{ARITHMETIC HIERARCHY CHEAT SHEET}

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Σ^0_n, Π^0_n AND Δ^0_n SETS

Definitions

Let $n \geq 1$.
A set $A \subset \mathbb{N}$ is Σ^0_n if, for some computable $R \subset \mathbb{N}^{n+1}$,

$$A = \{y : \exists x_1 \forall x_2 \cdots Q_n x_n (x_1, \cdots x_n, y) \in R\},$$

where Q_n is $\exists$ if n is odd, $\forall$ else.
A set $A \subset \mathbb{N}$ is Π^0_n if, for some computable $R \subset \mathbb{N}^{n+1}$,

$$A = \{y : \forall x_1 \exists x_2 \cdots Q_n x_n (x_1, \cdots x_n, y) \in R\},$$

where Q_n is $\forall$ if n is odd, $\exists$ else.
A set $A \subset \mathbb{N}$ is Δ^0_n if it is both Π^0_n and Σ^0_n.

Stability

The class of Σ^0_n sets is stable under finite unions, finite intersections, uniform bounded quantifications, and uniform countable unions.

The class of Π^0_n sets is stable under finite unions, finite intersections, uniform bounded quantifications, and uniform countable intersections.

Computability

A set is Σ^0_1 iff it is computably enumerable.
A set is Δ^0_1 iff it is computable.

MANY-ONE DEGREES

Reduction

Let A, B be two sets. We write $A \leq_m B$ if there exists a total computable function f such that

$$n \in A \iff f(n) \in B$$

If $A \leq_m B$ and $B \leq_m A$, we write $A \equiv_m B$. This is an equivalence relation, and we call *many-one degrees* its classes.

Compatibility with the hierarchy

Let $A \subset \mathbb{N}$ be a Σ^0_n (resp. Π^0_n) set, and let $B \leq_m A$. Then B is Σ^0_n (resp. Π^0_n).

Completeness of the halting set

A set A is Σ^0_1 iff $A \leq_m \emptyset'$.

POST'S THEOREM

Definition

Given a set X, we define recursively on $n \geq 0$:

- $X^{(0)} = X$,
- $X^{(n+1)} = X^{(n)'}$.

Completeness

A set A is Σ^0_n*-hard* (resp. Π^0_n*-hard*) if, for any Σ^0_n (resp. $\Pi^0_n(X)$) set B, we have $B \leq_m A$.

A set A is Σ^0_n*-complete* (resp. Π^0_n*-complete*) if it is Σ^0_n and Σ^0_n-hard (resp. Π^0_n and Π^0_n-hard).

The set $X^{(n)}$ is $\Sigma^0_n(X)$-complete. In particular, for all $n > 0$, the set $\emptyset^{(n)}$ is Σ^0_n-complete.

A set A is $\Sigma^0_n(X)$ iff $A \leq_m X^{(n)}$.

Post's theorem

Let A be a set and $n \geq 0$.

- A is Σ^0_{n+1} iff A $\Sigma^0_1(\emptyset^{(n)})$, iff A is $\emptyset^{(n)}$-c.e.
- A is Δ^0_{n+1} iff A $\Delta^0_1(\emptyset^{(n)})$, iff $A \leq_T \emptyset^{(n)}$-c.e.

Corollary

The arithmetic hierarchy is strict. In other words,

- for all $n > 0$, there exists a Σ^0_n set which is not Π^0_n and a Π^0_n set which is not Σ^0_n.
- for all $n > 0$, there exists a Δ^0_{n+1} set which is neither Σ^0_n nor Π^0_n.

RICE'S THEOREM

Index set

An *index set* is a set $A \subset \mathbb{N}$, such that for all $x, y \in \mathbb{N}$,

$$x \in A \wedge \phi_x = \phi_y \implies y \in A.$$

Index set theorem

If A is a non-trivial (i.e. not $\emptyset$ nor $\mathbb{N}$) index set, then either $\emptyset' \leq_m A$ or $\emptyset' \leq_m \overline{A}$.

Rice's theorem

Let $\mathcal{C}$ be a class of partial computable functions $\mathbb{N} \to \mathbb{N}$. Then, the set $A = \{x : \phi_x \in \mathcal{C}\}$ is non-computable unless $\mathcal{C} = \emptyset$ or $\mathcal{C}$ is all partial computable functions.

ARITHMETIC CODES

Δ^0_1 code

A Δ^0_1 *code* of a computable set A is an integer e such that $\phi_e = A$ (i.e. $\forall n\, \phi_e(n) \downarrow = A(n)$).

Uniform computability

A sequence $X_0, X_1, \cdots$ of sets is *uniformly computable* iff there exists a computable sequence $e_0, e_1, \cdots$ such that e_s is a Δ^0_1 code of X_s for all s.

Σ^0_{n+1} and Δ^0_{n+1} codes

A Σ^0_{n+1} (resp. Δ^0_{n+1}) *code* of a set A is an integer e such that $W_e(\emptyset^{(n)}) = A$ (resp. $\Phi_e(\emptyset^{(n)}) = A$).

Lowness code

A *lowness code* of a set A is an integer e such that $\Phi_e(\emptyset') = A'$.

Canonical code

The *canonical code* of a finite set F is the natural number $\sum_{i \in F} 2^i$.

THE ARITHMETIC HIERARCHY