c_0 + \omega^{\beta_1}c_1 + \dots + \omega^{\beta_{k-1}}c_{k-1}$, where $c_0, \dots, c_{k-1}$ are non-zero natural numbers, and and $\beta_0 > \beta_1 > \dots > \beta_{k-1} > 0$ are ordinals. Based on this normal form, every ordinal less than ϵ_0[44] can be represented by a finite tree of

44: Recall that ϵ_0 is the least fixpoint of the operation $\alpha \mapsto \omega^\alpha$. In particular,

$$\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

coefficients. To simplify manipulation, it is more convenient to work with *regular trees*, that is, finite trees such that the set of immediate successors of a node is an initial segment of $\mathbb{N}$, together with an evaluation map which associates to each node a coefficient. Using this representation, the map $(\vec{\beta}, \vec{c}) \mapsto \sum \omega^{\beta_i} c_i$ and the order $\leq$ are provably Δ^0_1 in $\mathsf{Q} + \mathsf{I}\Sigma^0_1$. See Hájek and Pudlák [41, p. II.3] for a formal development of ordinals over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

Given an ordinal $\alpha \leq \epsilon_0$, let $\mathsf{WF}(\alpha)$ be the statement "α is well-founded", that is, there is no infinite decreasing sequence of ordinals smaller than α. Proving that α is well-founded for some large ordinals requires some non-trivial amount of induction.[45] Actually, $\mathsf{WF}(\omega^\omega)$ is equivalent to BME_* over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

45: The statement

$$\forall a(\mathsf{WF}(\omega^a) \to \mathsf{WF}(\omega^{a+1}))$$

is provable over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$. It follows that in any model $\mathcal{M} = (M, S) \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$, the set $I = \{a \in M : \mathcal{M} \models \mathsf{WF}(\omega^a)\}$ is a cut. Actually, in such models, I is an additive cut, that is, if $a \in I$, then $a + a \in I$, but there exists non-standard models of $\mathsf{Q} + \mathsf{I}\Sigma^0_1$ in which $I = \sup\{a \cdot n : n \in \omega\}$ for some non-standard integer a. In such models, I does not have any better closure property than additivity.

Theorem 7.6.12 (Kreuzer and Yokoyama [62])
$\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \mathsf{WF}(\omega^\omega) \to \mathsf{BME}_*$.

Proof. Given a k-bounded finite coded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we define an ranking $\zeta_T : T \to \omega^k$ inductively as follows:

$$\zeta_T(\sigma) = \begin{cases} 0 & \text{if } |\sigma| = k \\ \omega^{k-|\sigma|} & \text{if } \sigma \text{ is a leaf in } T \text{ and } |\sigma| < k \\ \sum_{\sigma \cdot a \in T} \zeta_T(\sigma \cdot a) & \text{if } \sigma \text{ is not a leaf.} \end{cases}$$

Note that $\zeta_T(\epsilon) < \omega^\omega$ for any such tree T. Given a monotone enumeration of a k-bounded Σ^0_1 tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, if $T_{s+1} \neq T_s$, then $\zeta_{T_{s+1}}(\epsilon) < \zeta_{T_s}(\epsilon)$[46], so by $\mathsf{WF}(\omega^\omega)$, there are only finitely such stages. Letting s be larger than all such stages. Then $T_s = T$, so T is finite coded. ∎

46: Here, ϵ denotes the empty string, hence the root of the tree. It should not be confused with the ordinal ϵ_0.

Exercise 7.6.13 (Kreuzer and Yokoyama [62]). Fix $k \in \mathbb{N}$. Given a k-bounded finite coded tree T, let ζ_T be the function of Theorem 7.6.12.

1. Prove that for every ordinal $\alpha < \omega^k$, there is a k-bounded finite coded tree T such that $\zeta_T(\epsilon) = \alpha$.
2. Prove that for every k-bounded finite coded tree T and every $\alpha < \zeta_T(\epsilon)$, there is a a k-bounded finite coded tree $S \supseteq T$ which extends only leaves of T, and such that $\zeta_S(\epsilon) = \alpha$.
3. Deduce that $\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \mathsf{BME}_* \to \mathsf{WF}(\omega^\omega)$. ★

Working with a stronger base theory, namely, $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\alpha)$ for some ordinal $\alpha \leq \epsilon_0$, raises new complications, as one needs not only to prove a blocking lemma to control the jump, but also a blocking lemma to preserve $\mathsf{WF}(\alpha)$. For this, we shall use the natural (Hessenberg) sums and products over ordinals:

Definition 7.6.14 (Natural sum and product). Let α and β be two ordinals less than ϵ_0. Let $\alpha = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_k} n_k$ and $\beta = \omega^{\gamma_1} m_1 + \cdots + \omega^{\gamma_k} m_k$[47]. The *natural sum* $\alpha \dot{+} \beta$ is defined as

$$\omega^{\gamma_1}(n_1 + m_1) + \cdots + \omega^{\gamma_k}(n_k + m_k)$$

47: We allow the n_i and m_i to be equal to 0 in order to write α and β using the same exponents γ_i

Then, let $\alpha \dot{\times} k$ to be equal to be the natural sum of α with itself k times and $\alpha \dot{\times} \omega = \omega^{\gamma_1 + 1} n_1 + \cdots + \omega^{\gamma_k + 1} n_k$.[48]

48: Note that the natural product differs from the natural sum. Indeed,

$$\alpha \times \omega = \omega^{\gamma_1 + 1} n_1$$

Thankfully, Shore blocking for preserving $\mathsf{WF}(\alpha)$ comes for free, in the sense that for every $k \in \mathbb{N}$, one can define a Turing functional Γ_k such that if Φ^X_e is an infinite, decreasing sequence of ordinals less than α for some $e < k$, then Γ_k is an infinite, decreasing sequence of ordinals less than $\alpha \dot{\times} k$. Since for any model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{WF}(\alpha)$ and any $k \in M$, $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{WF}(\alpha \dot{\times} k)$, then the natural product overhead is not a problem.[49] In what follows, a code $\langle \alpha \rangle$ for an ordinal $\alpha < \epsilon_0$ is any fixed representation of α as an integer such that the various operations on it are provably Δ^0_1 over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

49: RCA_0 proves that the product of two well-orders is a well-order. Since $\alpha \dot{\times} k \leq \alpha \times \omega$ for every $k \in M$, it follows that $\mathsf{RCA}_0 \vdash \mathsf{WF}(\alpha) \to \mathsf{WF}(\alpha \times \omega)$.

Lemma 7.6.15 (Le Houérou, Levy Patey and Yokoyama [60]). Fix a model $\mathcal{M} = (M, S) \models \mathsf{Q}$. For every $k \in M$, there is a Turing functional Γ_k such that, letting $\alpha < \epsilon_0$ be the largest ordinal with $\langle \alpha \rangle < k$, for every $X \in 2^M$ such that $\mathcal{M} \cup \{X\} \models \mathsf{I}\Sigma^0_1$, if there is some $e < k$ such that Φ^X_e is an M-infinite decreasing sequence of elements smaller than α, then Γ^X_k is an M-infinite decreasing sequence of elements smaller than $\alpha \dot{\times} k$.

Moreover, an index of Γ_k can be found computably in k. ⋆

Proof. By twisting the Turing functionals, we can assume that for every $e, a \in M$, if $\Phi^\sigma_e(a) \downarrow$, then

(1) $a < |\sigma|$;
(2) $\Phi^\sigma_e(b) \downarrow$ for every $b < a$;
(3) $\Phi^\sigma_e(0), \Phi^\sigma_e(1), \dots, \Phi^\sigma_e(a)$ is a strictly decreasing sequence of elements smaller than α.

Given $\sigma \in 2^{<M}$ and $e < k$, let $\zeta(\sigma, e) = \Phi^\sigma_e(s)$ be the largest $s < |\sigma|$ such that $\Phi^\sigma_e(s) \downarrow$. If there is no such s, then $\zeta(\sigma, e) = \alpha$. Note that if $\sigma' \geq \sigma$, then $\zeta(\sigma', e) \leq \zeta(\sigma, e)$.

Let $\sigma_{-1} = \epsilon$. Let Γ_k be the Turing functional which, on oracle X and input a, searches for some $x > |\sigma_{a-1}|$ and some $\sigma_a \prec X$ such that $\Phi^{\sigma_a}_e(x) \downarrow$ for some $e < k$. If found, it outputs $\zeta(\sigma, 0) \dot{+} \dots \dot{+} \zeta(\sigma, k-1)$. Note that if $\Gamma^X_k(a) \downarrow$, then by (3), $\Gamma^X_k(a)$ is an ordinal smaller than $\alpha \dot{\times} k$.

Suppose that X is such that $\mathcal{M} \cup \{X\} \models \mathsf{I}\Sigma^0_1$ and there is an $e < k$ is such that Φ^X_e is total. Since $\mathcal{M} \cup \{X\} \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$, then by Exercise 7.3.1, $\mathcal{M}[X] \models \mathsf{RCA}_0$, so Γ^X_k is total.

Moreover, since $x > |\sigma_{a-1}|$, then for $e < k$ such that $\Phi^{\sigma_a}_e(x) \downarrow$, by (1) we have $\Phi^{\sigma_{a-1}}_e(x) \uparrow$. Thus, by (2) and (3), $\zeta(\sigma_{a+1}, e) < \zeta(\sigma_a, e)$, hence $\Gamma^X_k(a+1) < \Gamma^X_k(a)$. It follows that Γ^X_k is an M-infinite decreasing sequence of ordinals smaller than $\alpha \dot{\times} k$. ■

All the previous conservation theorems over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ also hold while preserving $\mathsf{WF}(\alpha)$ for any fixed ordinal $\alpha \leq \epsilon_0$. We give the details for formalized low basis theorem, and leave the other conservation theorems as exercises.

Theorem 7.6.16 (Le Houérou, Levy Patey and Yokoyama [60])
Fix $\alpha \leq \epsilon_0$. Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\alpha)$ be a countable model topped by a set Y and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $P \in [T]$ such that $(P \oplus Y)' \leq_T Y'$ and $\mathcal{M}[P] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\alpha)$.

Proof. The proof is very similar to Theorem 7.5.3, with an extra requirement for every $b \in \mathbb{N}$:

- $\mathcal{S}_b$: Let $\beta < \alpha$ be the $<_{\epsilon_0}$-largest ordinal with $\langle \beta \rangle < b$. For every $e < b$, $\Phi^{G \oplus Y}_e$ is not an infinite $<_{\epsilon_0}$-decreasing sequence of ordinals smaller than β.

For this, we need to prove a blocking lemma:

Lemma 7.6.17. Let (σ, T_1) be a condition. For every $b \in M$, letting Γ_b be the functional of Lemma 7.6.15, there is an extension $(\sigma, T_2) \leq (\sigma, T_1)$ and an $a \in M$ such that $(\sigma, T_2) \Vdash \Gamma_b^{G\oplus Y}(a)\uparrow$. ⋆

Proof. We have two cases.

Case 1: there exists some $a \in M$ such that the tree $T_2 = \{\tau \in T_1 : \Gamma_b^{\tau\oplus Y}(a)\uparrow\}$ is infinite. Note that the set T_2 is a primitive Y-recursive, as the set T_1 and the predicate $\Gamma_k^{\tau\oplus Y}(n)\uparrow$ are primitive Y-recursive. Then $(\sigma, T_2) \leq (\sigma, T_1)$ and $(\sigma, T_2) \Vdash \Gamma_k^{G\oplus Y}(a)\downarrow$.

Case 2: for every $a \in M$, there is some $\ell_a \in M$ such that for every $\tau \in T$ of length ℓ_a, $\Gamma_b^\tau(a)\downarrow$. For every $a \in M$, let

$$\alpha_a = \max\{\Gamma_b^\tau(a) : \tau \in T_1 \wedge |\tau| = \ell_a\}$$

We claim that for every $a \in M$, $\alpha_{a+1} <_{\epsilon_0} \alpha_a$. Indeed, for every $\tau \in T_1$ such that $|\tau| = \ell_{a+1}$, $\Gamma_b^\tau(a+1) <_{\epsilon_0} \Gamma_b^{\tau\upharpoonright\ell_a}(a)$, so

$$\max\{\Gamma_b^\tau(a+1) : \tau \in T_1 \wedge |\tau| = \ell_{a+1}\} <_{\epsilon_0} \max\{\Gamma_b^\tau(a) : \tau \in T_1 \wedge |\tau| = \ell_a\}$$

So $\mathcal{M} \not\models$ WF$(\alpha\dot{\times}b)$. However, since $\mathcal{M} \models$ BΣ_2^0+WF(α), then $\mathcal{M} \models$ WF$(\alpha\dot{\times}b)$. Contradiction. ∎

The construction is the same as in Theorem 7.5.3, except that there is a third type of stage, $\mathcal{S}$. Suppose a stage s is of type $\mathcal{S}$ and $s_0 < s$ is the latest stage at which we switched the stage type. If there exists some $\langle \tau, \hat{T}\rangle, a \leq s$ such that $(\tau, \hat{T}) \leq (\sigma_s, T_s)$ and $(\tau, \hat{T}) \Vdash \Gamma_{s_0}^{G\oplus Y}(a)\uparrow$, then let $\sigma_{s+1} = \tau$, $T_{s+1} = \hat{T}$, $\rho_{s+1} = \rho_s$ and let $s+1$ be of the next type. Otherwise, the elements are left unchanged and we go to the next stage. By Lemma 7.6.17, the construction eventually switches stage type.

The remainder of the proof is left unchanged. This completes the proof of Theorem 7.6.16. ∎

Exercise 7.6.18. Fix $\alpha \leq \epsilon_0$. Let $\mathcal{M} = (M, S) \models$ RCA$_0$ + BΣ_2^0 + WF(α) be a countable model topped by a set Y, and $A \subseteq M$ be a set such that $\mathcal{M}[A \oplus Y'] \models$ RCA$_0^*$. Adapt the proof of Theorem 7.5.6 to show the existence of a set $G \subseteq M$ such that $\mathcal{M}[G] \models$ RCA$_0$ + BΣ_2^0 + WF(α) and $A \oplus Y' \equiv_T (G \oplus Y)'$ ⋆

Exercise 7.6.19 (Le Houérou, Levy Patey and Yokoyama [60]). Fix $\alpha \leq \epsilon_0$. Let $\mathcal{M} = (M, S) \models$ RCA$_0$ + BΣ_2^0 + WF(α) be a countable topped model, and $\vec{R} = R_0, R_1, \dots$ be a uniform sequence in S. Adapt the proof of Theorem 7.5.10 to show the existence of an infinite $\vec{R}$-cohesive set $C \subseteq M$ such that $\mathcal{M}[C] \models$ RCA$_0$ + BΣ_2^0 + WF(α). ⋆

With a similar technique, but a much more involved disjunctive construction, Le Houérou, Levy Patey and Yokoyama [60] prove that RCA$_0$ + WF(ϵ_0) + RT$_2^2$ is a Π_1^1-conservative extension of RCA$_0$ + BΣ_2^0 + WF(ϵ_0).[50] The proof is based on the decomposition of RT$_2^2$ into COH and RT$_2^{1\prime}$. The proof of following theorem goes beyond the scope of this book:

50: Based on the equivalence between BME$_*$ and WF(ω^ω), one would expect to work with models of WF(ω^ω) instead of WF(ϵ_0). However, in order to preserve WF(ω_k^ω) in the extended model, one seems to need WF(ω_{k+1}^ω), where

$$\omega_0^\alpha = \alpha \text{ and } \omega_{k+1}^\alpha = \omega_k^{\omega^\alpha}$$

Theorem 7.6.20 (Le Houérou, Levy Patey and Yokoyama [60])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\epsilon_0)$ *be a countable topped model. For every* Δ^0_2 *set* $A \subseteq M$*, there is an infinite set* $H \subseteq A$ *or* $H \subseteq M \setminus A$ *such that* $\mathcal{M}[H] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\epsilon_0)$.