{Immunity and Function Growth}

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Immunity

Definition

An infinite set $A \subseteq \mathbb{N}$ is *immune* if it contains no infinite c.e. subset, or equivalently no infinite computable subset.

Degrees of immune sets

Every non-zero Turing degree contains an immune set : $A \equiv_T \{\sigma \in 2^{<\mathbb{N}} : \sigma \prec A\}$.

DNC

Effective immunity

A set $A \subseteq \mathbb{N}$ is *effectively immune* if $\forall e, |W_e| \geqslant h(e) \Rightarrow W_e \not\subseteq A$ for some computable function $h : \mathbb{N} \to \mathbb{N}$.

Diagonally non-computable

A function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally non-computable* (DNC) if $\forall n, f(n) \neq \Phi_n(n)$.

Fixpoint free

A function $f : \mathbb{N} \to \mathbb{N}$ is *fixpoint-free* (FPF) if $\forall n, \Phi_{f(n)} \neq \Phi_n$.

Equivalences

The following statements are equivalent:

- X computes a DNC function
- X computes a fixpoint free function
- X computes an effectively immune set
- X computes a function $h : \mathbb{N}^2 \to \mathbb{N}$ s.t. $\forall e, n, |W_e| \leqslant n \Rightarrow h(e,n) \notin W_e$

DNC degree

A Turing degree $\mathbf{d}$ is DNC if it contains a DNC function, or equivalently if it computes a DNC function.

Existence

There is a non-zero Δ^0_2 non-DNC degree.

Arslanov's completeness criterion

A c.e. degree is Turing complete iff it computes a DNC function.

Hyperimmunity

Hyperimmune set

Fix a canonical listing of all finite sets: $D_0, D_1, \dots$ A *c.e. array* is a sequence of mutually disjoint finite sets $\{D_{f(n)} : n \in \mathbb{N}\}$ where $f : \mathbb{N} \to \mathbb{N}$ is a computable function.

An infinite set $X \subseteq \mathbb{N}$ is *hyperimmune* if for every c.e. array $\{D_{f(n)} : n \in \mathbb{N}\}$, there is some $n \in \mathbb{N}$ such that $D_{f(n)} \cap X = \emptyset$.

Hyperimmune function

A function $g : \mathbb{N} \to \mathbb{N}$ *dominates* $f : \mathbb{N} \to \mathbb{N}$ if $\forall x\, g(x) \geq f(x)$.

A function $f : \mathbb{N} \to \mathbb{N}$ is *hyperimmune* is not dominated by any computable function.

Set vs function

An infinite set $X = \{x_0 < x_1 < \dots\}$ is hyperimmune iff its *principal* function $n \mapsto x_n$ is hyperimmune.

Hyperimmune degree

A Turing degree is *hyperimmune* iff it contains an hyperimmune function, or equivalently iff it computes a hyperimmune function.

Weak 1-genericity

A set $D \subseteq 2^{<\mathbb{N}}$ is *dense* if for every $\sigma \in 2^{<\mathbb{N}}$, there is some $\tau \succeq \sigma$ such that $\tau \in D$.

A set $X \in 2^{\mathbb{N}}$ is *weakly 1-generic* iff for every c.e. dense set $D \subseteq 2^{<\mathbb{N}}$, there is some $\sigma \prec X$ such that $\sigma \in D$.

Hyperimmunity vs genericity

A Turing degree is hyperimmune iff it computes a weakly 1-generic set.

Proposition

Every non-zero Δ^0_2 degree is hyperimmune. There is a non-zero Δ^0_3 non-hyperimmune degree.

Computably dominated

Definition

A set X is *computably dominated* or *hyperimmune-free* if for every X-computable function $f : \mathbb{N} \to \mathbb{N}$, there is a computable function $g : \mathbb{N} \to \mathbb{N}$ dominating f.

Existence

There exists a non-zero Δ^0_3 computably dominated degree.

Truth table reduction

A set Y is *truth-table reducible* to X ($Y \leqslant_{tt} X$) if there is a total functional Φ s.t. $\Phi(X) = Y$

Characterization

A set X is computably dominated iff for every set Y, $\forall Y \subseteq \mathbb{N}, Y \leqslant_{tt} X \Leftrightarrow Y \leqslant_T X$

High

High degree

A degree $\mathbf{d}$ is *high* if $\mathbf{d}' \geq \mathbf{0}''$.

Dominant function

A function $f : \mathbb{N} \to \mathbb{N}$ is *dominant* if it dominates, almost everywhere, all the computable functions.

Martin's domination theorem

The following statements are equivalent:

- X is high
- X computes a dominant function
- X computes a list containing exactly the computable sets (possibly with repetition)

High or DNC degree

A set is of high or DNC degree iff it computes a function which is different almost everywhere from any computable function.

Bounded DNC

DNC$_f$ degree

Given $f : \mathbb{N} \to \mathbb{N}$, a set is of DNC$_f$ degree iff it computes a DNC function g dominated by f.

A set is of DNC$_k$ degree iff it computes a k-valued DNC function.

Hierarchy theorem

Given $f : \mathbb{N} \to \mathbb{N}$, there exists $g : \mathbb{N} \to \mathbb{N}$ such that DNC$_f \subsetneq$ DNC$_g$

PA degree

A degree is DNC$_k$ iff it is PA, that is, computes a member of every non-empty Π^0_1 class.

Modulus

Definition

A function $f : \mathbb{N} \to \mathbb{N}$ is a *modulus* of a set $X \subseteq \mathbb{N}$ if every function dominating f computes X.

Δ^1_1 set

A set X is Σ^1_1 if $X = \{n : \exists Y\, R(Y,n)\}$ for an arithmetic predicate R. A set X is Π^1_1 if $X = \{n : \forall Y\, R(Y,n)\}$ for an arithmetic predicate R. A set X is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

Σ^1_1 singleton

A class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is Σ^1_1 if it can be written of the form $\mathcal{C} = \{X : \exists Y\, R(Y,X)\}$ for an arithmetic predicate R.

A set X is a Σ^1_1 *singleton* if $\{X\}$ is a Σ^1_1 class.

Computable encodability

A set $X \subseteq \mathbb{N}$ is *computably encodable* if for every infinite set $Y \subseteq \mathbb{N}$, there is an infinite subset $Z \subseteq Y$ such that $X \leq_T Z$.

Equivalences

The following statements are equivalent:

- X admits a modulus
- X is Δ^1_1
- X is a Σ^1_1 singleton
- X is computably encodable