Forcing design 8

As emphasized throughout the previous chapters, the computability-theoretic analysis of combinatorial theorems is closely related to the combinatorial features of the corresponding forcing questions. This analysis therefore depends on the choice of an appropriate notion of forcing to build solutions to the problem. So far, the preliminary step of designing a good notion of forcing was given for granted. In this chapter, we fill in the gap by explaining the key ideas behind the design of such notion of forcing. These core concepts will be exemplified with the analysis of the Erdős-Moser theorem and the free set theorem.

Prerequisites: Chapters 2 and 3

8.1 Core concepts

We focus on theorems coming from Ramsey theory. Indeed, as explained in Section 6.2, most theorems are equivalent in reverse mathematics to one of five systems of axioms with a well-understood computability-theoretic strength. The few exceptions to this empirical observation almost come exclusively from Ramsey theory, and require the design of a specific machinery. Ramsey theory deals with many kind of mathematical structures. Here, we consider statements about sets, that is, with no additional structure than cardinality. Furthermore, classical reverse mathematics being formulated in the language of second-order arithmetic, we shall focus on statements about the existence of an infinite subset of $\mathbb{N}$. [1]

1: The considerations in this section are rather abstract, and might make sense only after having considered a few examples. The reader might choose to skip this section, and directly learn by examples, with the Erdős-Moser and free set theorems.

The takeway of this discussion is that there is some tension between the structural properties imposed on the forcing conditions to build a solution to the instance of a combinatorial problem, and the necessity to add elements by block to the stem by satisfying only a Σ^0_1 predicate.

Stem. Turing functionals being continuous functions over Cantor space, computability-theoretic properties of the constructed object G are naturally forced by fixing initial segments of G. It follows that the forcing conditions usually contain a *stem*, represented as a finite binary string. This stem is supposed to grow over condition extension, and every sufficiently generic filter $\mathcal{F}$ will contain conditions with stems of arbitrary length, yielding a binary sequence $G_{\mathcal{F}}$ defined as the limit of these stems. The notion of focring with stems, partially ordered by the prefix relation, is nothing but Cohen forcing.

Structural properties. Given an instance I of a problem P, the goal is to build a P-solution to I. One therefore needs to impose structural constraints on the stem. The most basic such constraint is that the stem is a finite P-solution to I. For instance, in the case of Ramsey's theorem for pairs, one wants σ to code a finite homogeneous set. Thus, for every filter $\mathcal{F}$, the (finite or infinite) sequence $G_{\mathcal{F}}$ yields a homogeneous set.

Extendibility. One can think of a condition as an invariant property of the construction. Usually, being a finite P-solution to I is not a sufficiently strong invariant, in that some finite solution might not be extendible into an infinite solution. For instance, if P is Ramsey's theorem for pairs and two colors, given finite homogeneous set F for color 0, there might be an element $x \in F$ which, paired with cofinitely many other elements, has color 1. The extendibility constraint is usually formulated in terms of an infinite reservoir satisfying some additional structural properties. For instance, for Ramsey's theorem for pairs, one works with triples (σ_0, σ_1, X), where σ_0 and σ_1 are two stems, homogeneous for color 0 and 1, respectively, and $X \subseteq \mathbb{N}$ is an infinite reservoir with $\min X > |\sigma_i|$, such that for every $i < 2$, every $x \in \sigma_i$ and $y \in X$, $\{x, y\}$ has color i. To see that, given a condition (σ_0, σ_1, X), at least one of the stems is extendible into an infinite solution, apply Ramsey's theorem for pairs within X, to obtain an infinite homogeneous subset $Y \subseteq X$ for some color $i < 2$. Then, by the structural properties of the reservoir, $\sigma_i \cup Y$ is again homogeneous for color i.

Block extendibility. Extendibility yields a classical proof of the problem P, in that for every sufficiently generic filter $\mathcal{F}$, the set $G_{\mathcal{F}}$ is an infinite P-solution to I. However, in order to obtain a good forcing question for Σ^0_1-formulas, yielding a computationally weak solution, one must be able to add elements by block, and not only one by one. Indeed, the natural forcing question for Σ^0_1-formulas is of the form "Is there a block of elements from the reservoir such that, if I add them to the stem, it will satisfy the Σ^0_1-formula?" Because being a finite P-solution to I is usually not a sufficiently strong invariant to ensure extendibility, one must choose a block which will maintain the stronger extendibility property. The extendibility property being usually Π^0_1, the main difficulty lies in finding a sufficient Σ^0_1 property that must satisfy a block to preserve the extendibility property.

Computational properties. Because of the use of a reservoir, a Mathias condition is an infinite object. Given a Mathias-like condition (σ, X), the forcing question will ask for a finite subset $\rho \subseteq X$ with additional structural properties. It follows that the complexity of the forcing question involves the one of the reservoir. In order to obtain a diagonalization theorem such as Theorem 3.3.4, one must therefore impose some computational weakness to the reservoir. The usual requirement is that the reservoir satisfies the weakness property being studied. For instance, in cone avoidance of a set C, one will usually work with reservoirs $X \not\geq_T C$.

8.2 Erdős-Moser theorem

The Erdős-Moser was introduced and studied in Section 6.4, with a notion of forcing coming out of the blue. We recall the basic definitions, and give a step-by-step explanation of the process yielding to the design of its notion of forcing.

A *tournament* over an infinite domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation $T \subseteq D^2$ such that for every $a, b \in D$ with $a \neq b$, $T(a, b)$ iff $\neg T(b, a)$. The tournament T is *transitive* if for all $a, b, c \in D$, if $T(a, b)$ and $T(b, c)$ hold, then $T(a, c)$ also holds.[2] A *sub-tournament* of T is the restriction of T to a subdomain $D_1 \subseteq D$. Thus, given T, a sub-tournament is fully specified by the sub-domain D_1, and is therefore identified with it, and we say that D_1 is T-*transitive* if T is transitive on D_1. The Erdős-Moser theorem states that every infinite tournament admits an infinite transitive sub-tournament.

2: It is important to note that transitivity is a property over $[D]^3$. Thus, if a tournament is not transitive, then it is witnessed by a 3-tuple of elements of D.

Fix a computable tournament T over $\mathbb{N}$. In order to design a good notion of forcing to build an infinite T-transitive subtournament, one starts with Mathias forcing, that is, the notion of forcing whose conditions are pairs (σ, X), where $\sigma \in 2^{<\mathbb{N}}$ is the *stem*[3] and and $X \subseteq \mathbb{N}$ is an infinite *reservoir*. A condition (τ, Y) *extends* (σ, X) if $\sigma \preceq \tau$ (a longer initial segment of the solution is specified), $Y \subseteq X$ (the reservoir is restricted), and $\tau \setminus \sigma \subseteq X$ (the new elements of the stem come from the reservoir).

3: Think of the stem as an initial segment of the object being built.

Step 1: Extendibility. Of course, pure Mathias forcing does not produce infinite T-transitive sub-tournaments. One must therefore put a first restriction by asking the stem σ to be a finite T-transitive sub-tournament. This restriction structurally ensures that for every filter $\mathcal{F}$, the set $G_{\mathcal{F}}$ (defined as the limit of the stems of conditions in $\mathcal{F}$) is T-transitive. However, this restriction comes with a price: even with sufficiently generic filters $\mathcal{F}$, the set $G_{\mathcal{F}}$ might not be infinite. Indeed, there might be conditions (σ, X) where the stem is not extendible into an infinite solution. For instance, there might be some $x, y \in [\sigma]^2$ such that for all but finitely many $z \in X$, $\{x, y, z\}$ forms a 3-cycle. There might be an even more subtle situation: for almost every $z \in X$, there is some $x, y \in [\omega]^2$ (which depend on z) such that $\{x, y, z\}$ forms a 3-cycle.

One must therefore identify a stronger structural property which will ensure extendibility of the stem, and play the role of an invariant. Thankfully, there is a simple empirical criterion to identify this invariant: Given a condition (σ, X), by the classical Erdős-Moser theorem, there is an infinite T-transitive subset $Y \subseteq X$. The structural invariant is obtained by identifying sufficient hypothesis to ensure that $\sigma \cup Y$ is again T-transitive.

As mentioned, if $\sigma \cup Y$ is not T-transitive, then there exists a 3-cycle $\{x, y, z\} \in [\sigma \cup Y]^2$. Say $x < y < z$. Because σ and Y are T-transitive, one cannot have $x, y, z \in \sigma$ or $x, y, z \in Y$. There are only two possibilities remaining.

- Case 1: $x \in \sigma$ and $y, z \in Y$. This can be avoided by ensuring that each $x \in \sigma$ has the same behavior with respect to every element of X. We say that σ *is stabilized by* X if for every $x \in \sigma$, either $\forall y \in X$, $T(x, y)$, or $\forall y \in X$, $T(y, x)$. Given a condition (σ, X), one can always find an infinite X-computable subset $Y \subseteq X$ such that σ is stabilized by Y, as follows: Given a condition (σ, X), let $f : X \to 2^{|\sigma|}$ be defined by $f(y) = \rho$, where ρ is the binary string of length $|\sigma|$ such that for every $x < |\sigma|$, $\rho(x) = 1$ iff $T(x, y)$.[4] Since the pigeonhole principle is computably true, one can find an infinite X-computable f-homogeneous subset $Y \subseteq X$. One easily sees that σ is stabilized by Y. Thus, the condition (σ, Y) avoids every 3-cycle with one element in σ and two elements in Y.
- Case 2: $x, y \in \sigma$, $z \in Y$. This cannot be avoided for free by restricting the reservoir. One must therefore explicitely forbid this behavior. Because σ is T-transitive, one can equivalently ask that every element $y \in X$ is a *one-point extension*, that is, $\sigma \cup \{y\}$ is T-transitive.

4: Another way to see this is to consider each element x of σ, and successively apply RT^1_2 by considering the 2-partition $\{y \in X : T(x, y)\}$ and $\{y \in X : T(y, x)\}$. This yields a finite decreasing sequence of infinite sets, stabilizing the behavior of more and more elements of σ. The last set is the desired reservoir.

The previous analysis reveals two structural extendibility properties, the former being optional. A condition is a Mathias pair (σ, X) such that σ is stabilized by X, and every element of X is a one-point extension. In other words,

(a) $\forall x \in \sigma$, either $(\forall y \in X)T(x, y)$ or $(\forall y \in X)T(y, x)$
(b) $\forall y \in X$, $\sigma \cup \{y\}$ is T-transitive[5]

5: Note that this property encompasses the fact that σ is T-transitive. Thus, there is no need to add explicitly this constraint on the stem.

As mentioned, the first property is optional, as given a Mathias condition (σ, X), one can always find an infinite X-computable subset $Y \subseteq X$ such that (σ, Y) satisfies (a). On the other hand, the second property truly imposes a constraint on the stem σ. Because of this, one must check that property (b) can be preserved by adding new elements to the stem. The following extendibility lemma states that it is the case.

Lemma 8.2.1. Let (σ, X) be a condition, and $x \in X$. There is an X-computable infinite set $Y \subseteq X$ such that $(\sigma \cup \{x\}, Y)$ is a valid extension.[6] ★

6: Note how in this proof, the optional property (a) is useful to preserve property (b).

Proof. Fix $x \in X$ and let Y be either $\{y \in X : T(x, y)\}$ or $\{y \in X : T(y, x)\}$, depending on which one is infinite. We claim that $(\sigma \cup \{x\}, Y)$ is a valid extension. It is by construction a Mathias extension of (σ, X), so one only needs to check that properties (a) and (b) are satisfied. Property (a) of $(\sigma \cup \{x\}, Y)$ is satisfied by property (a) of (σ, X) and the choice of Y. We now prove (b). Suppose for the contradiction that $\sigma \cup \{x\} \cup \{y\}$ is not T-transitive, for some $y \in Y$. By definition, there is a 3-cycle $\{a, b, c\} \in [\sigma \cup \{x\} \cup \{y\}]^3$. Say $a < b < c$. Because of property (b) of (σ, X), one cannot have $\{a, b, c\} \in [\sigma \cup \{x\}]^3$ or $\{a, b, c\} \in [\sigma \cup \{y\}]^3$, so $a \in \sigma$, $b = x$ and $c = y$. In particular, a does not have the same behavior with respect to b and c, contradicting property (a) of (σ, X). ■

Step 2: Block extendibility. We now have a notion of forcing to build solutions to a given computable instance of the Erdős-Moser theorem. However, additional work is required to design a good forcing question for Σ^0_1-formulas. Consider the forcing question for Mathias forcing:

Definition 8.2.2. Given a Mathias condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \mathrel{?\vdash} \varphi(G)$ iff there is some finite set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◊

An Erdős-Moser condition being a Mathias condition, one should expect to have a similar forcing question, by replacing "finite set $\rho \subseteq X$" with "finite T-transitive set $\rho \subseteq X$". This definition raises two difficulties. First, one wants the forcing question for Σ^0_1-formulas to be Σ^0_1-preserving, but given a Mathias condition (σ, X), the forcing question for a Σ^0_1-formula is $\Sigma^0_1(X)$. We shall ignore this difficulty until Step 3. Second, the property (b) of a condition is not preserved by adding blocks simultaneously.

Example 8.2.3. Let (σ, X) be a condition, and $\rho = \{x, y\} \subseteq X$ be a finite set. The set ρ is vacuously T-transitive. Moreover, by choice of properties (a) and (b), $\sigma \cup \rho$ is again T-transitive. However, suppose that $T(x, y)$ holds, but for all but finitely many $z \in X$, $T(y, z)$ and $T(z, x)$ both hold. Then there is no infinite subset $Y \subseteq X$ such that $(\sigma \cup \rho, Y)$ satisfies property (b).

The previous example shows the importance of some "compatibility" property between the elements of ρ. Suppose first for simplicity that T is *stable*, that is, for every x, either $(\forall^\infty y)T(x, y)$, or $(\forall^\infty y)T(y, x)$. Such tournament induces a $\emptyset'$-computable coloring of singletons $f : \mathbb{N} \to 2$ defined by $f(x) = 1$ iff $(\forall^\infty y)T(x, y)$.[7]

7: One can see a tournament $T \subseteq \mathbb{N}^2$ as a function $h : [\mathbb{N}]^2 \to 2$ defined for $x < y$ by $h(x, y) = 1$ iff $T(x, y)$ and $h(x, y) = 0$ otherwise. The tournament is stable iff h is stable, and $f(x) = \lim_y h(x, y)$. is the limit function.

Definition 8.2.4. A set ρ is *f-compatible* if for every $x, y \in \rho$, if $T(x, y)$ holds, then $f(x) \geq f(y)$. ◊

Note that every f-homogeneous set is f-compatible. We leave as an exercise the fact that f-compatibility is a sufficient notion to preserve property (b).

Exercise 8.2.5. Suppose T is stable, with limit function $f : \mathbb{N} \to 2$. Let (σ, X) be a condition, and $\rho \subseteq X$ be a finite f-compatible set. Show that $(\sigma \cup \rho, X \cap (\max \rho, \infty))$ satisfies property (b). ⋆

Even among stable tournaments, the naive definition of the forcing question is too complex definitionally. Indeed, given a condition (σ, X), the following statement

"There is some finite f-compatible and T-transitive subset $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds."

is $\Sigma^0_1(X \oplus \emptyset')$, since the coloring f is $\emptyset'$-computable. In order to decrease the complexity of the statement, we use a standard trick of over-approximation by considering all the candidate limit colorings over an effectively compact space.

Definition 8.2.6. Given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let (σ, X) $?\vdash \varphi(G)$ iff for every coloring $g : \mathbb{N} \to 2$, there is some finite T-transitive and g-compatible set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◊

At first sight, this yields a statement of much stronger complexity, as it contains a universal second-order quantification. However, thanks to compactness, the statement is actually $\Sigma^0_1(X)$.

Exercise 8.2.7. Let (σ, X) be a condition and $\varphi(G)$ be a Σ^0_1-formula. Show that $(\sigma, X) ?\vdash \varphi(G)$ iff there is some $\ell \in \mathbb{N}$ such that for every coloring $g : \ell \to 2$, there is some finite T-transitive and g-compatible[8] set $\rho \subseteq X{\upharpoonright}_\ell$ such that $\varphi(\sigma \cup \rho)$ holds. ★

8: One can actually replace "g-compatible" with "g-homogeneous", and obtain a valid forcing question. Although less familiar, the notion of g-compatibilty is more natural in this context, as it contains the least necessary hypothesis to preserve property (b).

Because this forcing question is an over-approximation of the naive forcing question, if it holds, then there is an extension forcing the Σ^0_1-formula. On the other hand, if the forcing question does not hold, the witness of failure might be a function $g : \mathbb{N} \to 2$ which is not related to the true limit function $f : \mathbb{N} \to 2$. We shall then exploit the Ramseyan nature of the statements[9] by working with sets which are simultaneously f and g-compatible. With a little bit more work, one can actually show that this forcing question works even for non-stable tournaments, by stabilizing the set ρ *a posteriori*.

9: A common denominator of many Ramseyan statements is the existence, given multiple instances, of a singlet set which is simultaneously a solution to each instances. Consider Ramsey's theorem for example. Given two colorings $f : [\mathbb{N}]^n \to k$ and $g : [\mathbb{N}]^m \to \ell$, apply Ramsey's theorem to obtain an infinite f-homogeneous set $X \subseteq \mathbb{N}$. Then, within X, apply again Ramsey's theorem to obtain an infinite g-homogeneous subset $Y \subseteq X$. The set Y is simultaneously g-homogeneous and f-homogeneous.

Lemma 8.2.8. Let $p = (\sigma, X)$ be a condition and $\varphi(G)$ be a Σ^0_1-formula.

1. If $p ?\vdash \varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\varphi(G)$.
2. If $p ?\nvdash \varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\neg\varphi(G)$.

Moreover, every set P of PA degree over X computes such a set Y. ★

Proof. Suppose first $p ?\vdash \varphi(G)$. Then, by Exercise 8.2.7, there is some threshold $\ell \in \mathbb{N}$ such that for every coloring $g : \ell \to 2$, there is finite T-transitive and g-compatible set $\rho \subseteq X{\upharpoonright}_\ell$ such that $\varphi(\sigma \cup \rho)$ holds. Let $Y \subseteq X$ be an X-computable subset stabilizing $[0, \ell)$. This induces an X-computable coloring $g : \ell \to 2$ defined by $g(x) = 1$ iff $(\forall y \in Y)T(x, y)$. Let $\rho \subseteq X{\upharpoonright}_\ell$ be a finite T-transitive and g-compatible set such that $\varphi(\sigma \cup \rho)$ holds. We claim that $(\sigma \cup \rho, Y)$ is the desired extension. First, it is a Mathias condition, and by choice of Y, it satisfies property (a). By Exercise 8.2.5, it satisfies property (b). By choice of ρ, it forces $\varphi(G)$.

Suppose now $p ?\nvdash \varphi(G)$. Let $\mathscr{C}$ be the $\Pi^0_1(X)$ class of all $g : \mathbb{N} \to 2$ such that for every finite T-transitive and g-compatible set $\rho \subseteq X$, $\varphi(\sigma \cup \rho)$ does not hold. By assumption, the class $\mathscr{C}$ is non-empty. Pick any $g \in \mathscr{C}$ and let $Y \subseteq X$ be an infinite g-homogeneous subset. As mentioned, every g-homogeneous set is g-compatible, and the pigeonhole principle is computably true, so Y can be chosen $X \oplus g$-computably. The condition (σ, Y) is an extension of p forcing $\neg\varphi(G)$. Note that any PA degree over X computes member of $\mathscr{C}$, hence computes such a set Y. ■

Step 3: Computational property.

As mentioned, given a condition (σ, X), the forcing question for a Σ^0_1-formula is $\Sigma^0_1(X)$. In order to obtain a diagonalization theorem such as Theorem 3.3.4, one must impose some computational constraint on the reservoir X. In the most general case, one will add the following property to the definition of a condition (σ, X):

(c) $X \in \mathcal{W}$

where $\mathcal{W}$ is a weakness property[10] whose additional closure properties are identified by looking at the operations on the reservoir that appear in the use of the forcing question.

10: Recall from Section 6.1 that a *weakness property* is a class of sets downward-closed under the Turing reduction. The reader might be more familiar with the notion of Turing ideal, which is closed under effective join. However, most natural weakness properties, such as being low, avoiding a cone, or preserving hyperimmunies, are not closed under effective join.

In our case, all the operations on the reservoir are computable transformations (finite truncation, stabilization of the stem), except in the case where the forcing question does not hold. One then obtain a Π^0_1 class of 2-partitions, and take any infinite homogeneous set for any of these partitions as the new reservoir. Thus, the previous lemmas hold for any weakness property $\mathcal{W}$ preserved[11] by RT^1_2 and WKL.[12] The pigeonhole principle being computably true, it preserves every weakness property, so one can simply require $\mathcal{W}$ to be preserved by WKL, that is, for every $X \in \mathcal{W}$, there is some set $P \in \mathcal{W}$ of PA degree over X. In most cases, the weakness property $\mathcal{W}$ is nothing but the property that one wants the resulting set G to satisfy.

11: Recall that a problem P *preserves* a weakness property $\mathcal{W}$ if for every $Z \in \mathcal{W}$ and every Z-computable instance X, there is a solution Y to X such that $Z \oplus Y \in \mathcal{W}$.

12: One can actually be even more cautious, and only ask $\mathcal{W}$ to be closed under the Rasmey-type weak König's lemma (RWKL). However, over-optimization is not always desirable, and it sometimes yields unnecessary additional complexity.

Example 8.2.9. Suppose one wants to prove that EM admits cone avoidance. Any non-computable set C induces a weakness property $\mathcal{W}_C = \{Z : C \not\leq_T Z\}$. By the cone avoidance basis theorem (Theorem 3.2.6), $\mathcal{W}_C$ is closed under PA degrees, so one can impose $X \in \mathcal{W}_C$, in other words, $C \not\leq_T X$.

Exercise 8.2.10 (Wang ; Patey [72]). Recall that a problem P admits *strong cone avoidance*[13] if for every set Z and every non-Z-computable set C, every instance X of P admits a solution Y such that C is not $Z \oplus Y$-computable. Fix a non-computable set C and an arbitrary tournament $T \subseteq \mathbb{N}^2$. Consider the same notion of condition above, that is, pairs (σ, X) satisfying properties (a), (b) and (c).

13: The difference between cone avoidance and strong cone avoidance is that the instance X of P is not asked to be Z-computable in the latter case.

1. Use strong cone avoidance of RT^1_2 (Theorem 3.4.5) to prove that for every condition (σ, X) and $x \in X$, there is an infinite set $Y \subseteq X$ such that $(\sigma \cup \{x\}, Y)$ is a valid extension.

Given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \operatorname{?\vdash} \varphi(G)$ if for every tournament $S \subseteq \mathbb{N}^2$ and every coloring $g : \mathbb{N} \to 2$, there is some finite S-transitive and g-compatible set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds.

2. Show that the relation $(\sigma, X) \operatorname{?\vdash} \varphi(G)$ is $\Sigma^0_1(X)$.
3. Use strong cone avoidance of RT^1_2 to prove that if $(\sigma, X) \operatorname{?\vdash} \varphi(G)$, then there is an extension forcing $\varphi(G)$.
4. Use cone avoidance of EM and the cone avoidance basis theorem to prove that if $(\sigma, X) \operatorname{?\nvdash} \varphi(G)$, then there is an extension forcing $\neg\varphi(G)$.
5. Deduce that EM admits strong cone avoidance. ⋆

8.3 Free set theorem

The free set theorem is a combinatorial statement introduced by Friedman [73] which provides another good illustration of the forcing design process. Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, an infinite set $H \subseteq \mathbb{N}$ is *f-free* if for every $\sigma \in [\mathbb{N}]^n$, if $f(\sigma) \in H$, then $f(\sigma) \in \sigma$. The *free set theorem* for n-tuples (FS^n) is the problem whose instances are colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite f-free sets. This problem might seem artificial at first sight, but it can be reformulated as a strong version of the thin set theorem.[14] An infinite set $H \subseteq \mathbb{N}$ is *f-thin* if $f[H]^n \neq \mathbb{N}$, that is, at least one color does not appear on $[H]^n$.

14: Another way to think of the free set theorem is that any n-tuple $\sigma \in [\mathbb{N}]^n$ can optionally "choose" a forbidden element $f(\sigma)$, so that if σ belongs so the solution, then $f(\sigma)$ must be excluded. Setting $f(\sigma) \in \sigma$ is a way to refuse to choose.

Exercise 8.3.1. Let $f : [\mathbb{N}]^n \to \mathbb{N}$ be a coloring. Show that an infinite set $H \subseteq \mathbb{N}$ is f-free iff for every $x \in \mathbb{N}$, $H \setminus \{x\}$ is f-thin with witness color x. ⋆

Similar to Ramsey's theorem, the free set theorem induces a hierarchy of statements based on the size of the colored tuples. However, while Ramsey's theorem hierarchy collapses and is equivalent to ACA_0 for $n \geq 3$, Wang [13] surprisingly proved that the free set theorem admits strong cone avoidance for any size of tuples. The proof goes by induction over n.

In this section, we shall design a notion of forcing for computable instances of FS^3 with a Σ^0_1-preserving forcing question for Σ^0_1-formulas. This provides a good example of a statement which is not about colorings of pairs, but still admits a good first-jump control. For this, we follow the same steps as for the Erdős-Moser theorem. Fix a computable coloring $f : [\mathbb{N}]^3 \to \mathbb{N}$, and start with Mathias forcing.

Step 1: Extendibility. As before, we refine Mathias forcing by asking the stem to be a finite solution, that is, we work with Mathias conditions (σ, X) such that σ is a finite f-free set. Of course, there might be conditions (σ, X) such that the set σ is f-free, but not extendible into an infinite f-free set. For instance, it might be that for almost every $\{x, y, z\} \in [X]^3$, $f(x, y, z) \in \sigma$. There might also also be some $x \in \sigma$ such that for almost every $\{y, z\} \in [X]^2$, $f(x, y, z) \in \sigma \setminus \{x\}$. These are only a few examples of the possible issues.

In order to identify the stronger structural property ensuring extendibility, we apply the same criterion as before: Given a condition (σ, X), let $Y \subseteq X$ be an infinite f-free set. Suppose that $\sigma \cup Y$ is not f-free. There is therefore some $\{x, y, z\} \in [\sigma \cup Y]^3$ such that $f(x, y, z) \in (\sigma \cup Y) \setminus \{x, y, z\}$. Say $x < y < z$. Because σ and Y are both f-free, one cannot have x, y, z, and $f(x, y, z)$ in σ or Y. There are multiple possibilities remaining:

- Case 1: $x, y, z \in \sigma$; $f(x, y, z) \in Y$. This case can be simply avoided by removing the range of $f{\upharpoonright}[\sigma]^3$ from the reservoir. This range is finite, so this can be obtained for free by finite truncation of the reservoir.
- Case 2: $x, y \in \sigma$; $z, f(x, y, z) \in Y$. Fixing $\{x, y\} \in \sigma$ induces a coloring $f_{x,y} : \mathbb{N} \to \mathbb{N}$ defined by $f_{x,y}(z) = f(x, y, z)$. This coloring can be seen as an instance of FS^1. Given a condition (σ, X), one can use the induction hypothesis, and apply FS^1 on $f_{x,y}$ for every $\{x, y\} \in [\sigma]^2$ to obtain an infinite sub-reservoir $Y \subseteq X$ which is $f_{x,y}$-free simultaneously. Case 2 cannot happen with (σ, Y). It follows that Case 2 can be avoided without putting constraints to the stem σ.

- Case 3: $x, y, f(x, y, z) \in \sigma$; $z \in Y$. This cannot be avoided for free by restricting the reservoir. One must therefore explicitely forbid this behavior.
- Case 4: $x \in \sigma$; $y, z, f(x, y, z) \in Y$. This case is similar to Case 2. Fixing some $x \in \sigma$ induces a coloring $f_x : [\mathbb{N}]^2 \to \mathbb{N}$ defined by $f_x(y, z) = f(x, y, z)$. One can again use the induction hypothesis, and apply FS^2 finitely many times to avoid this case.
- Case 5: $x, f(x, y, z) \in \sigma$; $y, z \in Y$. This case is similar to Case 3. In particular, it cannot be avoided simply by restricting the reservoir, so this must be explicitly ruled out.
- Case 6: $f(x, y, z) \in \sigma$; $x, y, z \in Y$. This case is once again similar to Case 3 and Case 5.

These 6 cases can therefore be divided into two categories: the optional structural properties, which can be ensured by restricting the reservoir, with no constraint on the stem, and the required structural properties, which are really necessary to ensure extendibility. A condition is a Mathias pair (σ, X) satisfying the following two properties:

(a) $\forall\{x, y, z\} \in [\sigma \cup X]^3$ with $x \in \sigma$, $f(x, y, z) \notin X \setminus \{y, z\}$
(b) $\forall\{x, y, z\} \in [\sigma \cup X]^3$, $f(x, y, z) \notin \sigma \setminus \{x, y, z\}$.[15]

Property (a) encompasses f-freeness of σ together with the optional properties, namely, Case 1, Case 2 and Case 4, while property (b) covers Case 3, Case 5 and Case 6. We must now show that these structural properties provide a good invariant by proving an extendibility lemma. More precisely, the difficulty is to add new elements to the stem while preserving property (b). Given a condition (σ, X) and $x \in X$, property (b) on $(\sigma \cup \{x\}, X \setminus [0, x])$ is almost inherited from properties (a) and (b) on (σ, X), except one case: there might be some $\{a, b, c\} \in [X \setminus [0, x]]^3$ such that $f(a, b, c) = x$. This corresponds to Case 6, which must receive some special attention.

Given $x_0 \in X$, by Ramsey's theorem for triples, there is an infinite subset $Y \subseteq X$ such that either $(\forall\{a, b, c\} \in [Y]^3) f(a, b, c) \neq x_0$ or $(\forall\{a, b, c\} \in [Y]^3) f(a, b, c) = x_0$. In the former case, $(\sigma \cup \{x_0\}, Y)$ satisfies property (b), while in the latter case, for any $x_1 \in X$ with $x_0 \neq x_1$, $(\sigma \cup \{x_1\}, Y)$ satisfies property (b). Thus, combinatorially, it suffices to pick two elements in X, and at least one of them can be added to the stem while preserving the structural invariant. From a computational viewpoint however, Ramsey's theorem for triples is very strong, and is even applied of an f-computable coloring, which is of arbitrary complexity. Thankfully, one does not need the full power of Ramsey's theorem, and can weaken the statement by considering more than two elements in the reservoir.

Given $n, \ell \geq 1$, let $\mathsf{RT}^n_{<\infty,\ell}$ be the problem[16] whose instances are colorings $f : [\mathbb{N}]^n \to k$ for some $k \in \mathbb{N}$, and whose solutions are infinite sets $H \subseteq \mathbb{N}$ such that $\operatorname{card} f[H]^n \leq \ell$. In particular, $\mathsf{RT}^n_{<\infty,1}$ is nothing but Ramsey's theorem for n-tuples. Wang [13] proved that when ℓ is sufficiently large with respect to n, then $\mathsf{RT}^n_{<\infty,\ell}$ looses all its coding power and admits strong cone avoidance. In our case, fix some sufficiently large bound ℓ_n with respect to n so that $\mathsf{RT}^n_{<\infty,\ell_n}$ preserves our desired computational property.[17]

Lemma 8.3.2. Let (σ, X) be a condition, and $x_0, \dots, x_{\ell_3}$ be distinct elements of X. There is some $i \leq \ell_3$ and some infinite subset $Y \subseteq X$ such that $(\sigma \cup \{x_i\}, Y)$ is a valid extension. ⋆

15: As for the Erdős-Moser theorem, property (a) could be technically removed from the definition of a condition, and one would still obtain a structural invariant. However, property (a) is very convenient to preserve property (b), and can be added for free by restricting further the reservoir, so we include it in the definition.

16: This problem admits many names in the reverse mathematics literature. In Wang [13], it is called the *achromatic Ramsey theorem* and is written $\mathsf{ART}^n_{<\infty,\ell}$. In Dorais et al. [74] or Patey [75], it is considered as a strong version of the *thin set theorem*, and is written $\mathsf{TS}^n_{\ell+1}$. In Patey [76], it is seen as a generalization of Ramsey's theorem, and is written $\mathsf{RT}^n_{<\infty,\ell}$.

17: For $n = 1$, we can take $\ell_1 = 1$, as the pigeonhole principle is computably true, hence preserves any weakness property.

Proof. Let $g : [X \setminus \{x_0, \dots, x_{\ell_3}\}]^3 \to \{x_0, \dots, x_{\ell_3}\}$ be defined by

$$g(a,b,c) = \begin{cases} f(a,b,c) & \text{if } f(a,b,c) \in \{x_0, \dots, x_{\ell_3}\} \\ x_0 & \text{otherwise.} \end{cases}$$

By $\mathsf{RT}^3_{<\infty,\ell_3}$, there is some $i \leq \ell_3$ and an infinite subset $Z \subseteq X$ such that $x_i \notin g[Z]^3$. We claim that $(\sigma \cup \{x_i\}, Z)$ satisfies property (b). Indeed, let $\{a,b,c\} \in [\sigma \cup \{x_i\} \cup Z]^3$ be such that $f(a,b,c) \in (\sigma \cup \{x_i\}) \setminus \{a,b,c\}$. By property (b) of (σ, X), $f(a,b,c) \notin \sigma \setminus \{a,b,c\}$, hence $f(a,b,c) = x_i$ and $x_i \notin \{a,b,c\}$. By property (a) of (σ, X), if $a \in \sigma$, $f(a,b,c) \notin X \setminus \{b,c\}$, so $a \notin \sigma$, hence $a,b,c \in Y \setminus \{x_i\}$. But then, $g(a,b,c) = f(a,b,c) = x_i$, contradicting the choice of Z and x_i. Let $Y \subseteq Z$ be an infinite subset such that $(\sigma \cup \{x_i\}, Y)$ satisfies property (a). Then $(\sigma \cup \{x_i\}, Y)$ is the desired extension. ■

Step 2: Block extendibility. We now want to design a good forcing question for this notion of forcing. For this, we restart with the standard forcing question for Mathias forcing.

Definition 8.3.3. Given a Mathias condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \mathrel{?\vdash} \varphi(G)$ iff there is some finite set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◊

As for the Erdős-Moser theorem, one wants to modify this definition by asking for a finite f-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. Because of the combinatorics of the extendibility lemma, one needs to ask for $\ell_3 + 1$ many pairwise disjoint f-free sets $\rho_0, \dots, \rho_{\ell_3} \subseteq X$ such that for every $i \leq \ell_3$, $\varphi(\sigma \cup \rho_i)$ holds. However, even with this modification, property (b) might not hold over $(\sigma \cup \rho_i, Y)$ for any $i \leq \ell_3$ and any infinite set $Y \subseteq X$.

Example 8.3.4. Let (σ, X) be a condition, and $\rho = \{x,y,z\} \subseteq X$ be a finite set. The set ρ is vacuously f-free. Even putting aside Case 6, it might be that for all but finitely many $w \in X$, $f(x,y,w) = z$, or for all but finitely many $\{u,w\} \in [X]^2$, $f(x,u,w) = y$. Then there is no infinite subset $Y \subseteq X$ such that $(\sigma \cup \rho, Y)$ satisfies property (b).

One needs to find the appropriate notion of compatibility so that property (b) is preserved when adding blocks of elements. The issue usually comes from some hidden non-computable constraint between the elements of the block ρ and the limit behavior of the coloring. In order to reveal this constraint, one must first consider the appropriate notion of stability. In the case of the Erdős-Moser theorem, stability was obtained by multiple applications of the pigeonhole principle. In the case of the free set theorem, we shall use $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$.

Definition 8.3.5. An infinite set X *stabilizes* a finite set σ if there are finite sets $I \in [\sigma]^{\leq \ell_3}$, $\langle I_x \in [\sigma]^{\leq \ell_2} : x \in \sigma \rangle$ and $\langle I_{x,y} \in [\sigma]^{\leq \ell_1} : \{x,y\} \in [\sigma]^2 \rangle$ such that[18]

(i) $f[X]^3 \cap \sigma \subseteq I$;
(ii) for every $x \in \sigma$, $f_x[X]^2 \cap \sigma \subseteq I_x$;
(iii) for every $\{x,y\} \in [\sigma]^2$, $f_{x,y}[X]^1 \cap \sigma \subseteq I_{x,y}$.[19] ◊

18: Given a finite or infinite set Z and some $k \in \mathbb{N}$, we write $[Z]^{\leq k}$ for the collection of all subsets of Z of size at most k. In particular, $[Z]^{\leq k}$ contains the empty set.

19: Recall that $f_x : [\mathbb{N}]^2 \to \mathbb{N}$ and $f_{x,y} : \mathbb{N} \to \mathbb{N}$ are the functions obtained by fixing the parameters x and y.

We leave as an exercise the proof that every finite set can be stabilized by restricting the reservoir.

Exercise 8.3.6. Let σ be a finite set and $X \subseteq \mathbb{N}$ an infinite set. Use $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$ to show that there exists an infinite subset $Y \subseteq X$ stabilizing σ. ⋆

Suppose X stabilizes an initial segment $[0, k]$ for some $k \in \mathbb{N}$. Then this induces a coloring $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ defined by $g(\emptyset) = I$, $g(\{x\}) = I_x$ and $g(\{x, y\}) = I_{x,y}$. Note that for every $\nu \in [k]^{\leq 2}$, $\operatorname{card} g(\nu) \leq \ell_{3-|\nu|}$. A set $H \subseteq k$ is *g-free* if for every $\nu \in [H]^{\leq 3}$, $g(\nu) \cap H \subseteq \nu$.

Exercise 8.3.7. Let (σ, X) be a condition, and $Y \subseteq X$ be an infinite subset stabilizing some initial segment $[0, k]$. Let $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ be the corresponding limit function. Show that if $\rho \subseteq X$ is f-free and g-free, then $(\sigma \cup \rho, Y)$ satisfies property (b). ⋆

The previous exercise motivates the following definition of the forcing question.

> **Definition 8.3.8.** Given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let (σ, X) ?⊢ $\varphi(G)$ iff there is some $k \in \mathbb{N}$ such that for every coloring $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ such that for every $\nu \in [k]^{\leq 2}$, $\operatorname{card} g(\nu) \leq \ell_{3-|\nu|}$, there is some finite f-free and g-free set $\rho \subseteq X \upharpoonright_k$ such that $\varphi(\sigma \cup \rho)$ holds. ◊

Note that the previous definition is in explicit Σ^0_1 form. In order to handle the case where the forcing question does not hold, one would like to also state the same forcing question in the form of a second-order quantification. Let $\mathcal{F}$ be the class of all functions $g : [\mathbb{N}]^{\leq 2} \to [\mathbb{N}]^{<\mathbb{N}}$ such that for every $\nu \in [\mathbb{N}]^{\leq 2}$, $\operatorname{card} g(\nu) \leq \ell_{3-|\nu|}$. Contrary to the class of all tournaments, the class $\mathcal{F}$ is not compact. Thankfully, given a function $g \in \mathcal{F}$ and finite set ρ, the predicate "ρ is g-free" does not require to have a complete information about $g \upharpoonright [\rho]^{\leq 2}$, but only to decide $\{(\nu, z) : \nu \in [\rho]^{\leq 2}, z \in g(\nu)\}$. It follows that one can represent g by the relation $R_g = \{(\nu, z) : \nu \in [\mathbb{N}]^{\leq 2}, z \in g(\nu)\}$. Given such a set R_g and some ν, g-freeness is decidable, but one cannot know for example the cardinality of $g(\nu)$ in general. Let $\mathcal{R}$ be the class of all relations R over $[\mathbb{N}]^{\leq 2} \times \mathbb{N}$ such that for every $\nu \in [\mathbb{N}]^{\leq 2}$, $\operatorname{card}\{z : (\nu, z) \in R\} \leq \ell_{3-|\nu|}$. The class $\mathcal{R}$ forms an effectively compact set, and there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{R}$. Given a relation $R \in \mathcal{R}$, we write g_R for the corresponding function in $\mathcal{F}$.

Exercise 8.3.9. Let (σ, X) be a condition, and $\varphi(G)$ be a Σ^0_1-formula. Show that (σ, X) ?⊢ $\varphi(G)$ iff for every $R \in \mathcal{R}$, there is some finite f-free and g_R-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ⋆

We are now ready to prove that the forcing question meets its specification.

Lemma 8.3.10. Let $p = (\sigma, X)$ be a condition and $\varphi(G)$ be a Σ^0_1-formula.

1. If p ?⊢ $\varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\varphi(G)$.
2. If p ?⊬ $\varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\neg\varphi(G)$. ⋆

Proof. Suppose first p ?⊢ $\varphi(G)$. Let $k \in \mathbb{N}$ witness the definition of the forcing question. By Exercise 8.3.6, there is an infinite subset $Y_0 \subseteq X$ stabilizing $[0, k]$. Let $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ be the corresponding function, and let $\rho \subseteq X \upharpoonright_k$ be a finite f-free and g-free subset such that $\varphi(\sigma \cup \rho)$ holds. By Exercise 8.3.7, $(\sigma \cup \rho, Y_0)$ satisfies property (b). Let $Y \subseteq Y_0$ be an infinite subset such that $(\sigma \cup \rho, Y)$ satisfies property (a). Then $(\sigma \cup \rho, Y)$ is a valid extension forcing $\varphi(G)$.

Suppose now $p \operatorname{?\nvdash} \varphi(G)$. Let $\mathscr{C}$ be the $\Pi^0_1(X)$ class of all $R \in \mathscr{R}$ such that for every finite f-free and g_R-free set $\rho \subseteq X$, $\varphi(\sigma \cup \rho)$ does not hold. By Exercise 8.3.9, the class $\mathscr{C}$ is non-empty. Pick any $g \in \mathscr{C}$. By finitely many applications of FS^1 and FS^2, there is an infinite g-free subset $Y \subseteq X$. The condition (σ, Y) is an extension of p forcing $\neg\varphi(G)$. ■

Step 3: Computational property. As before, given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, the forcing question $(\sigma, X) \operatorname{?\vdash} \varphi(G)$ is $\Sigma^0_1(X)$. One must therefore impose some computability-theoretic constraints to the set X to obtain diagonalization theorems. A condition (σ, X) must therefore also satisfy the following property

(c) $X \in \mathscr{W}$

where $\mathscr{W}$ is a weakness property. Looking at the various lemmas, many preservation assumptions are used on $\mathscr{W}$: in the extendibility lemma, one used X-computable instances of FS^1 and FS^2 to satisfy property (a), and $\mathsf{RT}^3_{<\infty,\ell_3}$ to satisfy property (b). In the forcing question, one used X-computable instances of $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$ for stabilizing initial segments if the forcing question holds, and X-computable instances of WKL to pick a coloring $g : [\mathbb{N}]^{\leq 2} \to [\mathbb{N}]^{<\mathbb{N}}$ and $X \oplus g$-computable instances of FS^1 and FS^2 to thin out the reservoir and obtain an infinite g-free subset. Thus, overall, we required $\mathscr{W}$ to be preserved by FS^1, FS^2, $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$.

Note that there is some degree of freedom in the choice of ℓ_1, ℓ_2 and ℓ_3. These integers can be chosen to be arbitrarily large, depending on the property one wants to preserve.

Example 8.3.11. If one wants to prove cone avoidance, we shall use $\ell_1 = 1$, $\ell_2 = 1$ and $\ell_3 = 2$, as Wang [13] proved that these statements admit cone avoidance. If one wants to preserve k hyperimmunities simultaneously, we shall use larger values depending on k, based on Patey [43].

Exercise 8.3.12 (Wang [13]). Assume that for every $n \in \mathbb{N}$, there is some $\ell_n \in \mathbb{N}$ such that $\mathsf{RT}^n_{<\infty,\ell_n}$ admits cone avoidance.

1. Design a notion of forcing for FS^n.
2. Prove by induction on n that FS^n admits cone avoidance. ★

Exercise 8.3.13 (Wang [13]). A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is k-*bounded* if for every $c \in \mathbb{N}$, $f^{-1}(c)$ has size at most k. A set $H \subseteq \mathbb{N}$ is an f-*rainbow* if f is injective on $[H]^n$. The *rainbow Ramsey theorem* for n-tuples and k-bounded functions RRT^n_k is the problem whose instances are k-bounded colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite f-rainbows.

1. Design a notion of forcing for RRT^3_2.
2. Prove that RRT^3_2 admits cone avoidance.[20] ★

20: Actually, Wang proved that RRT^n_k is strongly computably reducible to FS^n, hence RRT^n_k admits strong cone avoidance for every $n, k \geq 2$.

Exercise 8.3.14 (Patey [43]). A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is *left (right) trapped* if for every $v \in [\mathbb{N}]^n$, $f(v) < \max v$ ($f(v) \geq \max v$). Fix a weakness property $\mathscr{W}$.

1. Show that if FS^n for left trapped and right trapped functions preserve $\mathscr{W}$, then so does FS^n.
2. Use Proposition 5.7.1 to show that for every right trapped function $f : [\mathbb{N}]^n \to \mathbb{N}$, every DNC function[21] relative to f computes an infinite f-free set.
2. Given a set X, construct a left trapped coloring $f : \mathbb{N} \to \mathbb{N}$ such that every infinite f-free set is effectively X-immune.
3. Deduce that if FS^n for left trapped functions preserves $\mathscr{W}$, then so does FS^n. ★

21: Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *DNC* relative to X if for every e, $f(e) \neq \Phi_e^X(e)$. This notion admits many computability-theoretic characterizations, in terms of effective X-immunity, and escaping bounded X-c.e. sets. See Sections 5.7 and 6.2.

Exercise 8.3.15. Given a coloring $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$, a set $H \subseteq \mathbb{N}$ if *f-free* if for every $v \in [H]^n$, $f(v) \cap H \subseteq v$. The coloring f is *h-constrained* for a function $h : \mathbb{N} \to \mathbb{N}$ if for every $v \in [\mathbb{N}]^n$, $\operatorname{card} f(v) \leq h(\min v)$. If h is the constant function k, we say that f is k-constrained.

1. Show that there exists an $(x \mapsto x)$-constrained coloring $f : \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}$ with no infinite f-free set.
2. Use FS^n to show that for every k-constrained coloring $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$, there is an infinite f-free set.

A coloring $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$ is *progressive* if for every $v \in [\mathbb{N}]^n$, $\min f(v) \geq \min v$.

3. Design a notion of forcing to build infinite f-free sets for $(x \mapsto x)$-constrained progressive colorings $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$. ★