Jump cone avoidance 8

From many perspectives, second-jump control is the same as first-jump control, *mutatis mutandis*: it consists of constructing sets G which controlling their $\Sigma^0_2(G)$ properties. To achieve this, one defines again a forcing question for the class of Σ^0_2 formulas, with the same abstract theorems. In the practice, however, there is a strong technical gap from first-jump control to second-jump control. This is merely due to the fact that, unlike Turing functionals, jump functional are not continuous functions in Cantor space. The forcing question therefore becomes a density statement, which often does not yield the appropriate definitional complexity. The main task of the design of a good second-jump control consists in finding the most effective notion of forcing to build solutions to a given problem. As a byproduct, this often yields insights about the structural nature of the problem.

Prerequisites: Chapters 2 to 4

8.1 Context and motivation

Second-jump control received much less attention than first-jump control in computability theory, and reverse mathematics in particular. One of the reasons is that the vast majority of statements studied in reverse mathematics could be separated using first-jump properties. Moreover, as we shall see in the next section, many second-jump properties can be obtained from effectivization of first-jump properties. Besides reverse mathematics, second-jump control can be used in computability theory to construct sets of low$_2$ degree. Such sets occur naturally in computability theory, but often using the following characterization, rather than directly using a second-jump control: a set X is of low$_2$ degree iff $\emptyset'$ is of high degree over X. There are however a few examples where second-jump control naturally occurs in reverse mathematics.

In the study of Ramsey's theorem and more generally combinatorial hierarchies, the cohesiveness principle quickly became an unavoidable tool, as a bridge between computable instances for $(n+1)$-tuples and arbitrary instances of n-tuples. For example, COH reduces computable instances of Ramsey's theorem for pairs to arbitrary instances of the pigeonhole principle (see Theorem 3.4.1). Recall from Section 3.4 that an infinite set $C \subseteq \mathbb{N}$ is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R}_n$, where $\subseteq^*$ means "included up to finite changes". The *cohesiveness principle* is the problem COH whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets. By Jockusch and Stephan [12] [1] proved that COH is equivalent to the problem "For every Δ^0_2 infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, there is a Δ^0_2-approximation of an infinite path." The cohesiveness principle is therefore a statement about jump computation and separating principles from COH over reverse mathematics requires to use second-jump control [65].

Ramsey's theorem for n-tuples induces a hierarchy of statements based on n. From a reverse mathematical perspective, this hierarchy is known to collapse at level 3 and RT^n_2 is equivalent to ACA_0 for every $n \geq 3$. [4, 14]. On the other hand, some consequences of Ramsey's theorem, such as the free set (FS^n) [66] and the rainbow Ramsey (RRT^n_2) [67] theorems are not known to

[1] 1: Jockusch and Stephan [12] actually proved that the sequence of all primitive recursive sets is maximally difficult for COH, and the degrees of its cohesive sets are exactly those whose jump is PA over $\emptyset'$. Brattka, Hendtlass and Kreuzer [64] refined it to obtain an instance-wise correspondence.

collapse [13]. The most promising approach to prove the strictness of these hierarchies is using iterated jump control [68].

In this section, we shall focus on the unability, for a given problem, to code a fixed set in the jump of its solutions. This is the notion of jump cone avoidance. This is one of the simplest applications of second-jump control, and already illustrates the core problematics of the techniques.

> **Definition 8.1.1.** A problem P admits *jump cone avoidance* if for every set Z and every non-$\Delta^0_2(Z)$ set C, every Z-computable instance X of P admits a solution Y such that C is not $\Delta^0_2(Z \oplus Y)$. ◇

Here again, one can drop the Z-computability restriction of the P-instance, to yield *strong jump cone avoidance*. By letting $Z = \emptyset$ and $C = \emptyset''$, if a problem P admits jump cone avoidance, then even computable instance admits a solution of non-high degree.

8.2 Use first-jump control

Second-jump control aims at proving theorems about the jump of solutions to mathematical problems. However, an effectivization of first-jump control is sometimes sufficient to obtain the same results. Indeed, if a problem admits a low basis, or a weakly low basis[2], it admits jump cone avoidance, a low$_2$ basis, and many other properties.

Proposition 8.2.1. If a problem P admits a weakly low basis, then it admits jump cone avoidance. ★

Proof. Fix a set Z, a non-$\Delta^0_2(Z)$ set C and a Z-computable instance X of P. By the low basis theorem relativized to Z' (see Theorem 4.4.6), there is a set Q of PA degree over Z' such that $C \not\leq_T Q$. Since P admits a weakly low basis, then there is a solution Y such that $(Y \oplus Z)' \leq_T Q$. In particular, C is not $\Delta^0_2(Z \oplus Y)$. ■

The strong technical gap between first-jump and second-jump control gives a strong incentive to use first-jump control to prove second-jump properties when possible. This should be the first consideration is the decisional process of the choice of jump-control techniques.

Exercise 8.2.2. A problem P admits *preservation of 1 jump hyperimmunity* if for every set Z and every Z-hyperimmune function f, every Z-computable instance X of P admits a solution Y such that f is $Y \oplus Z$-hyperimmune. Use the computably dominated basis theorem to prove that if P admits a weakly low basis, then it admits preservation of 1 jump hyperimmunity. ★

Exercise 8.2.3. A problem P admits *jump DNC avoidance* if for every set Z and every set D such that Z' is not of DNC degree over D, every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)'$ is not of DNC degree over D.

1. Show that if P admits a low basis, then it admits jump DNC avoidance.
2. Give an example of a problem which admits a weakly low basis, but not jump DNC avoidance. ★

[2] Recall that a problem P admits a *weakly low basis* if for every set Z every PA degree P over Z', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)' \leq_T P$. For example, Ramsey's theorem for pairs admits a weakly low basis.

8.3 Forcing and density

First-jump control using forcing constructions can be really thought of as straightforward generalization of the finite extension method. On the other hand, the full power of the forcing framework is unleashed when deciding properties at higher levels on the arithmetic hierarchy, and it already witnessed with Π^0_2 properties. Consider Cohen forcing for the sake of simplicity, that is, the set of finite binary strings $2^{<\mathbb{N}}$ partially ordered by the prefix relation $\preceq$. [3] The *interpretation* of a Cohen condition σ is the class $[\sigma] = \{X \in 2^{\mathbb{N}} : \sigma \prec X\}$, that is, the class of all infinite binary sequences starting with σ.

Intuitively, a condition p forces a property $\varphi(G)$ if p, seen as an approximation of the constructed set G, already contains the information that $\varphi(G)$ will hold. One would be therefore tempted to use the following definition:

> **Definition 8.3.1.** A condition p *strongly forces* a property $\varphi(G)$ if $\varphi(G)$ holds for every $G \in [p]$. ♢

In the case of Cohen forcing, σ strongly forces $\varphi(G)$ if $\varphi(G)$ holds for every infinite binary sequence starting with σ. The strong forcing relation ensures that whatever the remainder of the construction, even if the construction is very degenerate, then the property will hold. For example, if σ strongly forces $\varphi(G)$, then $\varphi(G)$ will hold even for $G = \sigma 00000\cdots$ or $G = \sigma 11111\cdots$, which can both be considered as very degenerate constructions since at any stage, one could decide to include any arbitrary finite binary sequence. This strong forcing relation is suitable for Σ^0_1 and Π^0_1 properties, and therefore sufficient for first-jump control.

Lemma 8.3.2. For every Σ^0_1 formula $\varphi(G)$, the set of all Cohen conditions strongly forcing either $\varphi(G)$ or $\neg\varphi(G)$ is dense. ★

Proof. Say $\varphi(G) \equiv (\exists x)\psi(G{\upharpoonright}x)$ for some Δ^0_0-formula ψ. Let σ be a Cohen condition. If there is some $\tau \succeq \sigma$ and some $x < |\tau|$ such that $\psi(\tau{\upharpoonright}x)$ holds, then for every $G \in [\tau]$, $\psi(G{\upharpoonright}x)$ holds, hence τ strongly forces $\varphi(G)$. Otherwise, for every $\tau \succeq \sigma$ and every $x < |\tau|$, $\neg\psi(\tau{\upharpoonright}x)$ holds, hence for every $G \in [\sigma]$ and every x, $\neg\psi(G{\upharpoonright}x)$ holds, so σ strongly forces $\neg\varphi(G)$. ■

The previous lemma can be thought of as stating the completeness of the strong forcing relation for Σ^0_1 and Π^0_1 formulas in Cohen forcing. In particular, it follows that every such property about the constructed set can be decided at a finite stage of the construction. We loose completeness of the strong forcing relation when dealing with Σ^0_2 and Π^0_2 formulas. Consider for example the Π^0_2 formula $\varphi(G) \equiv$ "G is infinite", which can be written as $\forall x \exists y(y > x \wedge y \in G)$. Then no Cohen condition σ strongly forces either $\varphi(G)$ or $\neg\varphi(G)$ since $[\sigma]$ contains the finite set $G = \sigma 00000\cdots$ and the infinite set $G = \sigma 11111\cdots$. On the other hand, there is an asymmetry between the two cases, as there are many ways to construct an infinite set, while any construction of a finite set must be degenerate. For every condition σ, there is an extension $\tau \succeq \sigma$ such that $\operatorname{card}\tau > \operatorname{card}\sigma$[4], hence every sufficiently generic filter yields an infinite set.

Let us now consider an arbitrary Σ^0_2 formula $\varphi(G) \equiv \exists x\psi(Gx)$, where ψ is a Π^0_1 formula. Given a Cohen condition σ, either there exists an extension $\tau \succeq \sigma$ strongly forcing $\psi(G, x)$ for some x, in which case τ forces $\varphi(G)$, or for every

[3] 3: Traditionally, the order relation is reversed in forcing, that is, a condition q *extends* p if $q \leq p$. This order is justified by the fact that the condition q seen as an approximation the constructed set G is more precise than p, hence the class $[q]$ of candidate sets satisfying the approximation q is a subclass of $[p]$.

In the case of Cohen forcing, the relation "σ is a prefix of τ" is denoted $\sigma \preceq \tau$, which might cause some confusion with the usual forcing notation. In particular, an infinite descending sequence of Cohen conditions is an infinite ascending sequence of strings $\sigma_0 \preceq \sigma_1 \preceq \dots$

[4] 4: Here, we distinguish the length $|\sigma|$ of a string σ, and the cardinality $\operatorname{card}\sigma$ which is the cardinality of the finite set $\{x < |\sigma| : \sigma(x) = 1\}$.

extension $\tau \geq \sigma$ such that for every x and every extension $\tau \geq \sigma$, τ does not strongly force $\psi(G, x)$. In the latter case, by Lemma 8.3.2, for every x and every $\tau \geq \sigma$, there is an extension ρ strongly forcing $\neg\psi(G, x)$. In other words, for every x, the set of conditions strongly forcing $\neg\psi(G, x)$ is dense below σ. Then, if $\mathcal{F}$ is a sufficiently generic filter containing σ, it will contain for every x a condition strongly forcing $\neg\psi(G, x)$, hence $(\forall x)\neg\psi(G_{\mathcal{F}}, x)$ will hold. This motivates the following definition of the forcing relation.

> **Definition 8.3.3.** A condition p *forces* a property $\varphi(G)$ if $\varphi(G_{\mathcal{F}})$ holds for every sufficiently generic filter $\mathcal{F}$ containing p. ◇

With this definition, every Cohen condition forces G to be infinite. For any reasonable notion of forcing, one can prove that for every arithmetic formula $\varphi(G)$, the set of conditions forcing either $\varphi(G)$ or $\neg\varphi(G)$ is dense.

The previous explanation induced a forcing question for Σ^0_2 formulas in Cohen forcing.

> **Definition 8.3.4.** Let σ be a Cohen condition, and $\varphi(G) \equiv \exists x\psi(G, x)$ be a Σ^0_2 formula. Define $\sigma \mathrel{?\vdash} \varphi(G)$ to hold if there exists some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that τ strongly forces $\psi(G, x)$, that is, for every $\rho \geq \tau$, $\psi(\rho, x)$ holds.[5] [6] ◇

A simple analysis on the definition of the forcing question shows that it is Σ^0_2-preserving. The existence of a Σ^0_2-preserving forcing question for Σ^0_2 formulas yields jump cone avoidance, with the same proof of Theorem 3.3.4, *mutatis mutandis*

Theorem 8.3.5
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_2-preserving forcing question. For every non-Δ^0_2 set C and every sufficiently generic filter $\mathcal{F}$, C is not $\Delta^0_2(G_{\mathcal{F}})$.

Proof. It suffices to prove the following lemma:

Lemma 8.3.6. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi_e^{G'} \neq C$. ★

Proof. Consider the following set[7]

$$U = \{(x, v) \in \mathbb{N} \times 2 : p \mathrel{?\vdash} \Phi_e^{G'}(x){\downarrow}= v\}$$

Since the forcing question is Σ^0_1-preserving, the set U is Σ^0_1. There are three cases:

- Case 1: $(x, 1-C(x)) \in U$ for some $x \in \mathbb{N}$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^{G'}(x){\downarrow}= 1 - C(x)$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. By Property (2) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^{G'}(x){\uparrow}$ or $\Phi_e^{G}(w){\downarrow}\neq C(x)$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_2 graph of the characteristic function of C, hence C is computable. This contradicts our hypothesis. ■

We are now ready to prove Theorem 8.3.5. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi_e^G \neq C$.. It follows from Lemma 8.3.6 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $C \not\leq_T G_{\mathcal{F}}$. This completes the proof of Theorem 8.3.5. ■

[5] Recall that Cohen forcing admits a Σ^0_1-preserving forcing question for Σ^0_1 formulas defined as $\sigma \mathrel{?\vdash} \varphi(G)$ if there is some $\tau \geq \sigma$ such that $\varphi(\tau)$ holds. It induces a forcing question for Π^0_1 formulas by taking its negation. In the following of this chapter, it might be better to think of the forcing question for a Σ^0_2 formula $\varphi(G) \equiv \exists x\psi(G, x)$ as $\sigma \mathrel{?\vdash} \varphi(G)$ if there is some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \mathrel{?\vdash} \psi(G, x)$.

[6] Note that with this forcing question, either there exists an extension strongly forcing $\varphi(G)$, or an extension forcing $\neg\varphi(G)$. In general, the forcing relation for Σ^0_2 formulas can be chosen to be the strong version, while the general definition is needed for Π^0_2 formulas.

[7] By Post's theorem, the property $\Phi_e^{G'}(x){\downarrow}= v$ is Σ^0_2, although the translation is not straightforward. It can be written as

$$\exists\rho\exists t[\Phi_e^{\rho}(x){\downarrow}= v \wedge \forall s\ \rho \leq G'_{t+s}]$$

where $\{G'_s\}_{s\in\mathbb{N}}$ is a fixed G-c.e. enumeration of G'.

In particular, since Cohen forcing admits a Σ^0_2-preserving forcing question for Σ^0_2 formulas, we obtain our first jump cone avoidance theorem using a direct second-jump control.

Theorem 8.3.7
Let C be a non-Δ^0_2 set. For every sufficiently Cohen generic filter $\mathcal{F}$, C is not $\Delta^0_2(G_{\mathcal{F}})$.

Exercise 8.3.8. Consider Cohen forcing. Recall from Section 3.6 that a forcing question is Σ^0_n*-compact* if for every $p \in \mathbb{P}$ and every Σ^0_n formula $\varphi(G, x)$, if $p \mathrel{?\!\vdash} \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ such that $p \mathrel{?\!\vdash} \exists x \in F\ \varphi(G, x)$.

1. Show that the forcing question for Σ^0_2 formulas is Σ^0_2-compact
2. Adapt Theorem 3.6.4 to prove that for every $\emptyset'$-hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and every sufficiently Cohen generic filter $\mathcal{F}$, the function f is $G'_{\mathcal{F}}$-hyperimmune. ⋆

8.4 Weak König's lemma

As explained in the previous section, the forcing relation for a Π^0_2 formula $\forall x \psi(G, x)$ is a density statement for a countable family of Σ^0_1 formulas $\{\psi(G, x) : x \in \mathbb{N}\}$. Density statements require to quantify over the partial order, which is not an issue when dealing with Cohen forcing, but can be very complicated is the partial order is not computable as it is often the case. One will then need to define a custom forcing question with the desired properties.

Our first non-trivial example concerns weak König's lemma, for which we prove it admits simultaneously cone and jump cone avoidance.[8]

Theorem 8.4.1 (Wang [70])
Let C be a non-computable set and D be a non-Δ^0_2 set. For every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists a member $G \in \mathcal{P}$ such that $C \not\leq_T G$ and $D \not\leq_T G'$.

Proof. Recall that Jockusch-Soare forcing is the notion of forcing whose conditions are infinite computable binary trees $T \subseteq 2^{<\mathbb{N}}$, partially ordered by the subset relation. In this proof, we shall actually restrict the partial order to infinite *primitive recursive* binary trees. Indeed, as mentioned before, the complexity of the partial order is relevant in second-jump control. The index set of all total computable sets is Π^0_2-complete, while all primitive recursive sets can be computably listed. The restriction to primitive recursive trees is without loss of generality, as shows the following lemma:

Lemma 8.4.2. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite co-c.e. tree. There is a primitive recursive tree $S \supseteq T$ such that $[S] = [T]$. ⋆

Proof. Say $T = \{\sigma \in 2^{<\mathbb{N}} : \Phi_e(\sigma)\uparrow\}$ for some partial computable function Φ_e. Let $S = \{\sigma \in 2^{<\mathbb{N}} : \forall s < |\sigma|\ \Phi_e(\sigma \upharpoonright s)[s]\uparrow\}$. Note that the predicate $\Phi_e(x)[s]\uparrow$ is primitive recursive, and primitive recursion is closed under bounded quantification. We first show that $S \supseteq T$. If $\sigma \in T$, then T being a tree, for every $s < |\sigma|$, $\sigma \upharpoonright s \in T$, so by definition of T, $\Phi_e(\sigma \upharpoonright s)[s]\uparrow$, hence $\sigma \in S$. Thus $S \supseteq T$, and in particular $[S] \supseteq [T]$. We now prove that $[S] \subseteq [T]$.

[8] By the cone avoidance basis theorem (Theorem 3.2.6), given a non-computable set C, every non-empty Π^0_1 class admits a member G such that $C \not\leq_T G$. By the low basis theorem (Theorem 4.4.6), given a non-Δ^0_2 set D, every non-empty Π^0_1 class admits a member G of low degree, in which case D is not $\Delta^0_2(G)$. One cannot however abstractly deduce from these theorems that WKL admits simultaneously cone and jump cone avoidance.

Lawton (see [69]) proved that one can actually combine the low and the cone avoidance basis theorem, by showing that if C is Δ^0_2 and non-computable, then every non-empty Π^0_1 class admits a member G of low degree such that $C \not\leq_T G$. The case where C is non-Δ^0_2 follows directly from the low basis theorem. Thus, as stated, Theorem 8.4.1 follows from Lawton's theorem, but its proof generalizes to countable cones avoidance, while Lawton's proof does not.

Let $P \in [S]$ and $\sigma \prec P$. Suppose for the contradiction that $\Phi_e(\sigma)\downarrow$. Then, letting $t > |\sigma|$ be such that $\Phi_e(\sigma)[t]\downarrow$, $P{\upharpoonright}t \notin S$, contradicting $P \in [S]$. It follows that $\Phi_e(\sigma)\uparrow$, and this for every $\sigma \prec P$, so $P \in [T]$. ■

In particular, there exists a primitive recursive tree T such that $[T] = \mathscr{P}$. The *interpretation* $[T]$ of a tree T is the class of its paths. Every sufficiently filter $\mathscr{F}$ for this notion of forcing induces a path $G_{\mathscr{F}}$ which is the unique element of $\bigcap\{[T] : T \in \mathscr{F}\}$. The forcing question for Σ^0_1 formulas of Exercise 3.3.7 also holds when working with primitive recursive trees.

Definition 8.4.3. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_1 formula $\varphi(G)$, define $T \operatorname{?\vdash} \varphi(G)$ to hold if there is some level $\ell \in \mathbb{N}$ such that $\varphi(\sigma)$ holds for every node σ at level ℓ in T. ◇

One easily sees that this forcing question is Σ^0_1-preserving.

Lemma 8.4.4. Let $T \subseteq 2^{<\mathbb{N}}$ be a condition and $\varphi(G)$ be a Σ^0_1 formula.

1. If $T \operatorname{?\vdash} \varphi(G)$, then T forces $\varphi(G)$
2. If $T \operatorname{?\nvdash} \varphi(G)$, then there is an extension $S \leq T$ forcing $\neg\varphi(G)$. ★

Proof. Suppose first $T \operatorname{?\vdash} \varphi(G)$. Let $\ell \in \mathbb{N}$ be the level witnessing it. For every $P \in [T]$, $P{\upharpoonright}\ell \in T$, so $\varphi(P{\upharpoonright}\ell)$ holds, hence $\varphi(P)$ holds. Thus T forces $\varphi(G)$. Suppose now $T \operatorname{?\nvdash} \varphi(G)$. Say $\varphi(G) \equiv \exists x\psi(G,x)$ for some Δ^0_0 formula ψ. Then $S = \{\sigma \in T : \forall x < |\sigma| \neg\psi(\sigma, x)\}$ is an infinite primitive recursive[9] subtree of T forcing $\neg\varphi(G)$. ■

Since this notion of forcing admits a Σ^0_1-preserving forcing question for Σ^0_1 formulas, by Theorem 3.3.4 for every sufficiently generic filter $\mathscr{F}$, $C \nleq_T G_{\mathscr{F}}$. Until now, the proof was only a rewriting of Theorem 3.2.6 with primitive recursive trees, using the more abstract framework of the forcing question. We now turn to second jump control.

Definition 8.4.5. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_2 formula $\varphi(G) \equiv \exists x\psi(G,x)$, define $T \operatorname{?\vdash} \varphi(G)$ to hold if there is some $x \in \mathbb{N}$ and an extension $S \leq T$ such that $S \operatorname{?\vdash} \psi(G,x)$.[10] [11] ◇

Looking at the complexity of the forcing question for Σ^0_2 formulas, the relation $S \operatorname{?\vdash} \psi(G,x)$ is Π^0_1 since it is the negation of the Σ^0_1-preserving forcing question for Σ^0_1 formulas. Being an infinite primitive recursive tree and being a subset of another primitive recursive tree is a Π^0_1 predicate, so the overall formula is Σ^0_2. We now show that this relation satisfies the specifications of a forcing question.

Lemma 8.4.6. Let $T \subseteq 2^{<\mathbb{N}}$ be a condition and $\varphi(G)$ be a Σ^0_2 formula.

1. If $T \operatorname{?\vdash} \varphi(G)$, then there is an extension $S \leq T$ forcing $\varphi(G)$
2. If $T \operatorname{?\nvdash} \varphi(G)$, then T forces $\neg\varphi(G)$. ★

Proof. Say $\varphi(G) \equiv \exists x\psi(G,x)$. Suppose first $T \operatorname{?\vdash} \varphi(G)$. Let $x \in \mathbb{N}$ and $S \leq T$ be such that $S \operatorname{?\vdash} \psi(G,x)$. By Lemma 8.4.4, there is an extension $S_1 \leq S$ forcing $\psi(G,x)$. In particular, $S_1 \leq T$ and S_1 forces $\varphi(G)$. Suppose now $T \operatorname{?\nvdash} \varphi(G)$. Let $x \in \mathbb{N}$. We claim that the set of all conditions forcing $\neg\psi(G,x)$ is dense below T. Indeed, given a condition $S \leq T$, $S \operatorname{?\nvdash} \psi(G,x)$, so by Lemma 8.4.4, there is an extension $S_1 \leq S$ forcing $\neg\psi(G,x)$. Thus, for every sufficiently generic filter $\mathscr{F}$ containing T and every $x \in \mathbb{N}$, there is a condition $S_1 \in \mathscr{F}$ forcing $\neg\psi(G,x)$, thus $\neg\varphi(G_{\mathscr{F}})$ holds. ■

[9] Every Δ^0_0 formula is primitive recursive. On this other hand, there exist primitive recursive predicates which are not Δ^0_0.

[10] In this definition, ψ is a Π^0_1 formula, so the relation $S \operatorname{?\vdash} \psi(G,x)$ is the forcing question for Π^0_1 formulas induced by the forcing question for Σ^0_1 formulas by taking the negation. Note the similarity with the forcing question for Σ^0_2 formulas in Cohen forcing.

[11] Although the partial order is not computable, the complexity of finding an extension is "absorbed" in the overall complexity of the forcing question for Σ^0_2 formulas, yielding a Σ^0_2-preserving forcing question. Because of this, the forcing questions at higher levels of the arithmetic hierarchy will be similar to the ones for Cohen forcing.

Since this notion of forcing admits a Σ^0_2-preserving forcing question for Σ^0_2 formulas, by Theorem 8.3.5 for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G'_{\mathcal{F}}$. To conclude the theorem, by Lemma 8.4.2, there is a condition T such that $[T] = \mathcal{P}$, so for every sufficiently generic filter $\mathcal{F}$ containing T, $G_{\mathcal{F}} \in \mathcal{P}$. This completes the proof of Theorem 8.4.1. ■

Exercise 8.4.7 (Le Houérou, Levy Patey and Mimouni [71]). Recall the notion of Σ^0_n-compactness from Section 3.6. Consider the Jockusch-Soare notion of forcing restricted to primitive recursive trees (Theorem 8.4.1).

1. Show that the forcing questions for Σ^0_1 and Σ^0_2 formulas are Σ^0_1-compact and Σ^0_2-compact, respectively.
2. Fix a hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and a $\emptyset'$-hyperimmune function $g : \mathbb{N} \to \mathbb{N}$. Prove that every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ has a member G such that f is G-hyperimmune and g is G'-hyperimmune. ⋆

8.5 Cohesiveness principle

As mentioned before, because of its equivalence with the statement "every Δ^0_2 infinite binary tree admits a Δ^0_2-approximation of a path", the cohesiveness principle is a statement about jump computation. By Toswner's theorem (Theorem 7.3.8) Δ^0_2-approximations of a path can be added to a model without of RCA_0 without affecting its first-jump properties. Thus, one should expect from a natural notion of forcing for COH to have a trivial first-jump control, and a second-jump control resembling the one of weak König's lemma. This is actually the case.

Consider a uniformly computable sequence of sets $R_0, R_1, \dots$ The usual notion of forcing to build $\vec{R}$-cohesive sets with a good first-jump control is computable Mathias forcing, that is, Mathias forcing whose reservoirs are computable. The first-jump control of such a notion of forcing is very similar to Cohen forcing, and preserves the same first-jump properties. On the other hand, even when working with computable reservoirs, Mathias forcing does not admit a good second-jump control. In particular, every sufficiently generic filter for computable Mathias forcing yields a set of high degree. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *dominating* if it eventually dominates every total computable function. By Martin's domination theorem [72], a set X is of high degree iff it computes a dominating function.

Proposition 8.5.1. Let $\mathcal{F}$ be a sufficiently generic filter for computable Mathias forcing. Then the principal function of $G_{\mathcal{F}}$ is dominating, hence $G_{\mathcal{F}}$ is of high degree. ⋆

Proof. Let f be a total computable function. We can assume without loss of generality that f is strictly increasing. Let us shows that the class $\mathcal{D}_f$ of all computable Mathias conditions (τ, Y) forcing the principal function of G to eventually dominate f is dense. Fix a computable Mathias condition (σ, X), and say $X = \{x_0 < x_1 < \dots\}$. Let $a = \mathrm{card}\{x < |\sigma| : \sigma(x) = 1\}$. Then the set $Y = \{x_{f(a+s)} : s \in \mathbb{N}\}$ is a computable subset of X and (σ, Y) forces the principal function of G to eventually dominate f. ■

There are multiple ways to explain why computable Mathias forcing does not admit a good second-jump control, each of them yielding the same conclusion:

the problem comes from the permissiveness of the reservoirs, which can be arbitrary computable sets.[12]

1. *Sparsity of the reservoirs*. Proposition 8.5.1 shows that computable Mathias forcing allows to take extensions with sparse reservoirs and then produce dominant functions. However, the only operations needed to produce cohesive sets is to split the reservoir according to computable partitions and pick any infinite part. The first condition is $(\epsilon, \mathbb{N})$ with a non-sparse reservoir. Then, intuitively, if a reservoir X is not too sparse, then for every 2-partition $X_0 \sqcup X_1 = X$, at least one of the parts not too sparse either. One could therefore maintain non-sparsity as an invariant by asking the reservoirs to be boolean combinations of $R_0, R_1, \dots$
2. *Complexity of the partial order*. When trying to design a forcing question for Σ^0_2 formulas in computable Mathias forcing, one needs to quantify over the partial order, and therefore quantify over infinite computable subsets of the reservoir. This quantification is too complex and cannot be "absorbed" in the complexity of the general formula to produce a Σ^0_2-preserving question. One must therefore adopt a more efficient way to represent forcing conditions, such as only keeping track of the boolean choices of partitions induced by the sets $R_0, R_1, \dots$

In the following theorem, we restrict computable Mathias forcing to conditions obtained boolean combinations of computable partitions, and take advantage of this additional structure to design a forcing question with a good second-jump control. This yields that COH admits simultaneously cone and jump cone avoidance.

Theorem 8.5.2
Let C be a non-computable set and D be a non-Δ^0_2 set. For every uniformly computable sequence of sets $R_0, R_1, \dots$, there exists an infinite cohesive set G such that $C \not\leq_T G$ and $D \not\leq_T G'$.

Proof. Given $\rho \in 2^{<\mathbb{N}}$, let

$$R_\rho = \bigcap_{\rho(n)=0} R_n \bigcap_{\rho(n)=1} \overline{R}_n$$

and let $T = \{\rho \in 2^{<\mathbb{N}} : \exists x > |\rho| \; x \in R_\rho\}$. Note that T is a Σ^0_1 tree, and for every extendible node $\rho \in T$, R_ρ is infinite. By the cone avoidance basis theorem (Theorem 3.2.6) relativized to $\emptyset'$, there is a path $P \in [T]$ such that $D \not\leq_T P \oplus \emptyset'$.

Consider the notion of forcing whose conditions[13] are pairs (σ, n). One can think of such a condition as computable Mathias condition $(\sigma, R_{P\upharpoonright n})$. Note that since $P \in [T]$, $R_{P\upharpoonright n}$ is infinite. The *interpretation* of a condition (σ, n) is the interpretation of the associated computable Mathias condition, that is

$$[\sigma, n] = \{G : \sigma \preceq G \subseteq \sigma \cup R_{P\upharpoonright n}\}$$

A condition (τ, m) *extends* (σ, n) if $\sigma \preceq \tau$, $m \geq n$, and $\tau \setminus \sigma \subseteq R_{P\upharpoonright n}$. Every sufficiently generic filter $\mathcal{F}$ for this notion of forcing induces a path $G_\mathcal{F}$ defined as $\bigcup\{\sigma : (\sigma, n) \in \mathcal{F}\}$. Alternatively, $G_\mathcal{F}$ is the unique element of $\bigcap_{(\sigma,n)\in\mathcal{F}}[\sigma, n]$. The forcing question for Σ^0_1 formulas is induced from the forcing question in computable Mathias forcing:

[12] The general takeway of this discussion is that when trying to design a notion of forcing with a good second-jump control, consider a notion of forcing with a good first-jump control, then restrict the partial order to be the less permissive possible, allowing only the conditions produced by the first-jump control. This usually yields a partial order with better complexity, and hopefully enables to define a Σ^0_2-preserving forcing question.

[13] Note the similarity with the notion of forcing in Theorem 3.2.4. In both cases, we build a cone avoiding set G whose jump computes a fixed degree. Indeed, if G is $\vec{R}$-cohesive, then for every n, there is exactly one ρ of length n such that $G \subseteq^* R_\rho$, and such a ρ can be found G'-computably. By construction, $\rho \prec P$, so $G' \geq_T P$.

Definition 8.5.3. Given a condition (σ, n) and a Σ^0_1 formula $\varphi(G)$, define $(\sigma, n) \operatorname{?\vdash} \varphi(G)$ to hold if there is some $\tau \in [\sigma, n]$ such that $\varphi(\tau)$ holds. $\diamond$

One easily sees that this forcing question is Σ^0_1-preserving, although not uniformly in the condition, since one needs to hard-code the initial segment of P of length n.

Lemma 8.5.4. Let (σ, n) be a condition and $\varphi(G)$ be a Σ^0_1 formula.

1. If $(\sigma, n) \operatorname{?\vdash} \varphi(G)$, then there is an extension $(\tau, n) \leq (\sigma, n)$ forcing $\varphi(G)$;
2. If $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$, then (σ, n) forces $\neg\varphi(G)$. $\star$

Proof. Suppose first $(\sigma, n) \operatorname{?\vdash} \varphi(G)$. Let $\tau \in [\sigma, n]$ be such that $\varphi(\tau)$ holds. Then (τ, n) is a valid extension and for every $G \in [\tau, n]$, $\tau \preceq G$, so $\varphi(G)$ holds. It follows that (τ, n) forces $\varphi(G)$. Suppose now $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$. Then for every extension $(\tau, m) \leq (\sigma, n)$, $\tau \in [\sigma, n]$, so $\neg\varphi(\tau)$ holds. It follows that (σ, n) forces $\neg\varphi(G)$. ■

Since this notion of forcing admits a Σ^0_1-preserving forcing question for Σ^0_1 formulas, by Theorem 3.3.4 for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$. We now turn to second jump control.

Definition 8.5.5. Given a condition (σ, n) and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $(\sigma, n) \operatorname{?\vdash} \varphi(G)$ to hold if there is some $x \in \mathbb{N}$ and an extension $(\tau, m) \leq (\sigma, n)$ such that $(\tau, m) \operatorname{?\vdash} \psi(G, x)$.[14] $\diamond$

The extension relation $(\tau, m) \leq (\sigma, n)$ is computable uniformly in P. Moreover, the relation $(\tau, m) \operatorname{?\vdash} \psi(G, x)$ is Π^0_1 since the forcing question for Σ^0_1 formulas is Σ^0_1-preserving. It follows that the forcing question for Σ^0_2 formulas is $\Sigma^0_1(P \oplus \emptyset')$.

Lemma 8.5.6. Let (σ, n) be a condition and $\varphi(G)$ be a Σ^0_2 formula.

1. If $(\sigma, n) \operatorname{?\vdash} \varphi(G)$, then there is an extension $(\tau, m) \leq (\sigma, n)$ forcing $\varphi(G)$;
2. If $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$, then (σ, n) forces $\neg\varphi(G)$. $\star$

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$. Suppose first $(\sigma, n) \operatorname{?\vdash} \varphi(G)$. Then there exists some $x \in \mathbb{N}$ and an extension $(\tau, m) \leq (\sigma, n)$ such that $(\tau, m) \operatorname{?\vdash} \psi(G, x)$. By Lemma 8.5.4, there is an extension of (τ, m) forcing $\psi(G, x)$, hence forcing $\varphi(G)$. Suppose now $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$. Fix some $x \in \mathbb{N}$. We claim that the set of all conditions forcing $\neg\psi(G, x)$ is dense below (σ, n). Indeed, given a condition $(\tau, m) \leq (\sigma, n)$, $(\tau, m) \operatorname{?\nvdash} \psi(G, x)$, so by Lemma 8.5.4, there is an extension for (τ, m) forcing $\neg\psi(G, x)$. Thus, for every sufficiently generic filter $\mathcal{F}$ containing (σ, n) and every $x \in \mathbb{N}$, there is a condition in $\mathcal{F}$ forcing $\neg\psi(G, x)$, so $\neg\varphi(G_{\mathcal{F}})$ holds. ■

Exercise 8.5.7. Using the fact that the forcing question for Σ^0_2 formulas is $\Sigma^0_1(P \oplus \emptyset')$ and that $D \not\leq_T P \oplus \emptyset'$, adapt Theorem 3.3.4 to show that for every sufficiently generic filter $\mathcal{F}$, $D \not\leq_T G'_{\mathcal{F}}$. $\star$

Thus, for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$ and $D \not\leq_T G'_{\mathcal{F}}$. Since $P \in [T]$, then for every n, $R_{P \upharpoonright n}$ is infinite, hence for every sufficiently generic

[14] As before, ψ is a Π^0_1 formula, so we consider the forcing question for Π^0_1 induced by the forcing question for Σ^0_1 formulas by taking the negation.

filter $\mathcal{F}$, $G_{\mathcal{F}}$ is infinite. Last, for every condition (σ, n), the condition $(\sigma, n+1)$ is a valid extension, so for every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is cohesive for $R_0, R_1, \dots$ This completes the proof of Theorem 8.5.2. ■

The second-jump control in the proof of Theorem 8.5.2 was in two steps: first, one picked the sequence of boolean decisions $P \in [T]$ by a relativized first-jump control for WKL, then one built an infinite cohesive set G with a $\Sigma^0_1(P \oplus \emptyset')$ forcing question for Σ^0_2 formulas. One can actually define a notion of forcing doing both at once, as shows the following exercise.

Exercise 8.5.8 (Patey [73]). Fix a uniformly computable sequence of sets $R_0, R_1, \dots$ and define R_ρ and T as in Theorem 8.5.2. Consider the notion of forcing whose *conditions* are tuples (σ, ρ, S), where σ is a finite string, S is an infinite $\emptyset'$-primitive recursive subtree of T[15], and ρ is an extendible node in S. One can think of a condition as a computable Mathias condition (σ, R_ρ), together with a $\emptyset'$-primitive recursive Jockusch-Soare forcing condition S. A condition (τ, μ, V) *extends* a condition (σ, ρ, S) if $\sigma \preceq \tau$, $\rho \preceq \mu$, $V \subseteq S$ and $\tau \setminus \sigma \subseteq R_\rho$.

1. Define a Σ^0_1-preserving forcing question for Σ^0_1 formulas.[16]
2. Define a Σ^0_2-preserving forcing question for Σ^0_2 formulas.[17] ⋆

8.6 Partition regularity

Most theorems from Ramsey theory are proven using variants of Mathias forcing. However, as shows Proposition 8.5.1, generic Mathias filters tend to produce sets of high degree, even when working with computable reservoirs. In order to construct solutions to theorems from Ramsey theory with a good second-jump control, one must therefore refine this notion of forcing to be less permissive about reservoirs. In the case of the cohesiveness principle, the solution was restrict the reservoirs to boolean combinations of a a uniformly computable sequence of sets. In this section, we generalize the approach by allowing to split the reservoirs based on *any* finite partition of the integers. This yields the notion of partition regularity.

Definition 8.6.1. A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is *partition regular*[18] if

1. $\mathcal{P}$ is non-empty ;
2. For all $X \in \mathcal{P}$ and $Y \supseteq X$, $Y \in \mathcal{P}$;
3. For every $X \in \mathcal{P}$ and every 2-cover $Y_0 \cup Y_1 \supseteq X$, there is some $i < 2$ such that $Y_i \in \mathcal{P}$. ◊

There exist many examples of partition regularity statements in combinatorics.

Example 8.6.2. The following classes are partition regular:

1. $\{X : X \text{ is infinite }\}$ by the infinite pigeonhole principle ;
2. $\{X : n \in X\}$ for a fixed $n \in \mathbb{N}$;
3. $\{X : \limsup_{n\to\infty} \frac{|\{1,2,\dots,n\} \cap X|}{n} > 0\}$;
4. $\{X : \sum_{n\in X} \frac{1}{n} = \infty\}$.

[15] Note that by restricting the tree T, one restricts the possible reservoirs R_ρ with $\rho \in T$, so one restricts the forced negative information. Thus, the third component of a condition forces positive information. This shall be explained in the next section in further details.

[16] Note that given a condition (σ, ρ, S), the forcing question does not involve S, and the answers leave ρ and S unchanged. First-jump control can therefore "ignore" the components responsible of higher jump control.

[17] Hint: combine the forcing question for Σ^0_2 formulas in Definition 8.5.5 and the forcing question for Σ^0_1 formulas in Definition 8.4.3.

[18] By the upward-closure of a partition regular class, $\mathcal{P}$ is non-empty iff $\mathbb{N} \in \mathcal{P}$, and the last property can be restricted to 2-partitions of X, that is, where $Y_0 \cap Y_1 = \emptyset$ and $Y_0 \cup Y_1 = X$. By iterating the splitting, if $\mathcal{P}$ is partition regular, then for every k, for every $X \in \mathcal{P}$ and every k-cover $Y_0 \cup \cdots \cup Y_{k-1} \supseteq X$, there is some $i < k$ such that $Y_i \in \mathcal{P}$.

Among these examples, the second is considered as degenerate, as it contains finite sets. A partition regular class is *principal* if it is of the form $\{X : n \in X\}$ for a fixed $n \in \mathbb{N}$. We shall work only with partition regular classes containing only infinite sets. A class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *non-trivial* if it contains only sets with at least two elements. If $\mathcal{A}$ is partition regular, then it is non-trivial iff it contains only infinite sets.[19] The following operator is an easy way to define non-trivial partition regular classes:

Definition 8.6.3. Given an infinite set X, let $\mathcal{L}_X = \{Y : X \cap Y \text{ is infinite }\}$.◇

In the computability-theoretic realm, many statements of the form "Every set A has an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ satisfying some weakness property" can be rephrased in terms of partition regularity.

Example 8.6.4. The following classes are partition regular:

1. $\{X : \exists Y \in [X]^\omega\ Y \not\geq_T C\}$ for any $C \not\leq_T \emptyset$ (Theorem 3.4.5);
2. $\{X : \exists Y \in [X]^\omega\ Y$ is not of PA degree (Theorem 5.4.3);

One can think of non-trivial partition regular classes as generalizations of the notion of infinity, satisfying some basic operations that one expects of infinite sets, that is, if a set is infinite, then any superset is again infinite, and when splitting an infinite set in two parts, at least one of the parts is infinite.[20] Looking at the proof of strong cone avoidance of RT^1_2 (Theorem 3.4.5), splitting and finite truncation are the only operations on the reservoir to obtain a good first-jump control. One can therefore fix a partition regular class $\mathcal{P}$ and work with conditions whose reservoir belongs to $\mathcal{P}$.

Exercise 8.6.5 (Flood [75]). Adapt the proof of Theorem 3.4.5 to show that for every non-computable set C and every set A, there is a set $H \subseteq A$ or $H \subseteq \overline{A}$ such that $C \not\leq_T H$ and $\limsup_{n\to\infty} \frac{|\{1,2,\dots,n\}\cap X|}{n} > 0$. ★

Exercise 8.6.6. Let $\mathcal{P}$ be a non-trivial partition regular class. Show that if $X \in \mathcal{P}$ and $Y =^* X$, then $Y \in \mathcal{P}$. In other words, $\mathcal{P}$ is closed under finite changes. ★

Exercise 8.6.7 (Monin and Patey [74]). Let $\{\mathcal{P}_i\}_{i\in I}$ be an arbitrary union of partition regular classes. Show that $\bigcup_{i\in I} \mathcal{P}_i$ is partition regular. ★

Exercise 8.6.8. Given an infinite set X, let $\mathcal{L}_X = \{Z : Z \cap X \text{ is infinite }\}$. Prove that for every partition regular class $\mathcal{P}$, the following class is partition regular:

$$\{X : \mathcal{L}_X \cap \mathcal{P} \text{ is partition regular }\}$$

Positive and negative information. One can understand the restriction of the reservoirs to partition regular classes in terms of *positive* and *negative* information. In a Mathias condition (σ, X), the stem σ fixes an initial segment of the constructed set G. It specifies that G must contain $\{n : \sigma(n) = 1\}$ and must avoid $\{n : \sigma(n) = 0\}$. Thus, σ forces a finite amount of positive and negative information. On the other hand, the reservoir X forces an infinite amount of negative information since G must avoid any new element outside the reservoir, but does not force any positive information, as for every $n \in X$, one can construct a set G such that $n \notin G$.

[19] Note that a non-trivial partition regular class does not contain any principal partition regular subclass.

[20] Partition regular classes contain every "typical set". In particular, if $\mathcal{P}$ is partition regular and measurable, then its measure is 1 (see Monin and Patey [74]). Moreover, if $\mathcal{P}$ satisfies the Baire property, then it is co-meager.

It is useful to think as a Σ^0_1 property as a positive information and therefore a Π^0_1 property as a negative one. When constructing a set using a variant of Mathias forcing with the first-jump control, one usually increases the stem to force Σ^0_1 properties, and decrease the reservoir to force Π^0_1 properties. The situation becomes more complicated when forcing Π^0_2 properties $\forall x \psi(G, x)$, as it becomes a density statement about a countable collection of Σ^0_1 properties $\{\psi(G, x) : x \in \mathbb{N}\}$. It therefore requires to maintain some positive information over all future conditions. A partition regular class is therefore a "reservoir of reservoirs", as it restricts the possible choices of reservoirs, hence restricts the future negative information, which is a way of forcing positive information.

8.6.1 Largeness

One should expect from a notion of largeness that it is upward-closed under inclusion, that is, if $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is a largeness notion and $\mathcal{B} \supseteq \mathcal{A}$, then so is $\mathcal{B}$. The collection of all partition regular classes is not closed upward. For instance, pick any non-trivial partition regular class $\mathcal{P}$ which does not contain some infinite set X. Then the $\mathcal{P} \cup \{Z : Z \supseteq X\}$ is an upward-closed superset of $\mathcal{P}$, but is not partition regular. The following notion of largeness is more convenient to work with:

Definition 8.6.9. A class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *large*[21] if

1. For all $X \in \mathcal{A}$ and $Y \supseteq X$, $Y \in \mathcal{A}$;
2. For every $k \in \mathbb{N}$ and every k-cover $Y_0 \cup \cdots \cup Y_{k-1} = \mathbb{N}$, there is some $i < k$ such that $Y_i \in \mathcal{A}$. ◊

There exists a formal relationship between largeness and partition regularity: a class is large iff it contains a partition regular subclass. The union of a family of partition regular classes being again partition regular, every large class contains a maximal partition regular subclass for inclusion. This subclass admits the following explicit syntactic definition.

Proposition 8.6.10 (Monin and Patey [29]). Given a large class $\mathcal{A} \subseteq 2^{\mathbb{N}}$, the class

$$\mathcal{L}(\mathcal{A}) = \{X \in \mathcal{A} : \forall k \forall X_0 \cup \cdots \cup X_{k-1} \supseteq X\ \exists i < k\ X_i \in \mathcal{A}\}$$

is the maximal partition regular subclass of $\mathcal{A}$. ★

Proof. We first prove that $\mathcal{L}(\mathcal{A})$ is a partition regular subclass of $\mathcal{A}$. First, note that $\mathcal{L}(\mathcal{A})$ is upward-closed. Moreover, by definition of $\mathcal{A}$ being large, $\mathbb{N} \in \mathcal{L}(\mathcal{A})$, so $\mathcal{L}(\mathcal{A})$ is non-empty. Let $X \in \mathcal{L}(\mathcal{A})$ and $X_0 \cup \cdots \cup X_{k-1} \supseteq X$. Suppose for the contradiction that $X_i \notin \mathcal{L}(\mathcal{A})$ for every $i < k$. Then, for every $i < k$, there is some $k_i \in \mathbb{N}$ and some k_i-cover $Y^0_i \cup \cdots \cup Y^{k-1}_i \supseteq X_i$ such that $Y^j_i \notin \mathcal{A}$ for every $j < k_i$. hen $\{Y^j_i : i < k, j < k_i\}$ is a cover of X contradicting $X \in \mathcal{L}(\mathcal{A})$. Therefore, $\mathcal{L}(\mathcal{A})$ is partition regular. Moreover, $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ as witnessed by taking the trivial cover of X by itself.

We now prove that $\mathcal{L}(\mathcal{A})$ is the maximal partition regular subclass of $\mathcal{A}$. Let $\mathcal{B}$ be a partition regular subclass of $\mathcal{A}$. Then for every $X \in \mathcal{B}$, every $X_0 \cup \cdots \cup X_{k-1} \supseteq X$, there is some $i < k$ such that $X_i \in \mathcal{B} \subseteq \mathcal{A}$. Thus $X \in \mathcal{L}(\mathcal{A})$, so $\mathcal{B} \subseteq \mathcal{L}(\mathcal{A})$. ■

[21] Note that a large class is necessarily non-empty, as $\mathbb{N} \in \mathcal{A}$.

Recall that a class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *non-trivial* if it contains only sets with at least two elements. Note that contrary to partition regular classes, a non-trivial large class may contain finite sets, but its maximal partition regular subclass $\mathcal{L}(\mathcal{A})$ contains only infinite sets.

Exercise 8.6.11 (Monin and Patey [74] ; Mimouni).

1. Show that if $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is a non-trivial partition regular class and $X \in \mathcal{P}$, then $\mathcal{P} \cap \mathcal{L}_X$ is large.
2. Construct a non-trivial partition regular class $\mathcal{P}$ and a set $X \in \mathcal{P}$ such that $\mathcal{P} \cap \mathcal{L}_X$ is not partition regular. ⋆

Exercise 8.6.12 (Monin and Patey [74]). Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a non-trivial large class. Show that $\mathcal{L}(\mathcal{A}) = \{X : \mathcal{A} \cap \mathcal{L}_X \text{ is large }\}$. ⋆

Exercise 8.6.13 (Monin and Patey [29]). Show that if $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence of large classes, then $\bigcap_n \mathcal{A}_n$ is large. ⋆

8.6.2 Effective classes

The class of all infinite sets is Π^0_2. Actually, this is the first level of the effective Borel hierarchy containing a non-trivial partition regular class, as there is no non-trivial $\mathbf{\Sigma}^0_2$ partition regular class [74].[22] Moreover, Π^0_2 classes is the first level satisfying some stability, in the sense that if a Σ^0_1 class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is large, then $\mathcal{L}(\mathcal{A})$ is Π^0_2, while if $\mathcal{A}$ is Π^0_2, then so is $\mathcal{L}(\mathcal{A})$. Actually, we shall work with a slightly more general family of partition regular classes: arbitrary intersections of Σ^0_1 classes over a Scott ideal.

In what follows, fix a uniform sequence of all c.e. sets of strings $W_0, W_1, \dots \subseteq 2^{<\mathbb{N}}$. It induces an enumeration of all upward-closed Σ^0_1 classes $\mathcal{U}_0, \mathcal{U}_1, \dots$ defined by $\mathcal{U}_e = \{X \in 2^{\mathbb{N}} : \exists \rho \in W_e\ \rho \subseteq X\}$. These enumerations admit immediate relativizations to oracles. We therefore let $\mathcal{U}^Z_0, \mathcal{U}^Z_1, \dots$ be an enumeration of all upward-closed $\Sigma^0_1(Z)$ classes. From now on, fix a Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$ with Scott code M.[23] Given a set $C \subseteq \mathbb{N}^2$, we let

$$\mathcal{U}^{\mathcal{M}}_C = \bigcap_{(e,i) \in C} \mathcal{U}^{Z_i}_e$$

From now on, we shall work exclusively with classes of the form $\mathcal{U}^{\mathcal{M}}_C$, and give a particular focus on the complexity of the set C of indices. Thanks to Exercise 8.6.13, if $\mathcal{U}^{\mathcal{M}}_C$ is not large, then there is a finite set $F \subseteq C$ such that $\mathcal{U}^{\mathcal{M}}_F$ is not large either. Note that the latter class is $\Sigma^0_1(\mathcal{M})$. This pseudo-compactness phenomenon plays a key role in the computability-theoretic features of large classes.

Lemma 8.6.14 (Monin and Patey [68]). Let $C \subseteq \mathbb{N}^2$ be a set. The statement "$\mathcal{U}^{\mathcal{M}}_C$ is large" is $\Pi^0_1(C \oplus M')$ uniformly in C and M. ⋆

Proof. Let us first show that the statement "$\mathcal{U}^Z_e$ is large" is $\Pi^0_2(Z)$ uniformly in e and Z. Indeed, by compactness, $\mathcal{U}^Z_e$ is large iff for every $k \in \mathbb{N}$, there is some $\ell \in \mathbb{N}$ such that for every k-partition $Y_0 \cup \dots \cup Y_{k-1} = \{0, \dots, \ell\}$, there is some $i < k$ and some $\rho \in W_e$ such that $\rho \subseteq Y_i$. This statement is $\Pi^0_2(Z)$ uniformly in e and Z. Then, by Exercise 8.6.13, $\mathcal{U}^{\mathcal{M}}_C$ is large iff for every finite set $F \subseteq C$, $\mathcal{U}^{\mathcal{M}}_F$ is large. The resulting statement is therefore $\Pi^0_1(C \oplus M')$. ■

[22] We write boldface $\mathbf{\Sigma^0_n}$ for the levels of the Borel hierarchy, and lightface Σ^0_n for the levels of its effective hierarchy.

[23] Recall that a Scott ideal is a Turing ideal which satisfies weak König's lemma, that is, for every infinite binary tree $T \in \mathcal{M}$, then $[T] \cap \mathcal{M} \neq \emptyset$. A *Scott code* for a Turing ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$ is a set $M = \bigoplus_i Z_i$ such that the basic operations on the M-indices are computable.

The following lemma shows that classes of the form $\mathcal{U}_C^{\mathcal{M}}$ are robust, in the sense that if a large class is of this form, then so is its maximum partition regular subclass. Moreover, the translation of the index sets is computable.

Lemma 8.6.15 (Monin and Patey [68]). Let $C \subseteq \mathbb{N}^2$ be a set. Then there exists a set $D \subseteq \mathbb{N}^2$ computable uniformly in C such that $\mathcal{U}_D^{\mathcal{M}} = \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. ⋆

Proof. We first claim that $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$. Indeed, for some finite $F \subseteq C$, $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{U}_F^{\mathcal{M}}$, so $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$ is a partition regular subclass of $\mathcal{U}_F^{\mathcal{M}}$. By maximality of $\mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$, we have $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$. Since it is the case of every $F \subseteq_{\mathtt{fin}} C$, we have $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$.

We next claim that $\bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}}) \subseteq \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. Suppose that $X \notin \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. Then there is some k and some k-cover $Y_0 \cup \dots \cup Y_{k-1} = X$ such that for every $i < k$, $Y_i \notin \mathcal{U}_C^{\mathcal{M}}$. Then there is a finite set $F \subseteq_{\mathtt{fin}} C$ such that for every $i < k$, $Y_i \notin \mathcal{U}_F^{\mathcal{M}}$, so $X \notin \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$. This proves our claim.

For every $F \subseteq_{\mathtt{fin}} C$, let $h(F)$ be an M-index of the set $\bigoplus_{(e,i) \in F} Z_i$. For every $F \subseteq_{\mathtt{fin}} C$ and $k \in \mathbb{N}$, let $g(F,k)$ be an index of the $Z_{h(F)}$-c.e. set of all $\rho \in 2^{<\mathbb{N}}$ such that for every k-partition $\rho_0 \cup \dots \cup \rho_{k-1} = \rho$, there is some $i < k$ such that for each $(e,i) \in F$, $W_e^{Z_i}$ enumerates a subset of ρ_i. In other words,

$$\mathcal{U}_{g(F,k)}^{Z_{h(F)}} = \{X : \forall Y_0 \cup \dots \cup Y_{k-1} = X\ \exists i < k\ Y_i \in \mathcal{U}_F^{\mathcal{M}}\}$$

Then, letting $D = \{(g(F,k), h(F)) : k \in \mathbb{N}, F \subseteq_{\mathtt{fin}} C\}$, the class $\mathcal{U}_D^{\mathcal{M}}$ equals $\bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$, which is nothing but $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. ■

Exercise 8.6.16 (Monin and Patey [74]). Let $\mathcal{P}$ be a Π_2^0 large class and X be co-hyperimmune. Show that $X \in \mathcal{P}$. ⋆

8.6.3 $\mathcal{M}$-minimal classes

As mentioned above, to obtain a variant of Mathias forcing with a good second-jump control, one needs to maintain some positive information over all the reservoirs. This is achieved by restricting the reservoirs to a fixed partition regular class. Given the computability-theoretic nature of the $\Sigma_2^0(G)$ and $\Pi_2^0(G)$ statements needed to be forced, the appropriate partition regular class does not admit a nice explicit combinatorial definition. Seeing a partition regular class as a "reservoir of reservoirs", if $\mathbb{Q} \subseteq \mathcal{P}$ are two partition regular classes, $\mathbb{Q}$ will impose more restrictions on the possible choices of reservoirs than $\mathcal{P}$. Considering a reservoir forces negative information about the set G, $\mathbb{Q}$ will force more positive information than $\mathcal{P}$. With this intuition, minimal partition regular classes will ensure as much positive information as possible, while allowing the reservoirs to be split.

Definition 8.6.17. A large class $\mathcal{A}$ is *$\mathcal{M}$-minimal*[24] if for every set $X \in \mathcal{M}$ and $e \in \mathbb{N}$, either $\mathcal{A} \subseteq \mathcal{U}_e^X$, or $\mathcal{A} \cap \mathcal{U}_e^X$ is not large. ◇

Every large class containing a partition regular subclass, every $\mathcal{M}$-minimal large class of the form $\mathcal{U}_C^{\mathcal{M}}$ is also partition regular. There exists a natural greedy algorithm to build a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is non-trivial and $\mathcal{M}$-minimal.

[24] This notion of minimality is effective and not combinatorial, in the sense that there might exist large subclasses $\mathcal{B} \subsetneq \mathcal{A}$, but not of the form $\mathcal{U}_C^{\mathcal{M}}$.

Proposition 8.6.18 (Le Houérou, Levy Patey and Mimouni [71]). Let $D \subseteq \mathbb{N}^2$ be a set such that $\mathcal{U}_D^{\mathcal{M}}$ is large. Then $(D \oplus M')'$ computes a set $C \supseteq D$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-minimal. ★

Proof. By the padding lemma, there is a total computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, s \in \mathbb{N}$ and every set X, $\mathcal{U}^X_{g(e,s)} = \mathcal{U}^X_e$ and $g(e, s) > s$. By uniformity of the properties of a Scott code, there is another total computable function $h : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, s \in \mathbb{N}$ and every Scott code M, $h(e, s)$ and e are both M-indices of the same set, and $h(e, s) > s$.

We build a $(D \oplus M')'$-computable sequence of D-computable sets $C_0 \subseteq C_1 \subseteq \dots$ such that, letting $C = \bigcup_s C_s$, $\mathcal{U}^M_C$ is $\mathcal{M}$-minimal and for every s, $C \restriction s = C_s \restriction s$. Start with $C_0 = D$. Then, given a set $C_s \subseteq \mathbb{N}^2$ such that $\mathcal{U}^{\mathcal{M}}_{C_s}$ is large, and a pair (e, i), define $C_{s+1} = C_s \cup \{(g(e, s), h(i, s))\}$ if $\mathcal{U}^{\mathcal{M}}_{C_s} \cap \mathcal{U}^{Z_i}_e$ is large, and $C_{s+1} = C_s$ otherwise. The set $C = \bigcup_s C_s$ is the desired set. Note that by choice of g and h, in the former case, $\mathcal{U}^{\mathcal{M}}_{C_{s+1}} = \mathcal{U}^{\mathcal{M}}_{C_s} \cap \mathcal{U}^{Z_i}_e$. By Lemma 8.6.14, the statement "$\mathcal{U}^{\mathcal{M}}_{C_s} \cap \mathcal{U}^{Z_i}_e$ is large" is $\Pi^0_1(C_s \oplus M')$, so it can be decided $(D \oplus M')'$-computably since $C_s \leq_T D$. The use of g and h ensures that $C_{s+1} \restriction s = C_s \restriction s$. ■

Suppose M is of low degree by the low basis theorem (Theorem 4.4.6). One can start with a non-trivial class $\mathcal{U}^{\mathcal{M}}_D$ for some computable set D, and apply Proposition 8.6.18 to obtain a $\emptyset''$-computable set $C \supseteq D$ such that $\mathcal{U}^{\mathcal{M}}_C$ is $\mathcal{M}$-minimal. However, $\emptyset''$-computability is too complex for our purpose. Thankfully, one does not need to explicitly have access to the set of indices of the $\mathcal{M}$-minimal class, but only to be able to check that a class is "compatible" with it. This yields the notion of $\mathcal{M}$-cohesive class.

8.6.4 $\mathcal{M}$-cohesive classes

In general, if $\mathcal{A}$ and $\mathcal{B}$ are two large classes, then $\mathcal{A} \cap \mathcal{B}$ is not necessarily large. For instance, consider the class $\mathcal{A} = \mathcal{L}_X$ and $\mathcal{B} = \mathcal{L}_{\overline{X}}$ for some bi-infinite set X. Thus, in the algorithm of Proposition 8.6.18, the order in which one considers the pairs (e, i) matters. Therefore, there exist many $\mathcal{M}$-minimal classes of the form $\mathcal{U}^{\mathcal{M}}_C$, depending on the ordering of the pairs. The following notion of $\mathcal{M}$-cohesiveness is a way of choosing an $\mathcal{M}$-minimal class without explicitly giving its set of indices.

> **Definition 8.6.19.** A large class $\mathcal{A}$ is *$\mathcal{M}$-cohesive* if for every set $X \in \mathcal{M}$, either $\mathcal{A} \subseteq \mathcal{L}_X$, or $\mathcal{A} \subseteq \mathcal{L}_{\overline{X}}$. ◇

This definition may seem out of the blue, so let us start with a few manipulations which will give some intuition.

Exercise 8.6.20. Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be $\mathcal{M}$-cohesive.

1. Show that for every $X \in \mathcal{M}$, $X \in \mathcal{A}$ iff $\mathcal{A} \subseteq \mathcal{L}_X$.
2. Deduce that $\mathcal{A} \cap \mathcal{M}$ is an ultrafilter on $\mathcal{M}$. ★

The following exercise justifies the cohesiveness terminology.

Exercise 8.6.21 (Le Houérou, Levy Patey and Mimouni [71]). Recall that an infinite set H is *cohesive* for a sequence of sets $R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, either $H \subseteq^* R_n$, or $H \subseteq \overline{R}_n$. Show that for every infinite set H cohesive for the Turing ideal $\mathcal{M}$ seen as a sequence of sets, the class $\mathcal{L}_H$ is partition regular and $\mathcal{M}$-cohesive. ★

The following lemma is the most important combinatorial feature of $\mathcal{M}$-cohesive classes. It actually says that an $\mathcal{M}$-cohesive class already contains the information of an $\mathcal{M}$-minimal class, in the sans that in the greedy algorithm of Proposition 8.6.18, the ordering on the pairs does not matter.

Lemma 8.6.22 (Monin and Patey [68]). Let $\mathcal{U}_C^{\mathcal{M}}$ be an $\mathcal{M}$-cohesive class. Let $\mathcal{U}_D^{\mathcal{M}}$ and $\mathcal{U}_E^{M}$ be such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}}$ and $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ are both large. Then so is $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$.[25] ★

Proof. Suppose for the contradiction that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ is not large. Then, by Exercise 8.6.13, there are some finite sets $C_1 \subseteq C$, $D_1 \subseteq D$ and $E_1 \subseteq E$ such that $\mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$ is not large. For every $k \in \mathbb{N}$, let $\mathscr{C}_k$ be the collection of all sets $Y_0 \oplus \dots \oplus Y_{k-1}$ such that $Y_0 \sqcup \dots \sqcup Y_{k-1} = \mathbb{N}$ and for every $i < k$, $Y_i \notin \mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$. Note that for every k, $\mathscr{C}_k$ is $\Pi_1^0(\mathcal{M})$ since $\mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$ is $\Sigma_1^0(\mathcal{M})$. Moreover, there is some k such that $\mathscr{C}_k \neq \emptyset$. Since $\mathcal{M}$ is a Scott ideal, there is such a set $Y_0 \oplus \dots \oplus Y_{k-1} \in \mathscr{C}_k \cap \mathcal{M}$. Since $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive, there is some $i < k$ such that $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{Y_i}$. In particular, $Y_i \in \mathcal{U}_C^{\mathcal{M}}$, so either $Y_i \notin \mathcal{U}_D^{\mathcal{M}}$, or $Y_i \notin \mathcal{U}_E^{\mathcal{M}}$. Suppose $Y_i \notin \mathcal{U}_D^{\mathcal{M}}$, as the other case is symmetric. Since $Y_j \cap Y_i = \emptyset$ for every $j \neq i$, then $Y_j \notin \mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_i$ for every $j \neq i$. It follows that $Y_0, \dots, Y_{k-1}$ witnesses that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}}$ is not large. Contradiction. ■

It follows that every $\mathcal{M}$-cohesive class for the form $\mathcal{U}_C^{\mathcal{M}}$ admits a unique $\mathcal{M}$-minimal large subclass.

Lemma 8.6.23 (Monin and Patey [68]). For every $\mathcal{M}$-cohesive class $\mathcal{U}_C^{\mathcal{M}}$, there exists a unique $\mathcal{M}$-minimal large subclass:

$$\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \bigcap_{e \in \mathbb{N}, X \in \mathcal{M}} \{ \mathcal{U}_e^X : \mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^X \text{ is large } \}$$

Proof. We first prove that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is large. Let $(e_0, X_0), (e_1, X_1), \dots$ be an enumeration of all pairs $(e, X) \in \mathbb{N} \times \mathcal{M}$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^X$ is large. By induction on n, using Lemma 8.6.22, $\bigcap_{i<n} \mathcal{U}_{e_i}^{X_i}$ is large for every n. Thus, by Exercise 8.6.13, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is large. Next, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle \subseteq \mathcal{U}_C^{\mathcal{M}}$ as for every $(e, i) \in C$, $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^{Z_i}$ is trivially large. Last, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is $\mathcal{M}$-minimal by construction. ■

Contrary to $\mathcal{M}$-minimal classes, one can build a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive computably in any PA degree over M'.

Proposition 8.6.24 (Le Houérou, Levy Patey and Mimouni [71]). Let $D \subseteq \mathbb{N}^2$ be a set such that $\mathcal{U}_D^{\mathcal{M}}$ is large and non-trivial. Then any PA degree over $D \oplus M'$ computes a set $C \supseteq D$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive. ★

Proof. Fix P a PA degree over $D \oplus M'$.[26] First, consider two M-computable enumerations of sets $(E_n)_{n \in \mathbb{N}}$ and $(F_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\mathcal{U}_{E_n}^{Z_n} =$

[25] Note that in this proof, we exploit the fact that all these classes are intersections of $\Sigma_1^0(\mathcal{M})$ classes, and the fact that % is a Scott ideal.

[26] Recall that by Exercise 4.6.5, P is able to choose, among two $\Pi_1^0(D \oplus M')$ formulas such that at least one is true, a valid one.

$\mathscr{L}_{Z_n}$ and $\mathcal{U}_{F_n}^{Z_n} = \mathscr{L}_{\overline{Z}_n}$. By the padding lemma, one can suppose that $\min E_n$, $\min F_n \geq n$. The set C will be defined as $\bigcup_{n \in \mathbb{N}} C_n$ for $C_0 \subseteq C_1 \subseteq \dots$ a P-computable sequence of $M \oplus D$-computable sets satisfying:

- $C_0 = D$,
- $\mathcal{U}_{C_k}^{\mathscr{M}}$ is large for every $k \in \mathbb{N}$,
- $C_k \upharpoonright k = C \upharpoonright k$ for every $k \in \mathbb{N}$, and thus C will be P-computable.

Let $C_0 = D$, then, by assumption, $\mathcal{U}_{C_0}^{\mathscr{M}}$ is large.

Assume C_k has been defined for some $k \in \mathbb{N}$. Then, as $\mathcal{U}_{C_k}^{\mathscr{M}}$ is large, one of the two following $\Pi_1^0(D \oplus M')$ statements must hold: "$\mathcal{U}_{C_k}^{\mathscr{M}} \cap \mathscr{L}_{Z_k}$ is large" or "$\mathcal{U}_{C_k}^{\mathscr{M}} \cap \mathscr{L}_{\overline{Z}_k}$ is large". Hence, P is able to choose one that is true. If $\mathcal{U}_{C_k}^{\mathscr{M}} \cap \mathscr{L}_{Z_k}$ is large, let $C_{k+1} = C_k \cup E_k$, and if $\mathcal{U}_{C_k}^{\mathscr{M}} \cap \mathscr{L}_{\overline{Z}_k}$ is large, let $C_{k+1} = C_k \cup F_k$. By our assumption that $\min E_n, \min F_n \geq n$ for all n, the value of $C_k \upharpoonright k$ will be left unchanged in the rest of the construction. ■

8.7 Pigeonhole principle

By Jockusch and Dzhafarov's theorem (Theorem 3.4.5), RT_2^1 admits strong cone avoidance, the only sets that can be encoded by all the infinite subsets and co-subsets of an arbitrary set are the computable ones. Using the framework of largeness and partition regularity, we can now prove the counterpart for jump computation, known as strong jump cone avoidance of RT_2^1. It follows that for every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ of non-high degree.

Theorem 8.7.1 (Monin and Patey [29])
Let C be a non-Δ_2^0 set. For every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that C is not $\Delta_2^0(H)$.

Proof. Fix C and A. As in Theorem 3.4.5, we shall construct two sets $G_0 \subseteq A$ and $G_1 \subseteq \overline{A}$ using a disjunctive notion of forcing. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

By the low basis theorem (Theorem 4.4.6) and Theorem 4.3.2, there exists a set M of low degree coding a Scott ideal $\mathscr{M}$. By the cone avoidance basis theorem (Theorem 3.2.6) relativized to $\emptyset'$ and Theorem 4.3.2, there is a code N for a Scott ideal $\mathscr{N}$ containing $\emptyset'$ such that $C \not\leq_T N$. By Proposition 8.6.24, $\mathscr{N}$ contains a set $D \subseteq \mathbb{N}^2$ such that $\mathcal{U}_D^{\mathscr{M}}$ is an $\mathscr{M}$-cohesive class.

The two sets G_0 and G_1 will be constructed using a variant of Mathias forcing whose conditions are triples (σ_0, σ_1, X), where

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_i \subseteq A_i$; $X \in \langle \mathcal{U}_D^{\mathscr{M}} \rangle$;
3. $X \in \mathscr{N}$.[27]

One must really think of a condition as a pair of Mathias conditions which share a same reservoir. The *interpretation* $[\sigma_0, \sigma_1, X]$ of a condition (σ_0, σ_1, X) is the class

$$[\sigma_0, \sigma_1, X] = \{(G_0, G_1) : \forall i < 2\ \sigma_i \preceq G_i \subseteq \sigma_i \cup X\}$$

[27] This notion of forcing ressembles the one of Theorem 3.4.5, with two main differences. First, the reservoir must belong to the $\mathscr{M}$-minimal partition regular subclass of $\mathcal{U}_D^{\mathscr{M}}$, which ensures that it maintains a lot of positive information. Second, one usually requires that the reservoir satisfies the desired property, that is, C is not $\Delta_2^0(X)$. However, because of the forcing question for Σ_2^0 formulas, the reservoir only satisfies that $C \not\leq_T X \oplus D \oplus \emptyset'$. In particular, X can compute $\emptyset'$, or can even be of PA degree over $\emptyset'$.

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X) for each $i < 2$. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$. Note that $(G_{\mathcal{F},0}, G_{\mathcal{F},1}) \in \bigcap\{[\sigma_0, \sigma_1, X] : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$.

The goal is therefore to build two infinite sets G_0, G_1, satisfying the following requirements for every $e_0, e_1 \in \mathbb{N}$:

$$\mathcal{R}_{e_0,e_1} : \Phi_{e_0}^{G_0'} \neq C \vee \Phi_{e_1}^{G_1'} \neq C$$

If every requirement is satisfied, then an easy pairing argument shows that either $C \not\leq_T G_0$, or $C \not\leq_T G_1$. However, in general, it is not possible to ensure that G_0 and G_1 are both infinite. For example, A could be finite or co-finite.

In the proof of Theorem 3.4.5, we used as a hypothesis that there is no set satisfying the statement of the theorem, which implies in particular that for every reservoir X, both $X \cap A$ and $X \cap \overline{A}$ are infinite. In this proof, we will need to consider a stronger property.

Definition 8.7.2. We say that part i of (σ_0, σ_1, X) is *valid* if $X \cap A_i \in \mathcal{U}_D^{\mathcal{M}}$. Part i of a filter $\mathcal{F}$ is *valid* if part i is valid for every condition in $\mathcal{F}$. ◊

Since $X \in \langle \mathcal{U}_D^{\mathcal{M}} \rangle$, then by partition regularity, either $A_0 \cap X$ or $A_1 \cap X$ belongs to $\langle \mathcal{U}_D^{\mathcal{M}} \rangle$. It follows that every condition has at least a valid part.[28] Moreover, if q extends p and part i of q is valid, then so is part i of p. Thus, every filter admits a valid part.

We shall first prove that for every sufficiently generic filter $\mathcal{F}$ with valid part i, not only $G_{\mathcal{F},i}$ is infinite, but it furthermore belongs to $\langle \mathcal{U}_D^{\mathcal{M}} \rangle$.

Lemma 8.7.3. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and let $\mathcal{V} \supseteq \langle \mathcal{U}_D^{\mathcal{M}} \rangle$ be a large $\Sigma_1^0(\mathcal{M})$ class. There is an extension (τ_0, τ_1, Y) of p such that $[\tau_i] \subseteq \mathcal{V}$. ★

Proof. Since part i of p is valid, then $X \cap A_i \in \langle \mathcal{U}_D^{\mathcal{M}} \rangle \subseteq \mathcal{V}$. Moreover, $\mathcal{V}$ is $\Sigma_1^0(\mathcal{M})$, so there is some $\rho \subseteq X \cap A_i$ such that $[\rho] \subseteq \mathcal{V}$. Last, by upward-closure of $\mathcal{V}$, $[\sigma_i \cup \rho] \subseteq \mathcal{V}$, so letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, (τ_0, τ_1, Y) is the desired extension. ■

We now design a forcing question for Σ_1^0 formulas. Note that this forcing question is not Σ_1^0-preserving, and therefore does not yield a good first-jump control. This is due to the fact that the reservoir X is too complex, so the only way to access it is to approximate it by a large class, yielding a $\Pi_1^0(\mathcal{N})$ statement. On the bright side, the forcing question is not disjunctive, and can be applied on every valid part.

Definition 8.7.4. Given a string $\sigma \in 2^{<\mathbb{N}}$ and a Σ_1^0 formula $\varphi(G)$, define σ ?⊢ $\varphi(G)$ to hold if the following class is large[29]:

$$\mathcal{U}_D^{\mathcal{M}} \cap \{Z : \exists \rho \subseteq Z\ \varphi(\sigma \cup \rho)\}$$

By Lemma 8.6.14, the forcing question is $\Pi_1^0(D \oplus M')$ uniformly in σ and φ. Since M is of low degree, $M' \in \mathcal{N}$ and by assumption, $D \in \mathcal{N}$, so the forcing question is $\Pi_1^0(\mathcal{N})$.

[28] Also note that by Exercise 8.6.6, if part i is valid in $p = (\sigma_0, \sigma_1, X)$ and $q = (\tau_0, \tau_1, Y) \leq p$ with $Y =^* X$, then part i is valid in q.

[29] Note that this forcing question is not defined over conditions, but over strings. Given a condition (σ_0, σ_1, X), it is intended to be applied on σ_0 or σ_1, depending on which part is valid. Also note that, surprisingly, since the forcing question does not involve the reservoir, its answer only depends on the stem.

Lemma 8.7.5. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and $\varphi(G)$ be a Σ^0_1 formula.

1. If $\sigma_i \mathrel{?\vdash} \varphi(G)$, then there is an extension of p forcing $\varphi(G_i)$;
2. If $\sigma_i \mathrel{?\nvdash} \varphi(G)$, then there is an extension of p forcing $\neg\varphi(G_i)$. ★

Proof. Let $\mathcal{V} = \{Z : \exists \rho \subseteq Z\ \varphi(\sigma_i \cup \rho)\}$.

Suppose first $\sigma_i \mathrel{?\vdash} \varphi(G)$. Then $\mathcal{U}^{\mathcal{M}}_D \cap \mathcal{V}$ is large, so by Lemma 8.6.23, $\langle \mathcal{U}^{\mathcal{M}}_D \rangle \subseteq \mathcal{V}$. Since part i of p is valid, then $A_i \cap X \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle \subseteq \mathcal{V}$. Unfolding the definition of $\mathcal{V}$, there is some $\rho \subseteq A_i \cap X$ such that $\varphi(\sigma_i \cup \rho)$ holds. Letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, (τ_0, τ_1, Y) is an extension forcing $\varphi(G_i)$.

Suppose now $\sigma_i \mathrel{?\nvdash} \varphi(G)$. Then $\mathcal{U}^{\mathcal{M}}_D \cap \mathcal{V}$ is not large, so by Exercise 8.6.13, there is a finite set $F \subseteq D$ such that $\mathcal{U}^{\mathcal{M}}_F \cap \mathcal{V}$ is not large. For every k, let $\mathcal{C}_k$ be the $\Pi^0_1(\mathcal{M})$ class of all sets $Z_0 \oplus \dots \oplus Z_{k-1}$ such that $Z_0 \cup \dots \cup Z_{k-1} = \mathbb{N}$ and for every $j < k$, $Z_i \notin \mathcal{U}^{\mathcal{M}}_F \cap \mathcal{V}$. By assumption, $\mathcal{C}_k \neq \emptyset$ for some $k \in \mathbb{N}$, so since $\mathcal{M}$ is a Scott ideal, there is such a set $Z_0 \oplus \dots \oplus Z_{k-1}$ in $\mathcal{C}_k \cap \mathcal{M}$. By partition regularity of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$, there is some $j < k$ such that $X \cap Z_j \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle$. In particular, $X_j \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle \subseteq \mathcal{U}^{\mathcal{M}}_F$ so $X_j \notin \mathcal{V}$. Letting $Y = X \cap Z_j$, $q = (\sigma_0, \sigma_1, Y)$ is an extension such that for every $\rho \subseteq Y$, $\neg\varphi(\sigma_i \cup \rho)$ holds. It follows that q forces $\neg\varphi(G_i)$. ■

We now turn to second-jump control. The forcing relation for Σ^0_1, Π^0_1 and Σ^0_2 formulas is the usual one. It will be convenient to work with the following syntactic forcing relation for Π^0_2 formulas.

Definition 8.7.6. Let $p = (\sigma_0, \sigma_1, X)$ be a condition, $i < 2$ be a part and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. Let $p \Vdash \varphi(G_i)$ hold if for every $\rho \subseteq X$ and every $x \in \mathbb{N}$, $\sigma_i \cup \rho \mathrel{?\vdash} \psi(G, x)$.[30]. ◇

One easily proves that this syntactic forcing relation is closed under condition extension. The following lemma states that, for every sufficiently generic filter $\mathcal{F}$ with valid part i, if $p \Vdash \varphi(G_i)$ for some $p \in \mathcal{F}$, then p forces $\varphi(G_i)$.

Lemma 8.7.7. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. If $p \Vdash \varphi(G_i)$, then for every $x \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\psi(G_i, x)$. ★

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G_i)$, then in particular, for $\rho = \emptyset$, $\sigma_i \mathrel{?\vdash} \psi(G, x)$. By Lemma 8.7.5, there is an extension of p forcing $\psi(G_i, x)$. ■

The notion of forcing admits a Σ^0_2-preserving disjunctive forcing question for Σ^0_2 formulas, but which satisfies its specification only if *both parts* of the condition are valid.

Definition 8.7.8. Given a condition $p = (\sigma_0, \sigma_1, X)$ and a pair of Σ^0_2 formulas $\varphi_0(G)$ and $\varphi_1(G)$, with $\varphi_i(G) \equiv \exists x \psi_i(G, x)$, define $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ to hold if for every 2-partition $Z_0 \cup Z_1 = X$, there is some $i < 2$, some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$.[31] ◇

By compactness, this forcing question holds iff there is a level $\ell \in \mathbb{N}$ such that for every 2-partition $Z_0 \cup Z_1 = X{\upharpoonright}_\ell$, there is some $i < 2$, some $x \in \mathbb{N}$ and

[30] Assuming the forcing question for Σ^0_1 formulas meets its specification, this forcing relation says that for every x and every future extension of the stem, there will be an extension forcing $\psi(G_i, x)$. Thus, this forcing question states, for each x, the density below p of the set of conditions forcing $\psi(G_i, x)$. Since the forcing question for Σ^0_1 formulas meets its specification on valid parts, then this syntactic forcing relation implies the true forcing relation one the parts which remain valid in the future.

[31] As usual, the formula ψ_i being Π^0_1, we use here the forcing question for Π^0_1 formulas obtained by taking the negation of the forcing question for Σ^0_1 formulas.

some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$. The formula $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$ is $\Sigma^0_1(\mathcal{N})$ uniformly in σ_i, ρ and ψ_i, thus the overall forcing question is $\Sigma^0_1(\mathcal{N})$ uniformly in p, φ_0 and φ_1.

Lemma 8.7.9. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with both valid parts and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_1 formulas.

1. If $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension of p forcing $\varphi(G_i)$ for some $i < 2$;
2. If $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension q of p with $q \Vdash \neg\varphi(G_i)$ for some $i < 2$. ★

Proof. Say $\varphi_i(G) \equiv \exists x \psi_i(G, x)$.

Suppose first $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$. Then, letting $Z_0 = X \cap A_0$ and $Z_1 = X \cap A_1$, there is some $i < 2$, some $x \in \mathbb{N}$ and some $\rho \subseteq X \cap A_i$ such that $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$. In particular, letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, $q = (\tau_0, \tau_1, Y)$ is an extension such that both parts are valid. By Lemma 8.7.5, there is an extension of q forcing $\psi_i(G_i, x)$, hence forcing $\varphi(G_i)$.

Suppose now $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$. Let $\mathcal{C}$ be the $\Pi^0_1(\mathcal{N})$ class of all Z such that, letting $Z_0 = Z$ and $Z_1 = \overline{Z}$, for every $i < 2$, every $x \in \mathbb{N}$, and every $\rho \subseteq X \cap Z_i$, $\sigma_i \cup \rho \mathrel{?\nvdash} \psi_i(G, x)$. Since $\mathcal{N}$ is a Scott ideal, there is such a set $Z \in \mathcal{C} \cap \mathcal{N}$. By partition regularity of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$, there is some $i < 2$ such that $X \cap Z_i \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle$. The condition $q = (\sigma_0, \sigma_1, X \cap Z_i)$ is an extension of p such that $q \Vdash \neg\varphi_i(G_i)$. ■

In most cases, for sufficiently Cohen generic or sufficiently random sets A, both parts of every conditions will be valid. Unfortunately, in some degenerate cases, there might be some condition $p = (\sigma_0, \sigma_1, X)$ with only one valid part, say part 0, and the disjunctive forcing question may not work because it would yield an extension deciding the formula on part 1. In this case, for every extension of p, part 1 will stay invalid, and part 0 will be valid. We will therefore make a degenerate construction in the valid part.

If some part of a condition is not valid, then it is witnessed by a large $\Sigma^0_1(\mathcal{M})$ superclass of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$ in the following sense.

> **Definition 8.7.10.** A *witness of invalidity* of part i of a condition $p = (\sigma_0, \sigma_1, X)$ is a $\Sigma^0_1(\mathcal{M})$ large class $\mathcal{V} \supseteq \langle \mathcal{U}^{\mathcal{M}}_D \rangle$ such that $X \cap A_i \notin \mathcal{V}$. ◇

If part i of p is not valid, then by definition, $X \cap A_i \notin \langle \mathcal{U}^{\mathcal{M}}_D \rangle$, so by Lemma 8.6.23, there is some $\Sigma^0_1(\mathcal{M})$ class $\mathcal{V}$ such that $X \cap A_i \notin \mathcal{V}$. Thus, every invalid part admits a witness of invalidity. One can exploit this witness to design a non-disjunctive forcing question for Σ^0_2 formulas on the valid part with the good definitional properties.

> **Definition 8.7.11.** Let $p = (\sigma_0, \sigma_1, X)$ be a condition with witness of invalidity $\mathcal{V}$ on part $1 - i$, and let $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Define $p \mathrel{?\vdash^{\mathcal{V}}} \varphi(G_i)$ to hold if for every 2-partition $Z_0 \sqcup Z_1 = X$ such that $Z_{1-i} \notin \mathcal{V}$, there is some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$. ◇

Again, by compactness, this degenerate forcing question is $\Sigma^0_1(\mathcal{N})$. The following lemma shows that this forcing question meets its specification.

Lemma 8.7.12. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with witness of invalidity $\mathcal{V}$ on part $1 - i$, and let $\varphi(G)$ be a Σ^0_2 formula.

1. If $p \mathrel{?{\vdash}}^{\mathcal{V}} \varphi(G_i)$, then there is an extension of p forcing $\varphi(G_i)$.
2. If $p \mathrel{?{\nvdash}}^{\mathcal{V}} \varphi(G_i)$, then there is an extension $q \leq p$ such that $q \Vdash \neg\varphi(G_i)$. ★

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$.

Suppose first $p \mathrel{?{\vdash}}^{\mathcal{V}} \varphi(G_i)$. In particular, for $Z_0 = A_0 \cap X$ and $Z_1 = A_1 \cap X$, there is some $x \in \mathbb{N}$ and some $\rho \subseteq A_i \cap X$ such that $\sigma_i \cup \rho \mathrel{?{\vdash}} \psi_i(G, x)$. Letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, $q = (\tau_0, \tau_1, Y)$ is an extension such that part $1-i$ is invalid, hence part i is valid. By Lemma 8.7.5, there is an extension of q forcing $\psi_i(G_i, x)$, hence forcing $\varphi(G_i)$.

Suppose now $p \mathrel{?{\nvdash}}^{\mathcal{V}} \varphi(G_i)$. Let $\mathscr{C}$ be the $\Pi^0_1(\mathcal{N})$ class of all Z such that, letting $Z_0 = Z$ and $Z_1 = \overline{Z}$, then $Z_{1-i} \notin \mathcal{V}$ and for every $x \in \mathbb{N}$, and every $\rho \subseteq X \cap Z_i$, $\sigma_i \cup \rho \mathrel{?{\nvdash}} \psi_i(G, x)$. Since $\mathcal{N}$ is a Scott ideal, there is such a set $Z \in \mathscr{C} \cap \mathcal{N}$. By partition regularity of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$, since $X \cap Z_{1-i} \notin \mathcal{V} \supseteq \langle \mathcal{U}^{\mathcal{M}}_D \rangle$, then $X \cap Z_i \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle$. The condition $q = (\sigma_0, \sigma_1, X \cap Z_i)$ is an extension of p such that $q \Vdash \neg\varphi_i(G_i)$. ■

We are now ready to prove Theorem 8.7.1.

Suppose first there is a condition p with some invalid part $1 - i$. Let $\mathcal{F}$ be a sufficiently generic filter containing p and let $G_i = G_{\mathcal{F}, i}$. Then part i is valid in $\mathcal{F}$. By Lemma 8.7.7, the syntactic forcing relation for Π^0_2 formulas implies the true forcing relation on part i. By Lemma 8.7.12 and by adapting Theorem 8.3.5, for every Turing functional Φ_e, there is some condition $q \in \mathcal{F}$ forcing $\Phi_e^{G_i'} \neq C$, so C is not $\Delta^0_2(G_i)$.

Suppose now that for every condition, both parts are valid. Let $\mathcal{F}$ be a sufficiently generic filter, and let $G_i = G_{\mathcal{F}, i}$ for $i < 2$. By Lemma 8.7.7, the syntactic forcing relation for Π^0_2 formulas implies the true forcing relation on both parts. By Lemma 8.7.9 and by adapting Theorem 8.3.5, for every pair of Turing functionals Φ_{e_0}, Φ_{e_1}, there is some condition $q \in \mathcal{F}$ forcing $\Phi_{e_0}^{G_0'} \neq C \vee \Phi_{e_1}^{G_1'} \neq C$. By a pairing argument, there is some $i < 2$ such that C is not $\Delta^0_2(G_i)$. This completes the proof of Theorem 8.7.1. ■

Exercise 8.7.13 (Monin and Patey [29]). Let $f : \mathbb{N} \to \mathbb{N}$ be $\emptyset'$-hyperimmune. Adapt the proof of Theorem 8.7.1 and Theorem 3.6.4 to show that for every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that f us H'-hyperimmune. ★