ON COMBINATORIAL WEAKNESSES OF RAMSEYAN PRINCIPLES

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ABSTRACT. Avoidance properties such as cone avoidance or PA avoidance for a principle P in reverse mathematics shows the effectiveness weakness of P. Strong avoidance, namely, avoidance even for non-computable instances, expresses the combinatorial weakness of a principle. Some statements like Ramsey's theorem for pairs (RT^2_2) are effectively weak in the sense that every computable instance has a solution which does not compute the halting set. However RT^2_2 is combinatorially strong as there exists a (non-computable) stable 2-coloring of pairs such that every infinite homogeneous set computes the halting set.

In this paper we study various notions of avoidance related to closed sets. These properties generalize in particular cone avoidance and PA avoidance. We show that even very weak statements in reverse mathematics admit instances whose solutions compute a member of a special Π^0_1 class, thereby answering a question asked by Liu. We prove that many Ramsey-type theorems admit constant-bound enumeration (c.b-enum) avoidance, and deduce several new separations over RCA_0. In particular, we answer a question asked by Hirschfeldt by proving that none of the free set theorem (FS), the thin set theorem (TS) or the rainbow Ramsey theorem (RRT) imply weak weak König's lemma (WWKL_0). We use simultaneous c.b-enum avoidance to reprove many recent separation results. Therefore, c.b-enum avoidance can be seen as a powerful unifying framework for comparing the strength of statements in reverse mathematics.

1. INTRODUCTION

Reverse mathematics is a program of mathematics which aims to find optimal (in a computational sense) axioms necessary to prove theorems. It uses the framework of second-order arithmetic, with a base theory called RCA_0. RCA_0, standing for recursive comprehension axiom, contains the basic Peano axioms together with the Δ^0_1-comprehension and Σ^0_1 induction schemes. In this paper, we shall focus on statements provable in the arithmetic comprehension axiom (ACA_0) over RCA_0. See Simpson [40] for a general background on reverse mathematics and Hirschfeldt [17] for a gentle introduction to the reverse mathematics below ACA_0.

Models whose first-order part are the standard integers, ω-models, are of particular interest. They are characterized by their second-order part. An ω-model satisfies RCA_0 if and only if its second-order part is a Turing ideal, i.e., a set of reals $\mathscr{C} \subseteq 2^\omega$ closed under the effective join and downward-closed under the Turing reduction. RCA_0 admits a minimal ω-model whose second-order part is the set of the computable reals. [15] Therefore RCA_0 can be considered as representing *computational mathematics*.

Many principles studied in reverse mathematics are Π^1_2 statements of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X,Y)]$$

where Φ and Ψ are arithmetic formulas. A real X such that $\Phi(X)$ holds is called a P-*instance* and every real Y such that $\Phi(X,Y)$ holds is called a *solution* to X. Given two principles P and Q, proving that P does not imply Q over RCA_0 usually consists of creating a Turing ideal $\mathscr{I}$ such that every P-instance $X \in \mathscr{I}$ has a solution in $\mathscr{I}$, whereas there exists a Q-instance in $\mathscr{I}$ with no solution in $\mathscr{I}$. The process of building such a Turing ideal is usually the following.

1. Choose a particular Q-instance B admitting no B-computable solution.
2. Start with the Turing ideal $\mathscr{I}_0 = \{Z \in 2^\omega : Z \leq_T B\}$.
3. Given a Turing ideal $\mathscr{I}_n$ containing no solution to B, take any P-instance $X \in \mathscr{I}_n$ having no solution in $\mathscr{I}_n$ and add a solution Y to X. Let $\mathscr{I}_{n+1}$ be the closure of $\mathscr{I}_n \cup \{Y\}$ under the effective join and the Turing reduction.

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4. Repeat step 3 to obtain a Turing ideal $\mathscr{I} = \bigcup_n \mathscr{I}_n$ such that every P-instance in $\mathscr{I}$ admits a solution in $\mathscr{I}$.

The difficulty of such a construction is to avoid adding a solution to the instance B in $\mathscr{I}_{n+1}$ during step 3. One needs to ensure that every P-instance in $\mathscr{I}_n$ admits a solution Y such that $Y \oplus C$ *avoids* computing a solution to B for each $Y \in \mathscr{I}_n$. This leads to the following natural definition of *avoidance*.

Definition 1.1 (Avoidance) A Π^1_2 statement P admits $\mathscr{C}$ *avoidance* for set of sequences $\mathscr{C} \subseteq \omega^\omega$ closed upward under the Turing reduction if for every P-instance $X \notin \mathscr{C}$, there exists a solution Y to X such that $Y \oplus X \notin \mathscr{C}$.

The notion of $\mathscr{C}$ avoidance is extended to arbitrary sets of sequences $\mathscr{C} \subseteq \omega^\omega$ by taking their upward-closure under the Turing reduction. Avoidance reflects the *effective weakness* of P. If P admits $\mathscr{C}$-avoidance for a fixed computable Q-instance B and $\mathscr{C} = \{Z \in 2^\omega : Z \text{ is a solution to } B\}$, then we can build a standard model of P that is not a model of Q. However it sometimes happens that P is a principle so weak that even non-computable instances of P admit a solution which computes no solution to B. Such a weakness is not due to the effectiveness of the instances of P, but rather to the *structural weakness* of P. This leads to the notion of *strong avoidance*.

Definition 1.2 (Strong avoidance) A Π^1_2 statement P admits *strong $\mathscr{C}$-avoidance* for a set of sequences $\mathscr{C} \subseteq \omega^\omega$ closed upward under the Turing reduction if for every P-instance X (in $\mathscr{C}$ or not) and every sequence $C \notin \mathscr{C}$, there exists a solution Y to X such that $Y \oplus C \notin \mathscr{C}$.

Again, we extend the notion of strong $\mathscr{C}$ avoidance to any set of sequences $\mathscr{C} \subseteq \omega^\omega$ by taking their upward-closure under the Turing reduction. Beside the insights it gives over a principle, strong avoidance is of practical interest: Some statements like Ramsey's theorem are proven by induction over the size of the tuples. However, the induction hypothesis is applied over a non-effective instance. In this case, strong avoidance of the principle over n-tuples becomes useful for proving avoidance of the principle over $(n+1)$-tuples. In particular, avoidance notions for Ramsey's theorem for pairs are usually proven by considering first strong avoidance for Ramsey's theorem for singletons.

Some principles like SRT^2_2 are effectively weak in the sense that every computable instance has a solution avoiding the upper cone of the halting set [37]. However SRT^2_2 is combinatorially strong as there exists a (non-computable) stable 2-coloring of pairs such that every infinite homogeneous set computes the halting set [41].

Of course, if a principle P admits $\mathscr{C}$-avoidance for each class $\mathscr{C}$ then every computable instance of P has a computable solution, as witnessed by taking $\mathscr{C} = \{Z \in 2^\omega : Z \text{ is not computable}\}$. We then focus on particular classes of reals.

1.1. Constant-bound enumeration avoidance

C.b-enum avoidance has been introduced by Liu in [30] for separating RT^2_2 from weak weak König's lemma (WWKL_0), defined below. Liu deduced from this general notion of avoidance various strong results concerning computability theory, algorithmic randomness and reverse mathematics.

Definition 1.3 (Constant-bound enumeration avoidance)

1. A *k-enumeration* (or k-enum) of a class $\mathscr{D} \subseteq 2^\omega$ is a sequence $D_0, D_1, \dots$ such that for each $n \in \omega$, $|D_n| \leq k$, $(\forall \sigma \in D_n)|\sigma| = n$ and $\mathscr{D} \cap [D_n] \neq \emptyset$ where D_n is seen as a clopen set of reals in the Cantor space. A *constant-bound enumeration* (or c.b-enum) of $\mathscr{D}$ is a k-enum of $\mathscr{D}$ for some $k \in \omega$.
2. A principle P admits *(strong) c.b-enum avoidance* if it admits (strong) $\{Z : Z \text{ is a c.b-enum of} \mathscr{D}\}$ avoidance for every class $\mathscr{D} \subseteq 2^\omega$.

C.b-enum avoidance is a unifying avoidance notion generalizing cone avoidance [37] and PA avoidance [29].

Definition 1.4 (Weak (weak) König's lemma) A tree $T \subseteq 2^{<\omega}$ is of *positive measure* if

$$\lim_s \frac{|\{\sigma \in T : |\sigma| = s\}|}{2^s} > 0$$

WKL_0 is the statement "Every infinite subtree of $2^{<\omega}$ has an infinite path" and WWKL_0 is the restriction of WKL_0 to trees of positive measure.

Despite its seemingly artificial definition, WWKL_0 has very natural characterizations in terms of algorithmic randomness. As shown by Avigad et al. in [2], it is equivalent over RCA_0 to the statement "For every set X, there exists a Martin-Löf random real relative to X". A particular kind of computable tree of positive measure is the tree T_c whose paths are the c-incompressible sequences:

$$[T_c] = \{Z : (\forall n) K(Z \upharpoonright n) \geq n - c\}$$

where K is the prefix-free Kolmogorov complexity. Of course, every path through T_c computes a 1-enum of $[T_c]$, and Liu proved that $[T_c]$ has no computable c.b-enum. Therefore every principle admitting c.b-enum avoidance has an ω-model that is not a model of WWKL_0. Among Ramseyan principles, the most famous are Ramsey theorems for tuples.

Definition 1.5 (Ramsey's theorem) RT^n_k is the statement "Given a function $f : [\omega]^n \to k$, there exists an infinite set $X \subseteq \omega$ such that f is constant on $[X]^n$. $\mathsf{RT}^n_{<\infty}$ is the statement $(\forall k)\mathsf{RT}^n_k$.

When considering colorings of integers $\mathsf{RT}^1_{<\infty}$, the statement can be reformulated into "Every finite partition of ω, has an infinite subset of one of its parts". Liu proved in [30] that $\mathsf{RT}^1_{<\infty}$ admits strong c.b-enum avoidance and that both COH and RT^2_2 admit c.b-enum avoidance. RT^2_2 does not admit strong c.b-enum avoidance as there exists a $\emptyset'$-computable coloring of pairs into two colors such that every infinite homogeneous set computes the halting set.

Hirschfeldt asked in [17] whether some consequences of Ramsey's theorem like the free set theorem (FS) and thin set theorem (TS) imply WKL_0 over RCA_0. We answer negatively by proving the existence of an ω-model of simultaneously the Erdős Moser theorem (EM), FS, TS, the rainbow Ramsey theorem RRT and the thin set theorem (TS), which is not a model of WWKL_0. Strong c.b-enum avoidance being an avoidance schema, we use the same notion to prove the existence of an ω-model of $\mathsf{RT}^2_2 \wedge \mathsf{TS} \wedge \mathsf{FS} \wedge \mathsf{WWKL}_0$ that is not a model of WKL_0.

Theorem 1.6 COH, FS, TS, RRT, EM and TS admit strong c.b-enum avoidance.

One may naturally want to strengthen this notion of avoidance by requiring to avoid computing a 1-enum of a class $\mathscr{C}$ which has no computable one.

1.2. **1-enumeration avoidance**

This notion, as we shall see in section 3, subsumes c.b-enum avoidance and coincides with member avoidance for a lot of very natural classes of reals, namely homogeneous classes.

Definition 1.7 (1-enumeration avoidance) A principle P admits *(strong) 1-enum avoidance* if its admits (strong) $\{Z : Z \text{ is a 1-enum of } \vec{\mathscr{C}}\text{-enum}\}$ avoidance for each class $\mathscr{C} \subseteq 2^\omega$.

Liu asked in [30] whether every computable instance of RT^2_2 has a solution which does not compute a member in a closed set of reals $\mathscr{C}$ whenever $\mathscr{C}$ has no computable member. We answer negatively using the notion of homogeneous class of reals.

Definition 1.8 A class $\mathscr{C} \subseteq 2^\omega$ is *homogeneous* if for every $\sigma, \tau \in 2^{<\omega}$ such that $|\sigma| = |\tau|$, $[\sigma] \cap \mathscr{C} \neq \emptyset$ and $[\tau] \cap \mathscr{C} \neq \emptyset$, $[\sigma] \cap \mathscr{C} = [\tau] \cap \mathscr{C}$.

In the case of homogeneous classes $\mathscr{C}$, computing a 1-enum of $\mathscr{C}$ is equivalent to compute a member of $\mathscr{C}$. We construct a homogeneous class $\mathscr{C}$ which admits a computable 2-enum, but no computable 1-enum. It follows immediatly that RT^1_2 does not admit strong 1-enum avoidance as every 2-enum induce a partition of the integers such that every infinite subset in one of its parts computes a 1-enum. With a more careful analysis of the computable power needed to construct $\mathscr{C}$, we prove the existence of a Δ^0_2 set such that every infinite subset in either it or its complement computes a 1-enum, hence a member, of our class $\mathscr{C}$. A systematic study of other Ramseyan principles gives the following theorem:

Theorem 1.9

1. The rainbow Ramsey theorem for singletons (RRT^1_2) and the diagonally non-recursive principle (DNR) admit strong 1-enum avoidance
2. $\mathsf{RT}^1_{<\infty}$, cohesiveness (COH), the rainbow Ramsey theorem for pairs (RRT^2_2), the thin set theorem for singletons ($\mathsf{TS}(1)$), and the free set theorem for singletons ($\mathsf{FS}(1)$) admit 1-enum avoidance but not strong 1-enum avoidance.
3. The stable ascending descending sequence principle (SADS), the stable thin set theorem for pairs ($\mathsf{STS}(2)$), and the rainbow Ramsey theorem for triples (RRT^3_2) do not admit 1-enum avoidance.

The case of Erdős Moser theorem remains open. We prove that if the Ramsey-type weak König's lemma (RWKL) introduced by Flood in [13] admits 1-enum avoidance, then so does EM. Proving 1-enum avoidance of RWKL would provide another proof of separation of EM from SRT^2_2, after the initial forcing from Lerman et al. in [28] and the notion of preservation of Δ^0_2 definitions studied by Wang in [42].

Question 1.10 Does RWKL admit 1-enum avoidance?

Before going into technical details, we present a few other avoidance notions subsumed by c.b-enum avoidance.

1.3. **Cone avoidance**

ACA_0, standing for arithmetic comprehension axiom, is the comprehension axiom restricted to arithmetic formulas with parameters. It is known to be equivalent to the statement "for every X, the jump of X exists" over RCA_0. In particular every model of ACA_0 contains the halting set. Given an instance I of a principle P, cone avoidance states the existence of a solution to I which avoids the upper cone of a fixed non I-computable set A. This property has been successfully used within reverse mathematics for separating principles from ACA_0 by constructing ω-models of P avoiding the halting set.

Definition 1.11 (Cone avoidance) A principle P admits *(strong) cone avoidance* if it admits (strong) $\{A\}$ avoidance for each set A.

We prove in section 3 that a set computes a c.b-enum of the singleton class $\{A\}$ if an only if it computes A. Therefore (strong) cone avoidance is equivalent to (strong) c.b-enum avoidance for $\{A\}$ for each set A. Seetapun proved in [37] that $\mathsf{RT}^2_{<\infty}$ admits cone avoidance. Dzhafarov and Jockusch proved in [12] that $\mathsf{RT}^1_{<\infty}$ admits strong cone avoidance. Wang proved in [43] that FS – hence RRT –, TS^n_d for sufficiently large d's and COH admit strong cone avoidance and in [41] that SRT^2_2 does not. The equivalence between cone avoidance and $\{A\}$-enum avoidance enables us to reprove Seetapun's theorem, strong cone avoidance of $\mathsf{RT}^1_{<\infty}$, COH and FS and to obtain strong cone avoidance of EM which was unknown hitherto.

Remark that full Seetapun's theorem states that the upper cone of a countable collection of non-computable sets can be simultaneously avoided. We prove in section 6 that simultaneous cone avoidance is still a consequence of c.b-enum avoidance, even though c.b-enum avoidance is strictly weaker than simultaneous c.b-enum avoidance.

1.4. **PA avoidance**

WKL_0 states that every infinite tree $T \subseteq 2^{<\omega}$ admits an infinite path. It is a consequence of ACA_0 incomparable with RT^2_2 over RCA_0. The question of implication of WKL_0 by RT^2_2 over RCA_0 has been a long-standing open problem until Liu answered it negatively by proving that every infinite set X admits an infinite subset of non-PA degree in either X or $\overline{X}$ [29]. The forcing technique has later been reused by Wang to build an ω-model of RRT^3_2 which is not a model of WKL_0 [41].

Definition 1.12 (PA avoidance) A principle P admits *(strong) PA avoidance* if it admits (strong) $\mathscr{C}$ avoidance for $\mathscr{C} = \{Z \in 2^\omega : (\forall e)\Phi_e(e) \downarrow \rightarrow Z(e) = \Phi_e(e)\}$.

As proven in section 3, a set computes a c.b-enum of the class $\mathscr{C}$ defined above if and only if it is of PA degree. Liu proved in [29] that $\mathsf{RT}^1_{<\infty}$ admits strong PA avoidance. Wang proved in [41] that COH and RRT^2_2 admit strong PA avoidance and deduced that RRT^3_2 admits PA avoidance. We reprove through strong c.b-enum avoidance of $\mathsf{RT}^1_{<\infty}$ and COH their strong PA avoidance, and obtain as new results strong PA avoidance of FS, TS, RRT and EM.

1.5. Notation

String, sequence. Fix a function $h : \omega \to \omega$. A *string* (over h) is an ordered tuple of integers $a_0, \dots, a_{n-1}$ (such that $a_i < h(i)$ for every $i < n$). A *sequence* (over h) is an infinite listing of integers $a_0, a_1, \dots$ (such that $a_i < h(i)$ for every $i \in \omega$). We denote by $\preceq$ the prefix relation between two strings or between a string and a sequence. For $s \in \omega$, h^s is the set of all the strings of length s over h, $h^{<s}$ is the set of all the strings of length $< s$ over h, $h^{<\omega}$ is the set of all finite strings over h and h^ω is the set of all sequences (i.e. infinite strings) over h. When h is the constant function k, we write k^s (resp. $k^{<s}, \dots$) for h^s (resp. $h^{<s}, \dots$). Given a string $\sigma \in h^{<\omega}$, we denote by $|\sigma|$ its length. Given two strings $\sigma, \tau \in h^{<\omega}$, σ is a *prefix* of τ (written $\sigma \preceq \tau$) if there exists a string $\rho \in h^{<\omega}$ such that $\sigma\rho = \tau$. Given a sequence X, we write $\sigma \prec X$ if $\sigma = X{\upharpoonright}n$ for some $n \in \omega$, where $X{\upharpoonright}n$ is the restriction of the sequence X to its n first numbers. A *real* is a sequence over 2^ω. We may identify a real with a set of integers by considering that the real is its characteristic function.

Tree, path. A *tree* $T \subseteq \omega^{<\omega}$ is a set downward-closed under the prefix relation. A tree T is *finitely branching* if $T \subseteq h^{<\omega}$ for a function $h : \omega \to \omega$. A tree T is *binary* if $T \subseteq 2^{<\omega}$. A *path* through a tree T is a sequence $P \in \omega^\omega$ such that $\sigma \in T$ for every $\sigma \prec P$. We denote by $[T]$ the set of all paths through T.

Baire space, Cantor space. The set of sequences ω^ω can be given a topology structure induced by the basic open sets $[\sigma] = \{Z \in \omega^\omega : \sigma \prec \omega\}$ where $\sigma \in \omega^{<\omega}$ and is called the *Baire space*. A set of sequences $\mathscr{C}$ is (effectively) *closed* if $\mathscr{C} = [T]$ for some (computable) tree $T \subseteq \omega^{<\omega}$. If moreover T is finitely branching, then $\mathscr{C}$ is *compact*. A set of sequences $\mathscr{C}$ is a *clopen* if $\mathscr{C} = \bigcup_{\sigma \in D}[\sigma]$ for some finite set $D \subset \omega^{<\omega}$. The restriction of the Baire space to the set of the reals $2^{<\omega}$ is the *Cantor space*. When it is clear that we are working in the Cantor space, we will denote by $[\sigma]$ the set of reals $\{Z \in 2^\omega : \sigma \prec Z\}$ where $\sigma \in 2^{<\omega}$.

Mathias forcing. Given two sets E and F, we denote by $E < F$ the formula $(\forall x \in E)(\forall y \in F)x < y$. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (F_1, X_1) *extends* (F, X) (written $(F_1, X_1) \leq (F, X)$) if $F \subseteq F_1$, $X_1 \subseteq X$ and $F_1 \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

Computable reduction. A principle P is *computably reducible* to Q (written $\mathsf{P} \leq_c \mathsf{Q}$) if for every instance I of P, there exists an I-computable instance J of Q such that for every solution X to J, $X \oplus I$ computes a solution to I. P is *strongly computably reducible* to Q (written $\mathsf{P} \leq_{sc} \mathsf{Q}$) if X computes a solution to I without using I as an oracle.

2. Avoiding closed sets

The results are proven. Redaction needed.

Thanks to Liu's theorem [30], Ramsey's theorem for pairs does not imply weak weak König's lemma over RCA_0. Liu asked whether whenever an *arbitrary* tree T has no computable member, any set A has an infinite subset in either in it or its complement which still does not compute a path throught T. In this section, we answer negatively and give a general classification of the theorems in reverse mathematics which admit such a property.

Definition 2.1 (Path avoidance) A Π^1_2 statement P admits (strong) *path avoidance* if it admits (strong) $\mathscr{C}$ avoidance for every closed set $\mathscr{C} \subseteq \omega^\omega$.

Unfolding the definition, a Π^1_2 statements P admits path avoidance if for every set C, every closed set $\mathscr{C} \subseteq \omega^\omega$ with no C-computable member, and every C-computable instance X, there is a solution Y to X such that $\mathscr{C}$ has no $Y \oplus C$-computable member. The notion of path avoidance is defined for every closed set of the Baire space. However, it happens that whenever a principle is shown not to admit path avoidance, the closed set witnessing the failure belongs to the Cantor space.

2.1. **Cohen genericity**

Cohen genericity has been introduced by Kleene and Post for exhibiting a degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. Initially introduced in terms of forcing, the following modern presentation becomes standard.

Definition 2.2 (Genericity) Fix a set of strings $S \subseteq 2^{<\omega}$. The set S is *dense* if every string has an extension in S. A real G *meets* S if it has some initial segment in S. A real G *avoids* S is it has an initial segment with no extension in S. Given an integer $n \in \omega$, a real is *n-generic* if it meets or avoids each Σ^0_n set of strings. A real is *weakly n-generic* if it meets each Σ^0_n dense set of strings.

We say that a real is sufficiently Cohen generic if it is n-generic for a sufficiently large n. There exists a profusion of literature around Cohen genericity. In particular, Kautz proved in [25] that the measure of oracles computing a 1-generic real is positive, whereas it becomes null when considering 2-generic reals. See section 2.24 of [11] for an introduction to Cohen generics.

Theorem 2.3 Fix a set C computing no member some closed set $\mathscr{C} \subseteq \omega^\omega$. If G is a real sufficiently Cohen generic, then $G \oplus C$ computes no member of $\mathscr{C}$.

Proof. Given a Turing index e, consider the $\Sigma^{0,\mathscr{C}}_2$ sets of strings

$$D_e = \{\sigma \in 2^{<\omega} : (\exists n)(\forall \tau \succeq \sigma)\Phi^{\tau \oplus C}_e(n) \uparrow\}$$
$$H_e = \{\sigma \in 2^{<\omega} : [\Phi^{\sigma \oplus C}_e] \cap \mathscr{C} = \emptyset\}$$

It suffices to prove that the set $D_e \cup H_e$ is dense. Let $\sigma \in 2^{<\omega}$. Suppose there exists no finite extension $\tau \in D_e$. Then for every extension $\tau \succ \sigma$ and every $n \in \omega$, there is an extension $\rho \succeq \tau$ such that $\Phi^{\rho \oplus C}_e(n) \downarrow$. Define a C-computable sequence of binary strings $\sigma_0 \prec \sigma_1 \prec \dots$ as follows. At stage 0, $\sigma_0 = \sigma$. At stage $s+1$, let σ_{s+1} be the first string extending σ_s such that $\Phi^{\sigma_{s+1} \oplus C}_e(s) \downarrow$. Such a string exists as $\sigma_s \succeq \sigma$ and therefore $\sigma_s \notin D_e$. We claim that $\sigma_s \in H_e$ for some stage $s \in \omega$. If this is not the case, let $G = \bigcup_s \sigma_s$. The real G is C-computable and $\Phi^{G \oplus C}_e$ is a member of $\mathscr{C}$, contradiction. □

Corollary 2.4 OPT, AMT and $\Pi^0_1\mathsf{G}$ admit path avoidance.

Proof. Hirschfeldt et al. [20] proved that OPT and AMT are both consequences of $\Pi^0_1\mathsf{G}$, which itself is a restricted notion of Cohen genericity. □

Note that in the case of effectively closed sets, the sets D_e and H_e are $\Sigma^{0,C}_2$. Thus for every weakly 2-generic real G relative to C, $G \oplus C$ computes no member of $\mathscr{C}$. We now prove that if we furthermore assume that $\mathscr{C}$ is in the Cantor space and admits no C-computable 1-enum, then we can forget about the set D_e and deduce that for every 1-generic real G relative to C, $G \oplus C$ computes no 1-enum of $\mathscr{C}$.

Theorem 2.5 Fix a real C computing no 1-enum of some effectively closed set $\mathscr{C} \subseteq 2^\omega$. If G is 1-generic real relative to C, then $G \oplus C$ computes no 1-enum of $\mathscr{C}$.

Proof. Fix a functional Γ and any real G such that $\Gamma^{G \oplus C}$ is a 1-enum of $\mathscr{C}$. Consider the following c.e. set of strings

$$W = \{\sigma \in 2^{<\omega} : (\exists n)\Gamma^{\sigma \oplus C}(n) \downarrow \notin 2^n \vee [\Gamma^{\sigma \oplus C}(n)] \cap \mathscr{C} = \emptyset\}$$

As $\Gamma^{G \oplus C}$ is a 1-enum of $\mathscr{C}$, W contains no initial segment of G. If there exists a $\sigma \prec G$ such that for every $\tau \succ \sigma$, $\tau \notin W$, then we can C-compute a 1-enum of $\mathscr{C}$ by searching on input n for a $\tau \succ \sigma$ such that $\Gamma^{\tau \oplus C}(n) \downarrow$. As $\tau \notin W$, $\Gamma^{\tau \oplus C}(n) \in 2^n$ and $[\Gamma^{\tau \oplus C}(n)] \cap \mathscr{C} \neq \emptyset$. Such a τ must exist as any sufficiently long initial segment of X satisfies the property. So G is not 1-generic. □

2.2. **The arithmetic hierarchy**

By Simpson's embedding lemma [39, Lemma 3.3] (see Corollary 2.12), there exists an effectively closed set $\mathscr{C} \subseteq 2^\omega$ with no computable member, and a set A such that every infinite subset in either A or its complement computes a member of $\mathscr{C}$. Therefore, every degree **d** such that A is c.e. or co-c.e. relative to **d** computes a member of $\mathscr{C}$. However, when considering Δ^0_2 approximations, we never have enough computational power to compute a member of $\mathscr{C}$, as states the following theorem.

Theorem 2.6 Fix a real C computing no member of some closed set $\mathscr{C} \subseteq \omega^\omega$. For every real A, there exists a real X such that A is $\Delta^{0,X}_2$ and $X \oplus C$ computes no member of $\mathscr{C}$.

Proof. For a given real A, we build a limit-computable function $f^\infty : \omega^2 \to 2$ such that $f^\infty \oplus C$ computes no member of $\mathscr{C}$ and $(\forall x)\lim_s f^\infty(x,s) = A(x)$. By Schoenfield's limit lemma, the jump of f^∞ computes A. Our forcing conditions are tuples (g,n) such that g is a finite partial approximation of f^∞ and n is an integer. A condition (h,m) *extends* (g,n) if

(a) $dom(g) \subseteq dom(h)$ and $(\forall (x,s) \in dom(g)) g(x,s) = h(x,s)$
(b) $m \geq n$ and $(\forall (x,s) \in dom(h) \setminus dom(g))[s < n \to h(x,s) = A(s)]$

Informally, property (a) states that h is a function extending g and property (b) forces the n first columns of g to converge to $A \upharpoonright n$. Therefore making n grow arbitrarily large will ensure that the constructed function f^∞ is a Δ^0_2 approximation of A. Our initial condition is $(\emptyset, 0)$. The following lemma states that we can force the constructed function f^∞ to be total.

Lemma 2.7 For every condition (g,n) and every $t \in \omega$, there exists an extension (h,m) such that $m > n$ and $[0,t]^2 \subseteq dom(g)$

Proof. Let h be the function over domain $[0,t]^2 \cup dom(g)$ defined by $h(x,s) = g(x,s)$ for $(x,s) \in dom(g)$ and $h(x,s) = A(x)$ for $(x,s) \notin dom(g)$. $(h, n+1)$ is a valid extension of (g,n). ☐

A function $f^\infty : \omega^2 \to 2$ *satisfies* a condition (g,n) if $(\forall (x,s) \in dom(g)) g(x,s) = f^\infty(x,s)$ and $(\forall (x,s) \in \omega^2 \setminus dom(g))[s < n \to f^\infty(x,s) = A(s)]$. In other words, for every finite approximation h of f^∞ such that $dom(h) \geq dom(g)$, (h,n) is a valid extension of (g,n). Note that f^∞ may not be limit-computable, and that if f^∞ satisfies (g,n) and $m < n$, then f^∞ satisfies (g,m). A condition c *forces $\Phi_e^{f^\infty \oplus C}$ to be partial* if $\Phi_e^{f^\infty \oplus C}$ is partial for every function f^∞ satisfying c.

Lemma 2.8 For every condition (g,n) and every $e \in \omega$, there exists an extension (h,m) forcing $\Phi_e^{f^\infty \oplus C}$ to be partial, or $[\Phi_e^{h \oplus C}] \cap \mathscr{C} = \emptyset$.

Proof. If there is an extension (h,m) forcing $\Phi_e^{f^\infty \oplus C}$ to be partial or such that $\Phi_e^{h \oplus C} \upharpoonright n = \sigma$ for some string $\sigma \in \omega^n$ such that $\mathscr{C} \cap [\sigma] = \emptyset$, then we are done. So suppose that it is not the case. We will describe how to C-compute a member of $\mathscr{C}$ and derive a contradiction. Define a C-computable sequence of conditions $(g,n) = (g_0,n) \geq (g_1,n) \geq \dots$ as follows: Given some condition (g_i,n), let (g_{i+1},n) be the least extension such that $\Phi_e^{g_{i+1} \oplus C}(i) \downarrow$. Such extension exists as otherwise (g_i,n) would force $\Phi_e^{f^\infty \oplus C}$ to be partial. Let $f^\infty = \bigcup_i g_i$. The function f^∞ has been constructed C-computably in such a way that $\Phi_e^{f^\infty \oplus C}$ is total and a member of $\mathscr{C}$. This contradicts the assumption that C does not compute a member of $\mathscr{C}$. ☐

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, 0)$, where $c_s = (g_s, n_s)$. The filter $\mathscr{F}$ yields a unique partial function $f^\infty = \bigcup_s g_s$. By Lemma 2.7, the function f^∞ is total, and by definition of a forcing condition, f^∞ is a Δ^0_2 approximation of the real A. By Lemma 2.8, $f^\infty \oplus C$ computes no member of $\mathscr{C}$. ☐

Corollary 2.9 COH admits path avoidance.

Proof. Fix a real C computing no member of some closed set $\mathscr{C} \subseteq \omega^\omega$ and let $R_0, R_1, \dots$ be a uniformly C-computable sequence of reals. By Theorem 2.6, there exists a real X such that $X \oplus C$

computes no member of $\mathscr{C}$ and the jump of X computes $\emptyset''$. Jockusch and Stephan [21] proved that if $R_0, R_1, \dots$ is a uniform sequence of reals, for real X whose jump if of PA degree relative to the jump of $\vec{R}$, $X \oplus \vec{R}$ computes an infinite $\vec{R}$-cohesive real. Therefore $X \oplus C$ computes an infinite $\vec{R}$-cohesive real. □

Corollary 2.10 For every real A and every non-computable real B, there exists a real X such that $A \in \Delta_2^{0,X}$ but $X \not\geq_T B$.

Proof. Apply Theorem 2.6 with $\mathscr{C} = \{B\}$ to obtain a real X such that $A \in \Delta_2^{0,X}$ and X computes no member of $\mathscr{C}$, hence $X \not\geq_T B$. □

We shall see in Corollary 2.17 that COH does not admit strong path avoidance since RT^1_2 does not.

2.3. **The embedding lemma**

The following application of Simpson's embedding lemma is very useful for proving that some principle does not admit path avoidance.

Lemma 2.11 If some principle P has a computable (resp. arbitrary) instance with no computable solution and such that its collection of solutions is a Σ^0_3 subset of ω^ω, then P does not admit (strong) path avoidance.

Proof. We prove it in the case of path avoidance; Let X be a computable P-instance with no computable solution, and let $\mathscr{C} \subseteq \omega^\omega$ be its set of solutions. By Lemma 3.3 in Simpson [39], there exists an effectively closed set class $\mathscr{D} \subseteq 2^\omega$ whose degrees are exactly the PA degrees and the degrees of members of $\mathscr{C}$. Since X has no computable solution, $\mathscr{D}$ has no computable member. Every solution to X is a member of $\mathscr{C}$ and thus computes a member of $\mathscr{D}$. Therefore P does not admit path avoidance. □

Note that the witness of failure of path avoidance is an effectively closed set.

Corollary 2.12 RT^1_2 does not admit strong path avoidance.

Proof. Let A be a Δ^0_2 bi-immune set. The collection of its infinite homogeneous sets a Π^0_2 subset of ω^ω:
$$\mathscr{C} = \{X \in \omega^\omega : (\forall i)[X(i) <_{\mathbb{N}} X(i+1) \wedge X(i) \in A \leftrightarrow X(i+1) \in A]\}$$
Apply Lemma 2.11. □

Of course, if $\mathsf{Q} \leq_c \mathsf{P}$ and Q does not admit path avoidance, then so does P. We therefore want to prove that very weak principles do not admit path avoidance to obtain the same conclusion for many statements belonging to the reverse mathematics zoo.

Corollary 2.13 DNR does not admit path avoidance.

Proof. The collection d.n.c. functions is a Π^0_1 subset of ω^ω with no computable member:
$$\mathscr{C} = \{f \in \omega^\omega : (\forall e, s)[\Phi_{e,s}(e) \downarrow \rightarrow \Phi_{e,s}(e) \neq f(e)]\}$$
Apply Lemma 2.11. □

Corollary 2.14 SADS does not admit path avoidance.

Proof. Tennenbaum [36] constructed a computable linear order of order type $\omega + \omega^*$ with no computable ascending or descending sequence. Given a linear order $\mathscr{L}$, the collection of its infinite ascending or descending sequence is a Π^0_1 subset of ω^ω:
$$\mathscr{C} = \{X \in \omega^\omega : (\forall i)[X(i) <_{\mathscr{L}} X(i+1)] \vee (\forall i)[X_i >_{\mathscr{L}} X(i+1)]\}$$
Apply Lemma 2.11. □

The following lemma shows that avoidance is closed downward under computable reducibilty. As many proofs of reductions in reverse mathematics are in fact computable reductions, this lemma has many applications.

Lemma 2.15 If P is (strongly) computably reducible to Q and Q admits (strong) $\mathscr{C}$ avoidance, then so does P.

Proof. We prove it in the case of computable reducibility. The strong case is similar. Let C be a real computing no member of $\mathscr{C}$ and let I be a C-computable instance of P. As $P \leq_c Q$, there exists an I-computable instance J of Q such that for every solution X to J, $X \oplus I$ computes a solution to I. By $\mathscr{C}$ avoidance of Q, there exists a solution X to J such that $X \oplus C$ computes no member of $\mathscr{C}$. $X \oplus C$ computes a solution Y to I, but computes no member of $\mathscr{C}$. ☐

Corollary 2.16 None of RT^2_2 ADS, CAC, EM, TS^2 RRT^2_2 admit path avoidance.

Proof. By Hirschfeldt et al. [18], $\mathsf{DNR} \leq_c \mathsf{SRT}^2_2$. By Hirschfeldt & Shore [19], $\mathsf{SADS} \leq_c \mathsf{ADS} \leq_c \mathsf{CAC}$. By Rice [35], $\mathsf{DNR} \leq_c \mathsf{TS}^2$. By Miller [31], $\mathsf{DNR} \leq_c \mathsf{RRT}^2_2$. By the author [33], $\mathsf{DNR} \leq_c \mathsf{EM}$. Conclude by Lemma 2.15, Corollary 2.13 and Corollary 2.14. ☐

Corollary 2.17 COH does not admit strong path avoidance.

Proof. Immediate by Corollary 2.12, Lemma 2.15 and the fact that $\mathsf{RT}^1_2 \leq_{sc} \mathsf{COH}$. ☐

2.4. **Simultaneous path avoidance**

The notion of path avoidance expresses the ability for a principle to avoid computing a member of a Π^0_1 set of the Baire space. We now see that the notion of avoidance for F_σ sets coincides with path avoidance.

Definition 2.18 (Simultaneous path avoidance) Fix a countable collection of closed sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq \omega^\omega$. A principle P admits *(strong) path avoidance* for $\vec{\mathscr{C}}$ if it admits (strong) $\bigcup_i \mathscr{C}_i$ avoidance. A principle P admits *(strong) simultaneous path avoidance* if it admits (strong) path avoidance for $\vec{\mathscr{C}}$ for every countable collection of closed sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq \omega^\omega$

Remark that $\bigcup_i \mathscr{C}_i$ is not a closed set in general, even if the $\mathscr{C}_i$'s are all closed. We start by proving that path avoidance and simultaneous path avoidance coincide.

Definition 2.19 (Muchnik reducibility) Let $\mathscr{C}$ and $\mathscr{D}$ be two classes of reals. $\mathscr{C}$ is *Muchnik reducible* to $\mathscr{D}$ (denoted by $\mathscr{C} \leq_w \mathscr{D}$) if for every $X \in \mathscr{D}$, there exists a $Y \in \mathscr{C}$ such that $Y \leq_T X$.

Lemma 2.20 Let $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq \omega^\omega$ be a countable collection of closed sets such that $\mathscr{C}_i$ has no computable member for each i. There exists a closed set $\mathscr{D} \subseteq \omega^\omega$ such that $\mathscr{D}$ and $\bigcup_i \mathscr{C}_i$ are Muchnik equivalent. Moreover, if the $\mathscr{C}$'s belong the Cantor space, then so does $\mathscr{D}$.

Proof. We may assume that some $\mathscr{C}_i$ is non-empty as otherwise, $\mathscr{D} = \emptyset$ is a trivial solution. Let X be a member of some $\mathscr{C}_i$ and define $\mathscr{D}$ as follows:

$$\mathscr{D} = \{\sigma^\frown(i+1 \mod 2)^\frown Z : \sigma^\frown i \prec X \wedge Z \in \mathscr{C}_{|\sigma|}\}$$

The set $\mathscr{D}$ is closed and Muchnik equivalent to $\bigcup_i \mathscr{C}_i$. ☐

Corollary 2.21 If a principle P admits (strong) path avoidance, then it admits (strong) simultaneous path avoidance.

Proof. We prove it in the case of path avoidance. Let C be a set computing no member of $\bigcup_i \mathscr{C}_i$ for some countable collection of sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq \omega^\omega$, and let X be a C-computable instance of P. By Lemma 2.20, there exists a closed set of reals $\mathscr{D}$ Muchnik equivalent to $\bigcup_i \mathscr{C}_i$. Therefore C computes no member of $\mathscr{D}$. By path avoidance of P, there is a solution Y to X such that $Y \oplus C$ computes no member of $\mathscr{D}$ and therefore computes no member of $\bigcup_i \mathscr{C}_i$. ☐

3. Avoiding the enumerations of closed sets

The results are proven. Redaction needed.

In this section, we relate various notions of enumeration avoidance and state a few basic lemmas which will be useful for the remainder of our developpment. The following notion of Medvedev reducibility is a uniform variant of Muchnik reducibility.

Definition 3.1 (Medvedev reducibility) Let $\mathscr{C} \subseteq \omega^\omega$ and $\mathscr{D} \subseteq \omega^\omega$ be two sets of sequences. We say that $\mathscr{C}$ is *Medvedev reducible* to $\mathscr{D}$ (denoted by $\mathscr{C} \leq_s \mathscr{D}$) if there exists a Turing functional Γ such that $\Gamma^X \in \mathscr{C}$ for every sequence $X \in \mathscr{D}$.

Lemma 3.2 Let $\mathscr{C} \subseteq \omega^\omega$ be a set of sequences Medvedev below a compact set of sequences $\mathscr{D} \subseteq \omega^\omega$. For every $k \in \omega$, every k-enum of $\mathscr{D}$ computes a k-enum of $\mathscr{C}$.

Proof. Let Γ be the Turing functional witnessing the Medvedev reduction from $\mathscr{C}$ to $\mathscr{D}$. We prove it by induction over k. Let $(D_i : i \in \omega)$ be a k-enum of $\mathscr{D}$. Suppose that there exists a $\sigma \in 2^{<\omega}$ such that $\mathscr{D} \cap [\sigma] = \emptyset$ and for infinitely many $i \in \omega$, $\sigma \preceq \tau$ for some $\tau \in D_i$. Then $k > 1$ and we can compute a $(k-1)$-enum of $\mathscr{D}$ by computably finding on input i a $j > i$ such that $\sigma \preceq \tau$ for some $\tau \in D_j$ and returning $E_i = \{\sigma \upharpoonright i : \sigma \in D_j \setminus \tau\}$. $\vec{E}$ is a $(k-1)$-enum of $\mathscr{D}$ and by induction hypothesis, it computes a $(k-1)$-enum of $\mathscr{C}$, so *a fortiori* a k-enum of $\mathscr{C}$.

So suppose there exists no such σ. This means that for every $i \in \omega$, there exists a $j > i$ such that $\mathscr{D} \cap [\sigma \upharpoonright i] \neq \emptyset$ for each $\sigma \in D_j$. As $\mathscr{C} \subseteq h^\omega$, by the pigeonhole principle Γ will produce arbitrarily large k-tuples of initial segments of members of $\mathscr{C}$. We compute a k-enum of $\mathscr{C}$ as follows: For each $i \in \omega$, let $E_i = \{\Gamma^\sigma \upharpoonright i : \sigma \in D_j\}$ for some j such that $\Gamma^\sigma \upharpoonright i$ is defined on each $\sigma \in D_j$. Such E_i has been shown to exist and can be found computably in $\vec{D}$. As $[\sigma] \cap \mathscr{D} \neq \emptyset$ for some $\sigma \in D_j$, $[\Gamma^\sigma \upharpoonright i] \cap \mathscr{C} \neq \emptyset$, hence $(\exists \tau \in E_i)\mathscr{C} \cap [\tau] \neq \emptyset$ hence and $\vec{E}$ is a valid k-enum of $\mathscr{C}$. □

3.1. Simultaneous enumeration avoidance

We can define a notion of simultaneous c.b-enum avoidance like we did for path avoidance. However, we shall see that the notions do not coincide in the case of c.b-enum avoidance.

Definition 3.3 (Simultaneous c.b-enum avoidance) Fix a countable collection of sets of reals $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$. A principle P admits *(strong) c.b-enum avoidance* for $\vec{\mathscr{C}}$ if it admits (strong) $\mathscr{D}$ avoidance, where $\mathscr{D} = \{Z : (\exists i) Z \text{ is a c.b-enum of} \mathscr{C}_i\}$. A principle P admits *(strong) simultaneous c.b-enum avoidance* if it admits (strong) c.b-enum avoidance for $\vec{\mathscr{C}}$ for every countable collection of sets of reals $\mathscr{C}_0, \mathscr{C}_1, \dots$ Given some $n \in \omega$, P admits *(strong) n c.b-enum avoidance* if it admits (strong) c.b-enum avoidance for $\vec{\mathscr{C}}$ for every sequence of n sets of reals $\mathscr{C}_0, \dots, \mathscr{C}_{n-1}$.

Simultaneous 1-enum avoidance and cone avoidance are defined similarly. Beware, a c.b-enum of a collection of sets of reals $\vec{\mathscr{C}}$ is a c.b-enum of $\mathscr{C}_i$ for some $i \in \omega$ and not a c.b-enum of $\bigcup_i \mathscr{C}_i$. Liu defined in [30] c.b-enum avoidance for any increasing sequence (in inclusion order) of sets of reals. In this subsection we prove that Liu's apparently stronger notion of avoidance is in fact equivalent to the avoidance of a single set of reals.

Lemma 3.4 Let C be real and $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ be a countable collection of sets of reals with no C-computable 1-enum of $\mathscr{C}_i$ for each i. There exists a set of reals $\mathscr{D} \subseteq 2^\omega$ Medvedev below each $\mathscr{C}_i$ such that $\mathscr{D}$ admits no C-computable 1-enum.

Proof. Fix a $X \not\leq_T C'$ and define $\mathscr{D}$ like in Lemma 2.20, that is,

$$\mathscr{D} = \{\sigma^\frown(1-i)^\frown Z : \sigma^\frown i \prec X \wedge Z \in \mathscr{C}_{|\sigma|}\}$$

$\mathscr{D}$ is Medvedev below each $\mathscr{C}_i$. Suppose for the contradiction that there is a C-computable 1-enum of $\mathscr{D}$ $(\tau_n : n \in \omega)$. If there exists a $\rho \not\prec X$ such that $\rho \preceq \tau_n$ for infinitely many n, then because $\vec{\tau}$ is a 1-enum of $\mathscr{D}$, there exists a $\sigma \in 2^{<\omega}$ and an $i \in \{0,1\}$ such that $\sigma^\frown i \prec X$ and $\sigma^\frown(1-i) \prec \rho$ and $[\tau_n] \cap \mathscr{C}_{|\sigma|} \neq \emptyset$ for every n such that $\rho \prec \tau_n$. We can C-compute a 1-enum of $\mathscr{C}_{|\sigma|}$ by C-effectively finding those n and removing the prefix σ to each string. If there exists no such ρ, then for every

$\rho \not\prec X$, there exists an n_0 such that $\rho \not\prec \tau_n$ for every $n \geq n_0$. Then the jump of C computes X, contradiction. □

Corollary 3.5 If a principle P admits (strong) 1-enum avoidance, then it admits (strong) simultaneous 1-enum avoidance.

Proof. We prove it in the case of 1-enum avoidance. Let C be a set computing no 1-enum of $\vec{\mathscr{C}}$ for some countable collection of sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$, and let X be a C-computable instance of P. By Lemma 3.4, there exists a set of reals $\mathscr{D}$ Medvedev below each $\mathscr{C}_i$ such that C computes no 1-enum of $\mathscr{D}$. By 1-enum avoidance of P, there is a solution Y to X such that $Y \oplus C$ computes no 1-enum of $\mathscr{D}$. By Lemma 3.2, $Y \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$. □

Corollary 3.6 For every set of reals $\mathscr{C} \subseteq 2^\omega$ admitting no computable c.b-enum, there exists a set of reals $\mathscr{D}$ whith no computable 1-enum, such that every c.b-enum of $\mathscr{C}$ computes a 1-enum of $\mathscr{D}$.

Proof. For each $k > 0$, let $\mathscr{D}_k = \{Z \in 2^\omega : Z \text{ codes a k-enum of } \mathscr{C}\}$. Any 1-enum of $\mathscr{D}_k$ for some $k > 0$ computes a c.b-enum of $\mathscr{C}$. Therefore there exists no computable 1-enum of $\mathscr{D}_k$ for any $k > 0$. By Lemma 3.4, there exists a class $\mathscr{D}$ Medvedev below $\mathscr{D}_k$ for each $k > 0$, such that $\mathscr{D}$ admits no computable 1-enum. It remains to show that every c.b-enum of $\mathscr{C}$ computes a 1-enum of $\mathscr{D}$. Fix any c.b-enum of $\mathscr{C}$. It is a k-enum of $\mathscr{C}$ for some $k > 0$, so computes a 1-enum of $\mathscr{D}_k$, and by Lemma 3.2 it computes a 1-enum of $\mathscr{D}$. □

The previous corollary shows that if a principle admits 1-enum avoidance, it admits c.b-enum avoidance. However the set of reals $\mathscr{D}$ constructed in Lemma 3.4 depends on a set $X \not\leq_T C'$. Therefore it does not show that for a fixed set of reals $\mathscr{C}$, there is a single set of reals $\mathscr{D}$ such that the degrees bounding a c.b-enum of $\mathscr{C}$ are exactly the degrees bounding a 1-enum of $\mathscr{D}$. We can recover this property if we use simultaneous avoidance.

Lemma 3.7 For every countable collection of set of reals $\vec{\mathscr{C}}$, there exists a countable collection of sets of reals $\vec{\mathscr{D}}$ such that the degrees bounding a c.b-enum of $\vec{\mathscr{C}}$ are exactly the degrees bounding a 1-enum of $\vec{\mathscr{D}}$.

Proof. Fix a countable collection of sets of reals $\vec{\mathscr{C}}$. For each $i, k \in \omega$, define

$$\mathscr{D}_{i,k} = \{Z \in 2^\omega : Z \text{ codes a } k\text{-enum of } \mathscr{C}_i\}$$

Fix a degree $\mathbf{d}$ bounding a c.b-enum of $\vec{\mathscr{C}}$. By definition, there exists an i and a $k \in \omega$ suc that $\mathbf{d}$ bounds a k-enum of $\mathscr{C}_i$. Therefore $\mathbf{d}$ bounds a member of $\mathscr{D}_{i,k}$, hence a 1-enum of $\mathscr{D}_{i,k}$. Conversely, suppose $\mathbf{d}$ bounds a 1-enum of $\mathscr{D}_{i,k}$ for some $i, k \in \omega$. Then it bounds a member of $\mathscr{D}_{i,k}$, so bounds a k-enum of $\mathscr{C}_i$ and therefore bounds a c.b-enum of $\vec{\mathscr{C}}$. □

Unlike 1-enum avoidance which has been proven equivalent to simultaneous 1-enum avoidance of an *arbitrary* sequence of sets of reals, we will only be able to prove that c.b-enum avoidance of a single set of reals is equivalent to simultaneous avoidance of an *increasing* sequence of sets of reals. This restriction will be proven necessary through Theorem 3.10.

Lemma 3.8 Let $\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \dots \subseteq 2^\omega$ be an increasing countable collection of sets of reals with no computable c.b-enum. There exists a set $\mathscr{D} \subseteq 2^\omega$ Medvedev below each $\mathscr{C}_i$ such that $\mathscr{D}$ has no computable c.b-enum.

Proof. Fix a set $X \not\leq_T \emptyset'$ and define $\mathscr{D}$ as in Lemma 3.4. $\mathscr{D}$ is Medvedev below each $\mathscr{C}_i$. We prove that there exists no computable c.b-enum of $\mathscr{D}$. Fix a computable k-enum $(D_i : i \in \omega)$ of $\mathscr{D}$. By thinning out $\vec{D}$, we can obtain a computable k-enum $(E_i : i \in \omega)$ of $\mathscr{D}$ together with a finite set of strings (with possible duplications) $\sigma_0, \dots, \sigma_{r-1}$ for some $r \leq k$ and a computable injective function $g : \omega \times r \to 2^{<\omega}$ such that

(i) $(\forall i < r)\sigma_i \not\prec X \wedge (\sigma_i \upharpoonright |\sigma_i| - 1) \prec X$
(ii) $(\forall i \in \omega)(\forall j < r)[g(i,j) \in E_i \wedge \sigma_j \prec g(i,j)]$

(iii) if $\sigma \not\prec X$ then there are finitely many i such that $\sigma \prec \tau$ for some $\tau \in E_i \setminus \{g(i,j) : j < r\}$.

If $r = k$ then let $n = max(\{|\sigma_j| : j < r\})$. For each $i \in \omega$ and $j < k$, let $f(i,j)$ be the unique string ρ of length i such that $\sigma_j^\frown \rho \prec g(n+i,j)$ and let $F_i = \{g(i,j) : j < k\}$. We claim that the sequence $\vec{F}$ is a k-enum of $\mathscr{C}_n$. Indeed, since $\vec{E}$ is a k-enum of $\mathscr{D}$, for each i, there exists a $\tau \in E_{i+n}$ such that $[\tau] \cap \mathscr{D} \neq \emptyset$. Since f is injective, there is some $j < k$ such that $\tau = g(i+n,j)$. By construction of $\mathscr{D}$, $[f(i,j)] \cap \mathscr{C}_{|\sigma_j|-1} \neq \emptyset$ so $[f(i,j)] \cap \mathscr{C}_n \neq \emptyset$ since $\mathscr{C}_n \supseteq \mathscr{C}_{|\sigma_j|-1}$.

If $r < k$ then consider for each i the non-empty set $F_i = E_i \setminus \{g(i,j) : j < r\}$. For every $m > max(|\sigma_j| : j < r)$, $(\forall^\infty i)(\forall \tau \in F_i)\tau\upharpoonright m \prec X$. Therefore we can $\emptyset'$-compute X, contradicting our choice of X. □

Lemma 3.8 is optimal in the sense that some principles admitting c.b-enum avoidance do not admit n c.b-enum avoidance (see Theorem 3.10). Although some principles do not admit c.b-enum avoidance of an arbitrary countable sequence of sets of reals, they can simultaneously avoid computing a c.b-enum of all effectively closed set with no computable c.b-enum.

Lemma 3.9 Let $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ be a countable collection of effectively closed sets with no computable c.b-enum. There exists a (non-effectively) closed set $\mathscr{D} \subseteq 2^\omega$ Medvedev below each $\mathscr{C}_i$ such that $\mathscr{D}$ has no computable c.b-enum.

Proof. Fix a set $X \not\leq \emptyset'$ and define $\mathscr{D}$ as in Lemma 3.4. $\mathscr{D}$ is Medvedev below each $\mathscr{C}_i$. We prove by induction over k that there exists no computable k-enum of $\mathscr{D}$. Fix a computable k-enum $(D_i : i \in \omega)$ of $\mathscr{D}$. If there exists a $\sigma \not\prec X$ and infinitely many i such that $\sigma \prec \tau$ for some $\tau \in D_i$. As $\sigma \not\prec X$, there exists $\rho, \nu \in 2^{<\omega}$ and $j \in \{0,1\}$ such that $\nu^\frown j \prec X$ and $\sigma = \rho^\frown(1-j)^\frown\nu$. If there exists infinitely many i such that $\sigma \prec \tau$ for some $\tau \in D_i$ and $\mathscr{C}_{|\rho|} \cap [\xi] = \emptyset$ where $\tau = \rho^\frown(1-j)^\frown\xi$, then we can computably find infinitely many such τ and compute a $(k-1)$-enum by enumerating $D_i \setminus \tau$ for each such i. If there are finitely many such i, then we can compute a 1-enum of $\mathscr{D}$ by enumerating each such τ. So suppose that for every $\sigma \not\prec X$, there exists finitely many i such that $\sigma \prec \tau$ for some $\tau \in D_i$. Then the jump of $\vec{D}$ computes X, contradicting $X \not\leq_T \emptyset'$. □

3.2. **Negative simultaneous avoidance**

We now prove that the notions of c.b-enum avoidance and simultaneous avoidance do not coincide. Moreover, there is a whole hierarchy of avoidance relations based on how many closed sets can be avoided simultaneously.

Theorem 3.10 There exists a countable collection of closed sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ together with a Δ^0_2 function $f : \omega \to \omega$ and a 1-enum $(\rho_i : i \in \omega)$ such that

(i) $\bigcup_{j \neq i} \mathscr{C}_j$ has no computable c.b-enum for each i
(ii) $[\rho_i] \cap \mathscr{C}_{f(i)} \neq \emptyset$ for each i

Proof. Fix a non-computable Δ^0_2 set X and a computable sequence $X_0, X_1, \dots$ of reals pointwise converging to X. We build the closed sets of reals $\vec{\mathscr{C}}$ by forcing. Our forcing conditions are tuples $(k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, E_0, \dots, E_{k-1})$ where

(a) $\bigcup_{j \neq i} \mathscr{C}_j$ are closed sets containing X and with no computable c.b-enum for each $i < k$
(b) E_i are finite sets of strings for each $i < k$
(c) $(\bigcup_{j \neq i} \mathscr{C}_j) \cap [E_i] = \emptyset$ for each $i < k$
(d) $(\forall s)(\exists i < k)([X_s \upharpoonright s] \not\subseteq [\bigcup_{j \neq i} E_j])$

A condition $(m, \tilde{\mathscr{C}}_0, \dots, \tilde{\mathscr{C}}_{m-1}, \tilde{E}_0, \dots, \tilde{E}_{m-1})$ *extends* a condition $(k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, E_0, \dots, E_{k-1})$ if $m \geq k$, $\mathscr{C}_i \subseteq \tilde{\mathscr{C}}_i$ and $E_i \subseteq \tilde{E}_i$ for each $i < k$. The set E_i is a forbidden open set for $\bigcup_{j \neq i} \mathscr{C}_j$. We forcing $\bigcup_{j \neq i} \mathscr{C}_j$ not to have computable c.b-enum, we shall put strings in it. Our initial condition is $(2, \{X\}, \{X\}, \emptyset, \emptyset)$ which is valid by Corollary 3.23. Note that given some condition $c = (k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, E_0, \dots, E_{k-1})$, the condition $(k+1, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, \{X\}, E_0, \dots, E_{k-1}, \emptyset)$ is a valid extension of c.

We want our forcing to be $\emptyset'$-effective to obtain a Δ^0_2 function $f : \omega \to \omega$ such that property (ii) holds. Given some condition $c = (k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, E_0, \dots, E_{k-1})$, a *code of* c is a tuple $\langle k, e_0, \dots, e_{k-1}, E_0, \dots, E_{k-1}\rangle$ such that for each $i < k$, $\Phi^{\emptyset'}_{e_i}$ is the characteristic function of the set of strings $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C}_i \neq \emptyset$. Note that a condition may not have a code in general, but our initial condition $(2, \{X\}, \{X\}, \emptyset, \emptyset)$ has one. We will show that we can $\emptyset'$-effectively find an infinite decreasing sequence of extensions having codes and forcing the desired properties.

Lemma 3.11 For every condition $c = (k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, E_0, \dots, E_{k-1})$ and $s \in \omega$, there exists an extension $d = (k, \tilde{\mathscr{C}}_0, \dots, \tilde{\mathscr{C}}_{k-1}, E_0, \dots, E_{k-1})$ and some $i < k$ such that $[X_s \upharpoonright s] \cap \tilde{\mathscr{C}}_i \neq \emptyset$. Moreover, one can $\emptyset'$-effectively find a code of d given a code of c.

Proof. By property (d) of the condition c there is some $i < k$ such that $[X_s \upharpoonright s] \not\subseteq [\bigcup_{j \neq i} E_j]$. Let $E = \bigcup_{j \neq i} E_j$. As E is finite, there exists a finite $\tau \succ X_s \upharpoonright s$ such that $[\tau] \cap [E] = \emptyset$. Moreover, those i and τ can be $\emptyset'$-effectively found. Let $\tilde{\mathscr{C}}_i = \mathscr{C}_i \cup \{\tau^\frown Z : Z \in \mathscr{C}_i\}$ and let $\tilde{\mathscr{C}}_j = \mathscr{C}_j$ for each $j \neq i$. The closed set $\tilde{\mathscr{C}}_i$ is Medvedev above $\mathscr{C}_i$. Therefore, for each $j < k$, $\bigcup_{r \neq j} \tilde{\mathscr{C}}_r$ is Medvedev above $\bigcup_{r \neq j} \mathscr{C}_r$ and by Lemma 3.2 and property (a) of condition c, it admits no computable c.b-enum. The condition $d = (k, \tilde{\mathscr{C}}_0, \dots, \tilde{\mathscr{C}}_{k-1}, E_0, \dots, E_{k-1})$ satisfies therefore properties (a), (b) and (d). We check property (c). If $(\bigcup_{r \neq j} \tilde{\mathscr{C}}_r) \cap [E_j] \neq \emptyset$ for some $j < k$, then by property (c) of the condition c, $j \neq i$ and $(\bigcup_{r \neq j} \mathscr{C}_r) \cap [E_j] = \emptyset$. As $(\bigcup_{r \neq j} \tilde{\mathscr{C}}_r) \subseteq (\bigcup_{r \neq j} \mathscr{C}_r) \cup [\tau]$, we obtain $[\tau] \cap [E_j] \neq \emptyset$, contradiction. Hence property (c) holds and d is a valid extension of c. The Turing index of the characteristic function of the strings extensible in $\tilde{\mathscr{C}}_i$ can be effectively found from the Turing index of the characteristic function of the strings extensible in $\mathscr{C}_i$. Therefore the condition d has a code, which can be $\emptyset'$-effectively found from a code of c. □

Lemma 3.12 For every condition $c = (k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, E_0, \dots, E_{k-1})$, every $i < k$ and every $e \in \omega$, there exists an extension $d = (k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, \tilde{E}_0, \dots, \tilde{E}_{k-1})$ such that if Φ_e is an e-enum then $(\exists n)\Phi_e(n) \subset \tilde{E}_i$. Moreover, one can $\emptyset'$-effectively find a code of d given a code of c.

Proof. Let $F = \bigcup_{j \neq i} E_j$ and let $u = max(|\sigma| : \sigma \in F)$. We can $\emptyset'$-effectively find some stage $t > u$ such that $X_t \upharpoonright u = X \upharpoonright u$. By Lemma 3.11, we can assume that for every $s < t$, there is some $j < k$ such that $[X_s \upharpoonright s] \cap \mathscr{C}_j \neq \emptyset$. As by property (a) of the condition c, $\bigcup_{j \neq i} \mathscr{C}_j$ admits no computable c.b-enum, there exists some $n > t + e$ such that either $\Phi_e(n) \uparrow$, or $[\Phi_e(n)] \cap \bigcup_{j \neq i} \mathscr{C}_j = \emptyset$. We can $\emptyset'$-decide in which case we are. In the first case, we take c as the desired extension. Set $\tilde{E}_i = E_i \cup \Phi_e(n)$ and $\tilde{E}_j = E_j$ for each $j \neq i$. Properties (a), (b) and (c) hold for the condition $d = (k, \mathscr{C}_0, \dots, \mathscr{C}_{k-1}, \tilde{E}_0, \dots, \tilde{E}_{k-1})$. We now check property (d).

Suppose for the contradiction that for some s, for every $j < k$, $[X_s \upharpoonright s] \subseteq [\bigcup_{r \neq j} \tilde{E}_r]$. In particular, $[X_s \upharpoonright s] \subseteq [F]$. In this case $s < t$, otherwise $[X_s \upharpoonright s] \subseteq [X \upharpoonright u]$. But then $[X \upharpoonright u] \cap [F] \neq \emptyset$ and as $u = max(|\sigma| : \sigma \in F)$, $[X \upharpoonright u] \subseteq [E_j]$ for some $j < k$, contradicting the fact that $X \in \bigcup_{r \neq j} \mathscr{C}_r$ and property (c) of the condition c. By property (d) of the condition c, there exists some $j < k$ such that $[X_s \upharpoonright s] \not\subseteq [\bigcup_{r \neq j} E_r]$. Let μ be the Lebesgue measure. Since $t > max(|\sigma| : \sigma \in F), t > s$ and $[X_s \upharpoonright s] \not\subseteq [\bigcup_{r \neq j} E_r]$, $\mu([X_s \upharpoonright s] \setminus [\bigcup_{r \neq j} E_r]) \geq 2^{-t}$. Since Φ_e is an e-enum and $n > t + e$, $\mu([\Phi_e(n)]) \leq e \times 2^{-t-e} < 2^{-t}$. Therefore, $\mu([X_s \upharpoonright s] \setminus ([\bigcup_{r \neq j} E_r] \cup [\Phi_e(e)])) > 0$ so $[X_s \upharpoonright s] \not\subseteq [\bigcup_{r \neq j} \tilde{E}_r]$, contradiction. □

Thanks to Lemma 3.11 and Lemma 3.12, we build an infinite $\emptyset'$-computable decreasing sequence of conditions $c_0 = (\{X\}, \{X\}, \emptyset, \emptyset) \geq c_1 \geq c_2 \geq \dots$ together with their codes, such that for each $s \in \omega$, assuming $c_s = (k_s, \mathscr{C}_{0,s}, \dots, \mathscr{C}_{k_s-1,s}, E_{0,s}, \dots, E_{k_s-1,s})$,

(i) $k_s \geq s$
(ii) If Φ_s is a total s-enum, then $(\forall i < k_s)(\exists n)\Phi_s(n) \subset E_{i,s}$
(iii) $[X_s \upharpoonright s] \cap \bigcup_{i < k_s} \mathscr{C}_{i,s} \neq \emptyset$

This way, taking $\mathscr{C}_i = \bigcup_{s \geq i} \mathscr{C}_{i,s}$, we obtain two closed sets admitting no computable c.b-enum by (ii) and such that $s \mapsto X_s \upharpoonright s$ is a computable 1-enum of $\bigcup_i \mathscr{C}_i$ by (ii). This completes the proof of Theorem 3.10. ☐

Corollary 3.13 TS^1_n does not admit strong n c.b-enum avoidance for every $n \geq 2$. In particular, RT^1_2 does not admit strong 2 c.b-enum avoidance. As well, TS^1 does not admit strong simultaneous c.b-enum avoidance.

Proof. Fix some $n \geq 2$. Let $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ be the collection of closed sets of reals, $f : \omega \to \omega$ be the function and $(\rho_i : i \in \omega)$ be the 1-enum constructed in Theorem 3.10. For each $i < n-1$, let $\mathscr{D}_i = \mathscr{C}_i$. As well, let $\mathscr{D}_{n-1} = \bigcup_{i \geq n-1} \mathscr{C}_i$. By (i) of Theorem 3.10, $\bigcup_{j \neq i} \mathscr{D}_j$ admits no computable c.b-enum for each $i < n$. Let $g : \omega \to n$ be defined by

$$g(i) = \begin{cases} f(i) & \text{if } f(i) < n-1 \\ n-1 & \text{otherwise} \end{cases}$$

By (ii) of Theorem 3.10, $[\rho_i] \cap \mathscr{D}_{g(i)} \neq \emptyset$ for each i. Every infinite g-thin set H with witness color $i < n$ will compute an 1-enum of $\bigcup_{j \neq i} \mathscr{D}_j$. Therefore TS^1_n does not admit strong n c.b-enum avoidance. The case of TS^1 is similar. ☐

Corollary 3.14 Neither RT^1_2, nor TS^1 admit strong 1-enum avoidance.

Proof. We prove it for TS^1, since the case of RT^1_2 follows from Lemma 2.15. Let $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ be the collection of closed sets of reals, $f : \omega \to \omega$ be the function and $(\rho_i : i \in \omega)$ be the 1-enum constructed in Theorem 3.10. For every i, $\bigcup_{j \neq i} \mathscr{C}_j$ admits no computable c.b-enum, and *a fortiori* no computable 1-enum. By Lemma 3.4, there is a set of reals $\mathscr{D}$ Medvedev below $\bigcup_{j \neq i} \mathscr{C}_j$ for each j with no computable 1-enum. Every infinite f-thin set with color i computes a 1-enum of $\bigcup_{j \neq i} \mathscr{C}_j$. By Lemma 3.2, every 1-enum of $\bigcup_{j \neq i} \mathscr{C}_j$ computes a 1-enum of $\mathscr{D}$. Hence TS^1 does not admit strong 1-enum avoidance. ☐

Using Schoenfield's limit lemma [38], we obtain negative avoidance results about stable colorings of pairs.

Corollary 3.15 STS^2_n does not admit n c.b-enum avoidance for every $n \geq 2$. In particular, SRT^2_2 does not admit 2 c.b-enum avoidance. As well, STS^2 does not admit simultaneous c.b-enum avoidance.

Corollary 3.16 Neither SRT^2_2 nor STS^2 admit 1-enum avoidance.

3.3. **Comparing notions of avoidance**

Simple coding arguments enables us to deduce that path avoidance implies 1-enum avoidance, which itself implies c.b-enum avoidance. We start our comparison with path avoidance and 1-enum avoidance. By a simple coding argument, if a principle admits path avoidance, then it admits 1-enum avoidance. As we have seen (Corollary 2.13), DNR does not admit path avoidance. On the other hand, DNR admits strong 1-enum avoidance by Theorem 4.36. Therefore the two notions are distinct.

Lemma 3.17 Fix a principle P.

(i) If P admits (strong) path avoidance, then it admits (strong) 1-enum avoidance.
(ii) If P admits (strong) 1-enum avoidance, then it admits (strong) simultaneous c.b-enum avoidance.

Proof. We prove (i) for path avoidance. The proof for strong path avoidance is similar. Fix a set of reals $\mathscr{C} \subseteq 2^\omega$ with no C-computable 1-enum for some real C. The set of reals $\mathscr{D} = \{Z \in 2^\omega : Z \text{ codes a 1-enum of } \mathscr{C}\}$ is closed and has no C-computable member. Fix any C-computable P-instance X. By path avoidance of P, there is a solution Y to X such that $Y \oplus C$ computes no member of $\mathscr{D}$ and therefore no 1-enum of $\mathscr{C}$.

We now prove (ii) for simultaneous 1-enum avoidance since by Corollary 3.5, (strong) 1-enum avoidance implies (strong) simultaneous 1-enum avoidance. Fix a countable collection of sets of reals $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ with no C-computable 1-enum for some real C. By Lemma 3.7, there exists a countable collection of sets of reals $\mathscr{D}_0, \mathscr{D}_1, \dots \subseteq 2^\omega$ such that the degrees bounding a c.b-enum of $\vec{\mathscr{C}}$ are exactly the degrees bounding a 1-enum of $\vec{\mathscr{D}}$. In particular C computes no 1-enum of $\vec{\mathscr{D}}$. Fix any C-computable P-instance X. By simultaneous 1-enum avoidance of P, there is a solution Y to X such that $Y \oplus C$ computes no 1-enum of $\vec{\mathscr{D}}$, and therefore computes no c.b-enum of $\vec{\mathscr{C}}$. □

The following lemma curiously shows that in the case of effectively closed sets of reals, the existence of a computable 1-enum is purely presentational if we are interested only in the Turing degrees.

Lemma 3.18 For every effectively closed set $\mathscr{C} \subseteq 2^\omega$, there exists an effectively closed set $\mathscr{D} \subseteq 2^\omega$ Medvedev above and Muchnik equivalent to $\mathscr{C}$ with a computable 1-enum.

Proof. Let T be a computable tree such that $[T] = \mathscr{C}$. The set $\mathscr{D} = \{\sigma^\frown Z : \sigma \in T \wedge Z \in \mathscr{C}\}$ is effectively closed, Medvedev above and Muchnik equivalent to $\mathscr{C}$. For every $\sigma \in T$, $[\sigma] \cap \mathscr{D} \neq \emptyset$, therefore we can compute a 1-enum of $\mathscr{D}$ by returning on input n a string of length n in T. □

Altough the notions of 1-enum and c.b-enum avoidance differ for arbitrary closed sets, they coincide at least in the case of effectively closed sets.

Lemma 3.19 Every c.b-enum of an effectively closed set computes a 1-enum.

Proof. We prove by induction over $k \geq 1$ that every k-enum of an effectively closed set $\mathscr{C} \subseteq 2^\omega$ computes a 1-enum. Suppose that every k-enum of $\mathscr{C}$ computes a 1-enum of $\mathscr{C}$. Let $(D_i : i < \omega)$ be a $(k+1)$-enum of $\mathscr{C}$. If for all but finitely many i, for each $\sigma \in D_i$, $\mathscr{C} \cap [\sigma] \neq \emptyset$, then we can trivially compute a 1-enum of $\mathscr{C}$ by choosing for almost all i an arbitrary member of D_i and hardcoding remaining cases. So suppose that the previous case does not hold. There are infinitely many i for which there exists a $\sigma \in D_i$ such that $\mathscr{C} \cap [\sigma] = \emptyset$. Because $\mathscr{C}$ is effectively closed, one can find an infinite subset of such i and compute a k-enum of $\mathscr{C}$ by removing all such σ. By induction hypothesis, every k-enum of $\mathscr{C}$ computes a 1-enum of $\mathscr{C}$. □

We have seen in Lemma 3.17 that if a principle P admits 1-enum avoidance, then it admits simultaenous c.b-enum avoidance as well. However, the proof transforms a countable sequence of closed sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ with no computable c.b-enum into another closed set $\mathscr{D} \subseteq 2^\omega$ admitting no computable 1-enum, but the degrees bounding a c.b-enum of $\vec{\mathscr{C}}$ and the degrees bounding a 1-enum of $\mathscr{D}$ do not necessarily coincide. Therefore, one cannot deduce from the conjunction of the sentences "P admits simultaneous c.b-enum avoidance" and "if P admits 1-enum avoidance for some closed set $\mathscr{C}$, then so does Q for $\mathscr{C}$ as well" that Q admits simultaneous c.b-enum avoidance. Thanks to Lemma 3.7, we can recover this property if we replace $\mathscr{C}$ by a countable sequence of closed sets. The next two lemmas formalize this reasoning and will be heavily used in the remainder of this paper.

Lemma 3.20 Let P and Q be two principles such that

(i) P admits c.b-enum (resp. simultaneous c.b-enum, n c.b-enum, 1-enum) avoidance,
(ii) For every closed set of reals $\mathscr{C} \subseteq 2^\omega$, if P admits path avoidance for $\mathscr{C}$ then so does Q

Then Q admits c.b-enum (resp. simultaneous c.b-enum, n c.b-enum, 1-enum) avoidance. The same statement holds if we replace avoidance by strong avoidance.

Proof. We prove it in the case of 1-enum avoidance. The cases of c.b-enum avoidance and of strong avoidances are similar. Fix a set of reals $\mathscr{C} \subseteq 2^\omega$. The set of reals

$$\mathscr{D} = \{Z \in 2^\omega : Z \text{ codes a 1-enum of } \mathscr{C}\}$$

is closed, and the degrees bounding a member of $\mathscr{D}$ are exactly the degrees bounding a 1-enum of $\mathscr{C}$. Fix a real C computing no member of $\mathscr{D}$, hence no 1-enum of $\mathscr{C}$, and consider a C-computable P-instance X. By (i), P admits 1-enum avoidance, so there is a solution Y to X such that $Y \oplus C$ computes no 1-enum of $\mathscr{C}$, hence no member of $\mathscr{D}$. Therefore P admits path avoidance for $\mathscr{D}$, thus by (ii), so does Q. Fix now a real C computing no 1-enum of $\mathscr{C}$, hence no member of $\mathscr{D}$ and consider any C-computable Q-instance X. By path avoidance of Q for $\mathscr{D}$, there is a solution Y to X such that $Y \oplus C$ computes no 1-enum of $\vec{\mathscr{D}}$, so $Y \oplus C$ computes no c.b-enum of $\mathscr{C}$. □

Lemma 3.21 Let P and Q be two principles such that

(i) P admits c.b-enum (resp. simultaneous c.b-enum, n c.b-enum) avoidance,
(ii) For every countable collection of sets of reals $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$, if P admits (strong) 1-enum avoidance for $\vec{\mathscr{C}}$ then so does Q

Then Q admits c.b-enum (resp. simultaneous c.b-enum, n c.b-enum) avoidance. The same statement holds if we replace avoidance by strong avoidance.

Proof. We prove it in the case of c.b-enum avoidance. The case of strong avoidance is similar. Fix a set of reals $\mathscr{C} \subseteq 2^\omega$. By Lemma 3.7, there exists a countable collection of sets of reals $\mathscr{D}_0, \mathscr{D}_1, \dots \subseteq 2^\omega$ such that the degrees bounding a c.b-enum of $\mathscr{C}$ are exactly the degrees bounding a 1-enum of $\vec{\mathscr{D}}$. Fix a real C computing no 1-enum of $\vec{\mathscr{D}}$, hence no c.b-enum of $\mathscr{C}$ and consider any C-computable P-instance X. By (i), P admits c.b-enum avoidance, so there is a solution Y to X such that $Y \oplus C$ computes no c.b-enum of $\mathscr{C}$. In particular $Y \oplus C$ computes no 1-enum of $\vec{\mathscr{D}}$. Therefore P admits 1-enum avoidance for $\vec{\mathscr{D}}$ and by (ii), so does Q. Fix now a real C computing no c.b-enum of $\mathscr{C}$, hence no 1-enum of $\vec{\mathscr{D}}$ and consider any C-computable Q-instance X. By 1-enum avoidance of Q for $\vec{\mathscr{D}}$, there is a solution Y to X such that $Y \oplus C$ computes no 1-enum of $\vec{\mathscr{D}}$, so $Y \oplus C$ computes no c.b-enum of $\mathscr{C}$. □

3.4. **Avoiding homogeneous closed sets**

In the case of homogeneous closed sets $\mathscr{C}$, every 1-enum of $\mathscr{C}$ computes a member of $\mathscr{C}$. Moreover, Lemma 3.19 shows that 1-enum avoidance and c.b-enum avoidance coincide in the case of effectively closed sets. Therefore, every statement which admits c.b-enum avoidance admits path avoidance for homogeneous effectively closed sets. In the case of non-effectively closed sets, the notions of 1-enum avoidance and c.b-enum avoidance differ, even when considering only homogeneous closed sets. However the notion coincide again when a homogeneous closed set is not Muchnik below the closed set of completions of Peano arithmetic, as states the following theorem.

Theorem 3.22 Let P be a set of PA degree and $\mathscr{C}$ be a homogeneous closed set with no P-computable member. There exists no computable c.b-enum of $\mathscr{C}$.

Proof. We prove by induction over k that every k-enum of $\mathscr{C}$ P-computes a member of $\mathscr{C}$. Case $k = 1$ follows from the fact that $\mathscr{C}$ is homogeneous. Suppose it holds for k and let $(D_i : i < \omega)$ be a $(k+1)$-enum of $\mathscr{C}$. If there exists an $n \in \omega$ and a $j \in \{0,1\}$ such that $\mathscr{C} \cap \{Z : Z(n) = j\} = \emptyset$ and $(\exists^\infty i)(\exists \tau \in D_i)\tau(n) = j$, then we can computably find infinitely many such i and by removing the corresponding τ, we obtain a k-enum. It then suffices to apply the induction hypothesis to deduce that $\vec{D} \oplus P$ computes a a member of $\mathscr{C}$.

So suppose there is no such n. P computes a $\{0,1\}$-valued d.n.c. function f. We f-compute a member of $\mathscr{C}$ by stages $\emptyset = \sigma_0 \prec \sigma_1 \prec \dots$ at follows. Suppose at stage s, σ_s is a string of length s and $\mathscr{C} \cap [\sigma_s] \neq \emptyset$. Let e be the Turing index of the program which on every input will search for a time t and a value $j \in \{0,1\}$ such that $\tau(s) = j$ for each $\tau \in D_t$ if such t exists, and returns $1 - j$. Otherwise the program does not halt. Set $\sigma_{s+1} = \sigma f(e)$ and go on next stage. We claim that $\mathscr{C} \cap [\sigma_{s+1}] \neq \emptyset$. Otherwise, by previous case, there will be a stage t at which $\tau(s) = j$ for each $\tau \in D_t$ and then $\Phi_e(e) \downarrow = 1 - j$. So $f(e) = 1 - \Phi_e(e) = j$. As $\vec{D}$ is a $(k+1)$-enum, there must be a $\tau \in D_t$ such that $\mathscr{C} \cap [\tau] \neq \emptyset$, and by homogeneity of $\mathscr{C}$, $\mathscr{C} \cap [\sigma_s j] \neq \emptyset$. □

The following corollary can be easily proven independently using Kolmogorov complexity. Indeed, Chaitin [6] proved that a set A is computable iff $C(A \upharpoonright n) \leq C(n) + O(1)$ for every n, where C denotes the plain Kolmogorov complexity.

Corollary 3.23 A set A is computable iff $\{A\}$ has a computable c.b-enum.

Proof. If A is computable, the enumeration of its initial segments for a computable c.b-enum of $\{A\}$. If A is not computable, by the cone avoidance basis theorem, there exists a set of PA degree which does not compute A. Therefore by Theorem 3.22 $\{A\}$ has no computable c.b-enum. □

Definition 3.24 Two disjoint sets A and B are *computably inseparable* if there exists no computable set S such that $A \subseteq S \subseteq \overline{B}$.

Theorem 3.25 There exists a 2-c.e. set A and a countable sequence of 2-c.e. sets $(B_i : i > 0)$ such that A and B_i are disjoint and computably inseparable for each $i > 0$, together with a Δ^0_2 partition $U_0 \cup U_1 \cup \dots = \omega$ such that every infinite set thin for the U's computes a separation of A and B_i for some $i > 0$.

Proof. We build our 2-c.e. set $A = \lim_s A_s$ and our countable sequence of sets $B_i = \lim_s B_{i,s}$ as well as sets $U_i = \lim_s U_{i,s}$ by a finite injury priority argument. Let $\Phi_0, \Phi_1, \dots$ be a computable enumeration of all $\{0,1\}$-valued functionals. The requirements to ensure that A and B_i are computably inseparable are the followings for each $e \in \omega$ and $i > 0$:

$$\mathscr{R}_{e,i} : \Phi_e \text{ total} \rightarrow [(\exists n \in A)\Phi_e(n) \downarrow= 0 \vee (\exists n \in B_i)\Phi_e(n) \downarrow= 1]$$

We also require that $A \cap B_i = \emptyset$ for each $i > 0$. The construction is done with a movable marker procedure. Each requirement $\mathscr{R}_{e,i}$ is given a marker $n_{e,i} = \lim_s n_{e,i,s}$ which may increase during the construction, but eventually stabilizes. The requirements are given the usual priority order ($\mathscr{R}_{e,i} < \mathscr{R}_{e',i'}$ if $\langle e,i \rangle <_{lex} \langle e',i' \rangle$). A strategy for $\mathscr{R}_{e,i}$ is *satisfied at stage* s if either $\Phi_{e,s}(n_{e,i,s}) \uparrow$ or one of the following holds

(i) $\Phi_{e,s}(n_{e,i,s}) \downarrow= 0$ and $n_{e,i,s} \in A_s$.
(ii) $\Phi_{e,s}(n_{e,i,s}) \downarrow= 1$ and $n_{e,s} \in B_{i,s}$.

A strategy for $\mathscr{R}_e$ *requires attention at stage* s if $e, i, n_{e,i,s} < s$, it is not satisfied at stage s and $\Phi_e(n_{e,i,s}) \downarrow$. At stage 0, $n_{e,i,0} = \langle e,i \rangle$, $A_0 = B_{i,0} = \emptyset$ for each $i > 0$ and $U_{j,0} = \emptyset$ for each $j \in \omega$. At stage $s+1$, if no strategy requires attention at stage s, set $n_{e,i,s+1} = n_{e,i,s}$, $A_{s+1} = A_s$, $B_{i,s+1} = B_{i,s}$, $U_{i,s+1} = U_{i,s}$ for each $e \in \omega$, $i > 0$, and $U_{0,s+1} = U_{0,s} \cup \{s\}$. Otherwise take the least such strategy (say for requirement $\mathscr{R}_{e,i}$). Set $n_{e',i',s+1} = n_{e',i',s}$ for each $\langle e',i' \rangle \leq \langle e,i \rangle$ and $n_{e',i',s+1} = \langle e',i' \rangle + s + 1$ for each $\langle e',i' \rangle > \langle e,i \rangle$.

(a) If $\Phi_{e,s}(n_{e,i,s}) \downarrow= 0$ then set $A_{s+1} = (A_s \cup \{n_{e,i,s}\}) \setminus [n_{e,i,s}+1, s]$, $B_{j,s+1} = B_{j,s} \setminus [n_{e,i,s}+1, s]$, $U_{j,s+1} = U_{j,s} \setminus [n_{e,i,s}, s]$ for each $j > 0$ and $U_{0,s+1} = U_{0,s} \cup [n_{e,i,s}, s]$.
(b) If $\Phi_{e,s}(n_{e,i,s}) \downarrow= 1$ then set $A_{s+1} = A_s \setminus [n_{e,i,s}, s]$, $B_{i,s+1} = (B_{i,s} \cup \{n_{e,i,s}\}) \setminus [n_{e,i,s}+1, s]$, $B_{j,s+1} = B_{j,s} \setminus [n_{e,i,s}+1, s]$ for each $j > 0$ such that $j \neq i$, $U_{i,s+1} = U_{i,s} \cup [n_{e,i,s}, s]$ and $U_{j,s+1} = U_{j,s} \setminus [n_{e,i,s}, s]$ for each $j \in \omega$ such that $j \neq i$.

This finishes stage $s+1$. An easy induction shows that each marker eventually stabilizes. Therefore each strategy has a finite number of injuries and is eventually satisfied. As movable markers are non-decreasing, the resulting sets A and B_i are 2-c.e. for each $i \in \omega$. Each element x changes of set U_i at most twice, so the U's are uniformly Δ^0_2. Moreover, at stage $s+1$, s enters $\bigcup_{i<s} U_{i,s}$ and an element is never removed from a set without being added to another one, so $\bigcup_i U_i = \omega$. Each step also ensures that $A \subseteq U_0$, $B_i \subseteq U_i$ for each $i > 0$ and $U_i \cap U_j = \emptyset$ for every $i \neq j$. Let $\mathscr{C}_i = \{Z : Z \text{ separates } A \text{ and } B_i\}$ for each $i > 0$.

Claim. Each $\mathscr{C}_i$ has a computable 2-enum.

Proof. On input s, return (σ_s, τ_s) where σ_s is the left-most string and τ_s the right-most string of length s such that for each $u < s$, if $u \in A_s$ then $\sigma_s(u) = \tau_s(u) = 1$ and if $u \in B_{i,s}$ then $\sigma_s(u) = \tau_s(u) = 0$. Suppose for the sake of absurd that there exists a least $s \in \omega$ such that $\mathscr{C}_i \cap [\sigma_s] = \mathscr{C}_i \cap [\tau_s] = \emptyset$. By definition, there must be two least $m_0, m_1 < s$ such that

$$m_0 \in (A \setminus A_s) \wedge \sigma_s(m_0) = 0 \vee m_0 \in (B_i \setminus B_{i,s}) \wedge \sigma_s(m_0) = 1$$

$$m_1 \in (A \setminus A_s) \wedge \tau_s(m_1) = 0 \vee m_1 \in (B_i \setminus B_{i,s}) \wedge \tau_s(m_1) = 1$$

Suppose that $m_0 \leq m_1$. The other case is symmetric. Let t_0 be the stage at which m_0 enters A or B_i. Then all the markers greater than m_0 are moved to a value greater than $t_0 > s$, contradicting the fact that $m_1 < s$ is in A or B_i. Therefore $m_0 = m_1$. As m_0 is in neither A_s nor $B_{i,s}$ and σ_n is left-most and τ_n is right-most, $\sigma_n(m_0) = 0$ and $\tau_n(m_0) = 1$ and so either m_0 is not a witness of $\mathscr{C}_i \cap [\sigma_s] = \emptyset$ or it is not a witness of $\mathscr{C}_i \cap [\tau_s] = \emptyset$ contradicting our choice of m_0. □

Claim. Every infinite set thin for the U's computes a member of some $\mathscr{C}_i$.

Proof. Let $(\sigma_s, \tau_s : s \in \omega)$ be the computable 2-enum of $\mathscr{C}_i$ built in the previous claim. It suffices to prove that $\overline{U_0} \subseteq \{s : \mathscr{C}_i \cap [\sigma_s] \neq \emptyset\}$ and $\overline{U_i} \subseteq \{s : \mathscr{C}_i \cap [\tau_s] \neq \emptyset\}$ for each $i > 0$. Indeed, if H is an infinite set thin for the U's with color 0 or some color $i > 0$, then it computes a 1-enum of $\mathscr{C}_i$, and therefore computes a member of $\mathscr{C}_i$ by homogeneity.

Let H be an infinite subset of $\overline{U_i}$ for some $i > 0$. The case $i = 0$ is similar. Fix any $x \in \overline{U_i}$ and let s be the last stage at which x enters some U_j for $j \neq i$. If no requirement caused this change, then no Turing machine with a marker smaller than x will ever halt on its marker after stage $x + 1$ and so by construction of the 2-enum, $\mathscr{C}_i \cap [\tau_x] \neq \emptyset$. So let $\mathscr{R}_{e,j}$ be the requirement causing x to enter in U_j for some $j \neq i$. Suppose for the sake of contradiction that $\mathscr{C}_i \cap [\tau_x] = \emptyset$. Then there exists a marker $n_{e',i} < x$ such that $n_{e',i} \in B_i$ and $n_{e',i}$ enters B_i at some stage $t \geq x$ and no strategy of higher priority injures $\mathscr{R}_{e',i}$ after stage t. By construction, every integer between $n_{e',i}$ and t enters U_i, so in particular x enters U_i. As every marker of smaller priorities are moved to a value greater than t, x never leaves U_i at a later stage, contradicting $x \in U_j$ for some $j \neq i$. □

This last claim finishes the proof. □

Corollary 3.26 STS^2 (resp. STS^2_k, SRT^2_2) does not admit simultaneous (resp. k, 2) 1-enum avoidance for homogeneous closed sets.

Proof. We first prove it for STS^2. Let $A, B_1, B_2, \dots$ and $U_0, U_1, \dots$ be as in Theorem 3.25 and let $\mathscr{C}_i = \{Z : Z \text{ separates } A \text{ from } B_i\}$ for each $i \geq 1$. Let $f : [\omega]^2 \to \omega$ be a stable computable function such that $\lim_s f(x, s) = i$ iff $x \in U_i$. Every f-thin set for color i is an infinite subset of $\overline{U_i}$, and therefore computes a member of any $\mathscr{C}_j$ if $i = 0$ and of $\mathscr{C}_i$ if $i > 1$. Since the $\mathscr{C}$'s are homogeneous closed sets with no computable 1-enum, STS^2 does not admit simultaneous 1-enum avoidance for homogeneous closed sets. The case of STS^2_k is similar and consists of defining $V_0, \dots, V_{k-1}$ by $V_i = U_i$ if $i < k-1$ and $V_{k-1} = \bigcup_{i \geq k-1} U_i$. SRT^2_2 is simply STS^2_2. □

4. The weakness of Ramsey's theorem for pairs

4.1. Cohesiveness

As we saw, RT^1_2 does not admit strong 1-enum avoidance. However, Lemma 3.21 shows that it can be useful to prove *relative* strong 1-enum avoidance theorems to deduce strong c.b-enum avoidance for other principles. For example, proving the theorem "if RT^1_2 admits strong 1-enum avoidance of $\vec{\mathscr{C}}$ for a countable collection of classes $\vec{\mathscr{C}}$, so does a principle P" enables to deduce strong c.b-enum avoidance of P. In all proofs of relative strong 1-enum avoidance, we could slightly modify the forcing notion to obtain a direct proof of strong c.b-enum avoidance, but this would weaken the statement as there may exist other classes for which RT^1_2 admits strong 1-enum avoidance.

Lemma 4.1 If RT^1_2 admits strong $\mathscr{C}$ avoidance for some class $\mathscr{C}$, then so does $\mathsf{RT}^1_{<\infty}$.

Proof. By induction over $n \geq 2$. Case $n = 2$ is the hypothesis. Let $f : \omega \to \{0, \dots, n\}$ be a function and C be a set computing no member of $\mathscr{C}$. Define the function $g : \omega \to \{0, \dots, n-1\}$ by $g(x) = max(f(x), n-1)$. By strong $\mathscr{C}$ avoidance of RT^1_n for $\mathscr{C}$, there exists an $i < n$ and an infinite set X such that $g(X) = i$ and $X \oplus C$ computes no member of $\mathscr{C}$. If $i < n-1$, then $f(X) = g(X)$ and X is f-homogeneous with color i. If $i = n-1$. Let $x_0 < x_1 < \dots$ be the elements of X. Define the coloring $h : \omega \to \{n-1, n\}$ by $h(m) = f(x_m)$. By strong $\mathscr{C}$ avoidance of RT^1_2 for $\mathscr{C}$, there exists a $j \in \{n-1, n\}$ and an infinite set Y such that $Y \oplus X \oplus C$ computes no member of $\mathscr{C}$ and $h(Y) = j$. Let $Z = \{x_m : m \in Y\}$. Z is $Y \oplus X$-computable, so $Z \oplus C$ computes no member of $\mathscr{C}$. $(\forall x_m \in Z) f(x_m) = h(m) = j$ so Z is f-homogeneous with color j. □

Corollary 4.2 (Liu in [30]) $\mathsf{RT}^1_{<\infty}$ admits strong c.b-enum avoidance.

Remark 4.3 From now on, we may apply freely strong $\mathscr{C}$ avoidance for $\mathscr{C}$ not only for instances of domain ω, but also on domain X for every infinite set X which computes no member of $\mathscr{C}$. It suffices to apply an X-computable bijection from X to ω to obtain an instance of domain ω and then take the reverse image of the solution by the bijection, as we did in the proof of Lemma 4.1.

Definition 4.4 (Cohesiveness) Given a sequence of sets $R_0, R_1, \dots$, an infinite set C is $\vec{R}$*-cohesive* if $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$ for every i. COH is the statement "Every uniform sequence of sets has a cohesive set".

COH is a consequence of RT^2_2 over RCA_0 [8]. It can be seen as a generalization of RT^1_2 stating the existence of a set eventually homogeneous for a countable collection of colorings of integers. COH is very useful for reducing a computable instance of RT^2_2 to a $\emptyset'$-computable instance of RT^1_2 [8]. Here, strong avoidance of RT^1_2 becomes of practical interest for proving avoidance of RT^2_2.

Since COH admits path avoidance (Corollary 2.9), then by Lemma 3.17 COH admits 1-enum avoidance. By Corollary 3.13, RT^1_2 does not admit strong simultaneous c.b-enum avoidance, and *a fortiori* does not admit strong 1-enum avoidance (Lemma 3.17). Using Lemma 3.21, the following theorem will enable us to prove that COH admits strong c.b-enum avoidance.

Theorem 4.5 If RT^1_2 admits strong path avoidance for some closed set $\mathscr{C} \subseteq \omega^\omega$, then so does COH.

Proof. Let $\mathscr{C} \subseteq \omega^\omega$ be a closed set with no C-computable member for some set C, and let $\vec{R}$ be a countable sequence of sets. Our forcing conditions are tuples (F, X) forming a Mathias condition, with the additional requirement that $\mathscr{C}$ has no $X \oplus C$-computable member. Our initial condition is $(\emptyset, \omega)$. We can easily force our satisfying sets to be infinite.

Lemma 4.6 For every condition $c = (F, X)$ and every $e, \in \omega$, there exists an extension $(\tilde{F}, \tilde{X})$ of c forcing $\Phi_e^{G \oplus C}$ not to be a member of $\mathscr{C}$.

Proof. Suppose for the sake of absurd that there is no extension of c forcing $\Phi_e^{G\oplus C}$ to be partial or $\Phi_e^{G\oplus C} \upharpoonright |\sigma| = \sigma$ for some $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C} = \emptyset$. We show how to $X \oplus C$-compute a member of $\mathscr{C}$. Define an $X \oplus C$-computable sequence of sets $F_0 \subseteq F_1 \subseteq \dots \subseteq X$ such that $\Phi_e^{(F \cup F_i)\oplus C}(i) \downarrow$ and $\forall x \in F_{i+1} \setminus F_i$, $x \geq max(F_i)$. Such a sequence exists since there is no extension of c forcing $\Phi_e^{G\oplus C}$ to be partial. We claim that the set Y defined by $Y(i) = \Phi_e^{(F\cup F_i)\oplus C}(i)$ is a member of $\mathscr{C}$. If not, then there is some i such that $\mathscr{C} \cap [Y \upharpoonright i] = \emptyset$. In this case, $(F \cup F_i, X \setminus [0, max(F_i)])$ is an extension of c forcing $\Phi_e^{G\oplus C} \upharpoonright |\sigma| = \sigma$ for some $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C} = \emptyset$, contradiction. ☐

Lemma 4.7 For every condition $c = (F, X)$ and every $e, i \in \omega$, there exists an extension $(\tilde{F}, \tilde{X})$ of c such that $\tilde{X} \subseteq R_i$ or $\tilde{X} \subseteq \overline{R_i}$.

Proof. Consider the coloring $f : X \to \{0, 1\}$ such that $f(x) = 1$ iff $x \in R_i$. By strong 1-enum avoidance of RT^1_2 for $\vec{\mathscr{C}}$, there exists an infinite subset $\tilde{X} \subseteq X$ such that $\tilde{X} \oplus C$ does not compute a 1-enum of $\vec{\mathscr{C}}$ and $\tilde{X} \subseteq R_i$ or $\tilde{X} \subseteq \overline{R_i}$. $(F, \tilde{X})$ is the desired extension. ☐

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \omega)$, where $c_s = (F_s, X_s)$. The filter $\mathscr{F}$ yields a unique infinite set $G = \bigcup_s F_s$. By Lemma 4.7, G is $\vec{R}$-cohesive and by Lemma 4.6, $\mathscr{C}$ has no $G \oplus C$-computable member. ☐

Corollary 4.8 COH admits strong c.b-enum avoidance.

Proof. By strong c.b-enum avoidance of RT^1_2, Theorem 4.5 and Lemma 3.20. ☐

Lemma 4.9 If COH admits $\mathscr{C}$ avoidance and RT^n_2 strong $\mathscr{C}$ avoidance, then RT^{n+1}_2 admits $\mathscr{C}$ avoidance.

Proof. Let C be a set computing no member of $\mathscr{C}$ and $f : [\omega]^{n+1} \to 2$ be a C-computable coloring function. For each $\sigma \in [\omega]^n$, let $R_\sigma = \{y : f(\sigma, y) = 1\}$. By $\mathscr{C}$ avoidance of COH applied to $\vec{R}$, there exists an infinite set U such that $X \oplus C$ computes no member of $\mathscr{C}$ and $\lim_{s \in X} f(\sigma, s)$ exists for each $\sigma \in [\omega]^n$. Let $\tilde{f} : [\omega]^n \to 2$ be the function defined by $f(\sigma) = \lim_{s \in X} f(\sigma, s)$. By strong $\mathscr{C}$ avoidance of RT^n_2, there exists an infinite set $Y \subseteq X$ and an $i \in \{0,1\}$ such that $(\forall \sigma \in [Y]^n)\tilde{f}(\sigma) = i = \lim_{s \in X} f(\sigma, s)$ and $Y \oplus X \oplus C$ computes no member of $\mathscr{C}$. $Y \oplus X \oplus C$ computes an infinite set H such that $f([H]^{n+1}) = i$. □

Corollary 4.10 If RT^1_2 admits strong path avoidance for some set $\mathscr{C} \subseteq \omega^\omega$, then RT^2_2 admits path avoidance for $\mathscr{C}$.

Proof. It follows from Lemma 4.9 and Theorem 4.5. □

Corollary 4.11 (Liu in [30]) RT^2_2 admits c.b-enum avoidance.

Proof. Apply Lemma 3.20 to Corollary 4.10, using strong c.b-enum avoidance of RT^1_2 (Liu [30]). □

4.2. The Ramsey-type weak König's lemma

Definition 4.12 (Ramsey-type weak König's lemma)

1. A set H is *homogeneous for* $\sigma \in 2^{<\omega}$ (resp. for $X \in 2^\omega$) if there is $v \in \{0,1\}$ such that for all $i \in H$, $\sigma(i) = v$ (resp. $X(i) = v$). Given an infinite tree T, we say that H is *homogeneous for a path through* T if the tree $T' = \{\sigma \in T : H$ is homogeneous for $\sigma\}$ is infinite.
2. RWKL is the statement "Every infinite tree T has an infinite set homogeneous for a path through T".

The complicated formulation of RWKL is for the purposes of reverse mathematics. One might think of RWKL as the statement "for every Π^0_1 class $\mathscr{D}$, there exists an infinite set which is homogeneous for a member of $\mathscr{D}$ seen as a 2-coloring". In this remaining of this section, we say that H is *$\mathscr{D}$-homogeneous* if it is homogeneous for a path through T where T is a computable tree such that $[T] = \mathscr{D}$.

RWKL has been introduced by Flood in [13] under the name RKL and proven to be a consequence of both SRT^2_2 and WKL_0 over RCA_0. Bienvenu & al. refined this result in [3] by proving that SEM implies RWKL over RCA_0. They built an ω-model of WWKL_0 not model of RWKL by constructing a computable instance of RWKL such that the measure of oracles computing a solution is null.

The following theorem will be useful for proving simultaneous c.b-enum for the Erdős-Moser theorem and n c.b-enum avoidance for the thin set theorem for pairs with $(n+1)$-colorings.

Theorem 4.13 RWKL admits simultaneous c.b-enum avoidance.

TODO

4.3. The Erdős-Moser theorem

Erdős Moser theorem provides, together with the ascending descending sequence principle, another proof of Ramsey's theorem for pairs. Due to its natural combination with ADS to obtain RT^2_2, many of the properties of EM can be deduced from known properties of RT^2_2 and ADS. For example, Kreuzer deduced in [27] the existence of a computable stable tournament with no low infinite transitive subtournament by combining the know existence of a computable instance of SRT^2_2 with no low solution and the existence for every computable instance of SADS of a low solution.

Definition 4.14 (Erdös Moser theorem)

1. A *tournament* T is an irreflexive binary relation on ω such that for all $x \neq y$, exactly one of $T(x,y)$ and $T(y,x)$ holds. T is *transitive* if for all x,y,z, if $T(x,y)$ and $T(y,z)$ hold then $T(x,z)$ holds. A tournament T is *stable* if for every x, either $(\forall^\infty y)T(x,y)$ holds or $(\forall^\infty y)T(y,x)$ holds.

2. EM is the statement "Every infinite tournament admits an infinite transitive subtournament". SEM is the restriction of EM to stable tournaments.

Bovykin and Weiermann proved in [5] that $\mathsf{EM}+\mathsf{ADS}$ is equivalent to RT^2_2 over RCA_0, equivalence still holding between their stable versions. Lerman & al. [28] proved over $\mathsf{RCA}_0+\mathsf{B}\Sigma^0_2$ that EM implies OPT and constructed ω-model of EM not model of STS(2). Wang proved in [24] that EM implies RRT^2_2 over RCA_0. The author proved in [33] that $\mathsf{RCA}_0 \vdash \mathsf{EM} \rightarrow [\mathsf{STS}(2) \vee \mathsf{COH}]$. Using Theorem 3.25, we deduce the existence of a closed set with no computable members $\mathscr{C}$ such that every model of $\mathsf{RCA}_0+\mathsf{EM}$ containing no path of $\mathscr{C}$ is also a model of COH.

4.3.1. *Enum avoidance of* EM. The question of 1-enum avoidance of EM is open. However we are able to prove it relatively to the 1-enum avoidance of the Ramsey-type weak König's lemma.

Theorem 4.15 If RWKL admits 1-enum avoidance for some countable sequence of sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ then so does EM.

Before proving Theorem 4.15, we introduce some terminology coming from the computable analysis of the Erdős-Moser theorem by Lerman, Solomon and Towsner [28].

Definition 4.16 (Minimal interval) Let T be an infinite tournament and $a,b \in T$ be such that $T(a,b)$ holds. The *interval* (a,b) is the set of all $x \in T$ such that $T(a,x)$ and $T(x,b)$ hold. Let $F \subseteq T$ be a finite transitive subtournament of T. For $a,b \in F$ such that $T(a,b)$ holds, we say that (a,b) is a *minimal interval of* F if there is no $c \in F \cap (a,b)$, i.e. no $c \in F$ such that $T(a,c)$ and $T(c,b)$ both hold.

The following notion of Erdős-Moser condition has been defined by the author in [32] and takes its inspiration from [28].

Definition 4.17 An *Erdős Moser condition* (EM condition) for an infinite tournament T is a Mathias condition (F,X) where

(a) $F \cup \{x\}$ is T-transitive for each $x \in X$
(b) X is included in a minimal T-interval of F.

The extension is the usual Mathias extension. EM conditions have good properties for tournaments as state following lemma. Given a tournament T and two sets E and F, we denote by $E \rightarrow_T F$ the formula $(\forall x \in E)(\forall y \in F)T(x,y)$ holds.

Lemma 4.18 (Patey [32]) Fix an EM condition $c=(F,X)$ for a tournament T, an infinite subset $Y \subseteq X$ and a finite T-transitive set $F_1 \subset X$ such that $F_1 < Y$ and $[F_1 \rightarrow_T Y \vee Y \rightarrow_T F_1]$. Then $d=(F\cup F_1, Y)$ is a valid extension of c.

Proof of Theorem 4.15. Since COH admits 1-enum avoidance, it suffices to prove the result for stable tournaments. Fix a countable sequence of sets $\mathscr{C}_0, \mathscr{C}_1, \dots \subseteq 2^\omega$ for which RWKL admits 1-enum avoidance. Let C be a set computing no 1-enum of $\mathscr{C}_i$ for any i, and let T be a C-computable tournament Our forcing conditions are EM conditions (F,X) for T such that the $\mathscr{C}$'s have no $X \oplus C$-computable 1-enum. A set G *satisfies* a condition (F,X) if it is T-transitive and satisfies the Mathias condition (F,X). Our initial condition is $(\emptyset, \omega)$. The first lemma shows that we can force the transitive subtournament to be infinite.

Lemma 4.19 For every condition $c=(F,X)$, there is an extension $(\tilde{F},\tilde{X})$ such that $|\tilde{F}|>|F|$.

Proof. Let $x \in X$. Since T is stable, there is some n such that $\{x\} \rightarrow_T X \cap [n,+\infty)$ or $X \cap [n,+\infty) \rightarrow_T \{x\}$. By Lemma 4.18, $d=(F\cup\{x\}, X\cap[n,+\infty))$ is a valid extension. ☐

Lemma 4.20 For every condition $c=(F,X)$ and every $e,i \in \omega$, there exists an extension $(\tilde{F},\tilde{X})$ of c forcing $\Phi_e^{G\oplus C}$ not to be a 1-enum of $\mathscr{C}_i$ where G is the forcing variable.

Proof. Suppose there exists a string $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C}_i = \emptyset$ and a finite set $E \subset X$ such that for every 2-partition $E_0 \cup E_1 = E$, there exists a finite T-transitive $F' \subseteq E_j$ for some $j < 2$ such such that $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \downarrow = \sigma$. Then consider the 2-partition $E_0 \cup E_1 = E$ defined by $E_0 = \{x \in E : (\forall^\infty s) T(x,s)\}$ and $E_1 = \{x \in E : (\forall^\infty s) T(x,s)\}$. Let $F' \subseteq E_i$ be such that $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \downarrow = \sigma$. In particular, there is some $n \in \omega$ such that $F' \rightarrow_T X \cap [n, +\infty)$ or $X \cap [n, +\infty) \rightarrow_T F'$, so by Lemma 4.18, the condition $(F \cup F', X \cap [n, +\infty))$ is a valid extension forcing $\Phi_e^{G \oplus C}$ not to be a 1-enum of $\mathscr{C}_i$.

So suppose there is no such $\sigma \in 2^{<\omega}$. For each $\sigma \in 2^{<\omega}$, let $\mathscr{T}_\sigma$ denote the collection of the sets Z such that for every finite T-transitive set $F' \subseteq Z$ or $F' \subseteq \overline{Z}$, $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \uparrow$ or $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \neq \sigma$. Note that $\mathscr{T}_\sigma$ are uniformly $\Pi_1^{0, X \oplus C}$ classes. Because the previous case does not hold, then by compactness $\mathscr{T}_\sigma \neq \emptyset$ for each σ such that $\mathscr{C} \cap [\sigma] = \emptyset$. The set $\{\sigma : \mathscr{T}_\sigma = \emptyset\}$ is $X \oplus C$-c.e. If for each $u \in \omega$, there exists a $\sigma \in 2^u$ such that $\mathscr{T}_\sigma = \emptyset$ then $X \oplus C$ computes a 1-enum of $\mathscr{C}$, contradicting our hypothesis. So there must be a u such that $\mathscr{T}_\sigma \neq \emptyset$ for each $\sigma \in 2^u$.

Thanks to 1-enum avoidance of RWKL for $\vec{\mathscr{C}}$, define a finite decreasing sequence $X = X_0 \supseteq \dots \supseteq X_{2^u - 1} = \tilde{X}$ such that for each $\sigma \in 2^u$

1. X_σ is homogeneous for a path in $\mathscr{F}_\sigma$.
2. $X_\sigma \oplus C$ computes no 1-enum of any of the $\mathscr{C}$'s.

We claim that $(F, \tilde{X})$ is an extension forcing $\Phi_e^{G \oplus C}(u) \uparrow$ or $\Phi_e^{G \oplus C}(u) \notin 2^u$. Suppose for the sake of contradiction that there exists a $\sigma \in 2^u$ and a set G satisfying $(F, \tilde{X})$ such that $\Phi_e^{G \oplus C}(u) \downarrow = \sigma$. By continuity, there exists a finite set $F' \subseteq G$ such that $\Phi_e^{(F \cup F') \oplus C}(u) \downarrow = \sigma$. The set F' is T-transitive by definition of satisfaction of $(F, \tilde{X})$. It suffices to show that $F' \subseteq Z$ or $F' \subseteq \overline{Z}$ for some $Z \in \mathscr{T}_\sigma$ to obtain a contradiction. This is immediate since $\tilde{X}$ is homogeneous for a path in $\mathscr{T}_\sigma$. ☐

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \omega)$, where $c_s = (F_s, X_s)$. The filter $\mathscr{F}$ yields a unique set $G = \bigcup_s F_s$. By definition of a condition, the set G is a transitive subtournament of T. By Lemma 4.19, G is infinite and by Lemma 4.20, $G \oplus C$ computes no 1-enum of $\mathscr{C}_i$ for any $i \in \omega$. ☐

Corollary 4.21 EM admits simultaneous c.b-enum avoidance.

Proof. Apply Lemma 3.21 to Theorem 4.15 and simultaneous c.b-enum avoidance for RWKL (Theorem 4.13). ☐

4.3.2. *Strong c.b-enum avoidance of* EM. The author proved in [32] that $\mathsf{RT}^1_2 \leq_{sc} \mathsf{EM}$. Therefore, by Lemma 2.15, EM admits neither strong 1-enum avoidance, nor strong 2 c.b-enum avoidance. Once again, we will use a relativized proof of strong 1-enum avoidance to deduce strong c.b-enum avoidance of EM.

Theorem 4.22 If RT^1_2 admits strong 1-enum avoidance for some countable sequence of classes $\mathscr{C}_0, \mathscr{C}_1, \dots$ then so does EM.

Corollary 4.23 EM admits strong c.b-enum avoidance.

Proof. Apply Lemma 3.21 to Theorem 4.22, knowing that RT^1_2 admits strong c.b-enum avoidance. ☐

We now turn to the proof of strong 1-enum avoidance of EM relative to 1-enum avoidance of RT^1_2.

Definition 4.24 A $\oplus_k$-*tournament* is a set $\vec{T} = T_0 \oplus \dots \oplus T_{k-1}$ such that each T_i is a tournament. One might think of a $\oplus_k$-tournament as a conjunction of tournaments. Thus notions over tournaments can be naturally extended to $\oplus_k$-tournaments – e.g. A set U is a *subtournament* of a $\oplus_k$-tournament $\vec{T}$ if it is a subtournament of T_i for each $i < k$ –.

Proof of Theorem 4.22. Fix a set C computing no 1-enum of $\vec{\mathscr{C}}$ for some countable collection of classes $\mathscr{C}_1, \mathscr{C}_2, \dots$ such that RT^1_2 admits strong 1-enum avoidance of $\vec{\mathscr{C}}$. Let T be an infinite tournament. Our forcing conditions are tuples $(k, F, X, \vec{U})$ such that

(a) $\vec{U}$ is a $\oplus_k$-tournament
(b) $X \oplus C$ does not compute a 1-enum of $\vec{\mathscr{C}}$
(c) (F, X) is an EM condition for each $U_i \in \vec{U}$

A condition $(m, F', X', \vec{U}')$ *extends* another condition $(k, F, X, \vec{U})$ if (F', X') Mathias extends (F, X), $m \geq k$ and $\{U_i : i < k\} \subseteq \{U'_i : i < m\}$. A set G *satisfies* a condition $(k, F, X, \vec{U})$ if it is $\vec{U}$-transitive and satisfies the Mathias condition (F, X). Our initial condition is $(1, \emptyset, \omega, T)$.

Lemma 4.25 For every condition $(k, F, X, \vec{U})$, there exists an extension $(k, \tilde{F}, \tilde{X}, \vec{U})$ such that $|\tilde{F}| > |F|$.

Proof. Take any $x \in X$. Let $f : X \to 2^k$ be the coloring defined by $f(y) = \sigma_y$ where $|\sigma_y| = k$ and for each $i < k$, $\sigma_y(i) = 1$ iff $U_i(x, y)$ holds. By strong 1-enum avoidance of $\mathsf{RT}^1_{<\infty}$ for $\vec{\mathscr{C}}$, there exists an infinite set $\tilde{X}$ and a $\sigma \in 2^k$ such that

$$(\forall i < k)(\forall y \in \tilde{X})(U_i(x, y) \text{ holds } \leftrightarrow \sigma(i) = 1)$$

and $\tilde{X} \oplus C$ does not compute a 1-enum of $\vec{\mathscr{C}}$. By Lemma 4.18, $(F \cup \{x\}, \tilde{X})$ is a valid EM extension for U_i for each $i < k$ so $(k, F \cup \{x\}, \tilde{X}, \vec{U})$ is a valid extension. □

Lemma 4.26 Fix a set C computing no 1-enum of $\vec{\mathscr{C}}$. Let X be an infinite C-computable set and $\vec{T}$ be a $\oplus_k$-tournament. For each finite subset $E \subseteq X$, there is a 2^k partition $E = \bigcup_{\sigma \in 2^k} E_\sigma$ and an infinite set $Y \subseteq X$ such that $E < Y$, $Y \oplus C$ does not compute a 1-enum of $\vec{\mathscr{C}}$ and for all $\sigma \in 2^k$ and $i < k$, if $\sigma(i) = 0$ then $E_\sigma \to_{T_i} Y$ and if $\sigma(i) = 1$ then $Y \to_{T_i} E_\sigma$.

Proof. Given a set E, define P_E to be the finite set or ordered 2^k-partitions of E, i.e.

$$P_E = \{\langle E_\sigma : \sigma \in 2^k \rangle : \bigcup_{\sigma \in 2^k} E_\sigma = E \text{ and } \sigma \neq \tau \to E_\sigma \cap E_\tau = \emptyset\}$$

Define the coloring $g : X \to P_E$ by $g(x) = \langle E^x_\sigma : \sigma \in 2^k \rangle$ where

$$E^x_\sigma = \{a \in E : (\forall i < k) T_i(a, x) \text{ holds iff } \sigma(i) = 0\}$$

By strong 1-enum avoidance of $\mathsf{RT}^1_{<\infty}$ for $\vec{\mathscr{C}}$, there exists an infinite set $Y \subseteq X$ homogeneous for g such that $X \oplus Y$ does not compute a 1-enum of $\vec{\mathscr{C}}$. Let $\langle E_\sigma : \sigma \in 2^k \rangle$ be the color. By removing finitely many elements of X, we can ensure that $E < Y$ and by definition of g, for all $\sigma \in 2^k$ and $i < k$, if $\sigma(i) = 0$ then $E_\sigma \to_{T_i} Y$ and if $\sigma(i) = 1$ then $Y \to_{T_i} E_\sigma$. □

Lemma 4.27 For every condition $(k, F, X, \vec{U})$ and every $e, i \in \omega$, there exists an extension $(m, \tilde{F}, \tilde{X}, \vec{V})$ forcing $\Phi_e^{G \oplus C}$ not to be a 1-enum of $\mathscr{C}_i$ where G is the forcing variable.

Proof. Suppose there exists a string $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C}_i = \emptyset$ and a finite set $E \subset X$ such that for each 2^k-partition $E = E_0 \cup \dots \cup E_{2^k - 1}$, there is a $j < 2^k$ and a $\vec{U}$ transitive set $F' \subseteq E_j$ such that $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \downarrow = \sigma$. Take the partition $E = E_0 \cup \dots \cup E_{2^k - 1}$ and the infinite set $\tilde{X} \subseteq X$ guaranteed by Lemma 4.26. Fix a $j < 2^k$ and an $\vec{U}$-transitive set $F' \subseteq E_j$ such that $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \downarrow = \sigma$. By Lemma 4.18, $(F \cup F', \tilde{X})$ is a valid EM condition for U_i for each $i < k$ so $(k, F \cup F', \tilde{X}, \vec{U})$ is a valid extension and forces $\Phi_e^{G \oplus C}$ not to be a 1-enum of $\mathscr{C}_i$.

So suppose there is no such $\sigma \in 2^{<\omega}$ and finite set $E \subset X$. For each $\sigma \in 2^{<\omega}$, let $\mathscr{T}_\sigma$ denote the collection of $\oplus_k$-tournaments $\vec{W}$ satisfying conditions (c) and (d) such that for each finite set $E \subset X$, there exists a 2^k-partition $E = E_0 \cup \dots \cup E_{2^k - 1}$ such that for every $j < 2^k$ and $\vec{W}$-transitive set $F' \subseteq E_j$, $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \uparrow$ or $\Phi_e^{(F \cup F') \oplus C}(|\sigma|) \neq \sigma$. Note that $\mathscr{T}_\sigma$ are uniformly $\Pi^{0, X \oplus C}_1$ classes. Because above case does not hold, $\vec{U} \in \mathscr{T}_\sigma$ for each σ such that $\mathscr{C}_i \cap [\sigma] = \emptyset$. The set $\{\sigma : \mathscr{T}_\sigma = \emptyset\}$ is $X \oplus C$-c.e.

If for each $u \in \omega$, there exists a $\sigma \in 2^u$ such that $\mathscr{T}_\sigma = \emptyset$ then $X \oplus C$ computes a 1-enum of $\mathscr{C}_i$, contradicting our hypothesis. So there must be a u such that $\mathscr{T}_\sigma \neq \emptyset$ for each $\sigma \in 2^u$.

Given a $\sigma \in 2^u$, let $\vec{V}_\sigma \in \mathscr{T}_\sigma$. Define the (non-computable) predicate $Q(E, E_0, \dots, E_{2^k-1})$ which holds iff for each $j < 2^k$ and $\vec{V}_\sigma$-transitive set $F' \subseteq E_j$, $\Phi_e^{(F \cup F') \oplus C}(u) \uparrow$ or $\Phi_e^{(F \cup F') \oplus C}(u) \neq \sigma$. For each $m \in \omega$, let $S(m)$ be the set of all 2^k-partitions $E_0 \cup \dots \cup E_{2^k-1}$ of the m first elements E of X such that $Q(E, E_0, \dots, E_{2^k-1})$ holds. By definition of $\mathscr{T}_\sigma$, $S(m)$ is non-empty for each $m \in \omega$. Moreover, if $Q(E, E_0, \dots, E_{2^k-1})$ holds then so does $Q(E \upharpoonright s, E_0 \upharpoonright s, \dots, E_{2^k-1} \upharpoonright s)$. Therefore S is an infinite finitely branching tree. Every infinite path in S is a 2^k-partition $X_0^\sigma \cup \dots \cup X_{2^k-1}^\sigma$ of X such that for every $j < 2^k$, and every $\vec{V}_\sigma$-transitive set $F' \subseteq X_j^\sigma$, $\Phi_e^{(F \cup F') \oplus C}(u) \uparrow$ or $\Phi_e^{(F \cup F') \oplus C}(u) \neq \sigma$. By strong 1-enum avoidance of $\mathsf{RT}^1_{<\infty}$ for $\vec{\mathscr{C}}$, there exists a $j < 2^k$ and an infinite set $X_\sigma \subseteq X_j$ such that $X_\sigma \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$.

By repeating the process for each $\sigma \in 2^u$, we obtain an infinite set $\tilde{X} \subseteq X$ such that $\tilde{X} \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$ and for every $(\bigoplus_{\sigma \in 2^u} \vec{V}_\sigma)$-transitive $F' \subseteq \tilde{X}$, $\Phi_e^{(F \cup F') \oplus C}(u) \uparrow$ or $\Phi_e^{(F \cup F') \oplus C}(u) \downarrow \notin 2^u$. $((2^u + 1)k, F, \tilde{X}, \vec{U} \bigoplus_{\sigma \in 2^u} \vec{V}_\sigma)$ is the desired extension. □

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(1, \emptyset, \omega, T)$, where $c_s = (k_s, F_s, X_s, \vec{U}_s)$. The filter $\mathscr{F}$ yields a unique set $G = \bigcup_s F_s$. By definition of a forcing condition, the set G is a transitive subtournament of T. By Lemma 4.25, G is infinite and by Lemma 4.27, $G \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$. □

4.3.3. *Negative strong cone avoidance of* SADS. EM together with ADS leads to a second proof of RT^2_2 [5]. A function $f : [\omega]^2 \to 2$ can be seen as a tournament, and every transitive subtournament is a linear order. EM and ADS are both incomparable consequences of RT^2_2 [28, 19].

Corollary 4.28 SADS does not admit strong cone avoidance.

Proof. As SRT^2_2 does not admit strong cone avoidance, there exists a stable coloring of pairs $f : [\omega]^2 \to \omega$, a set C and a non C-computable set A such that for every infinite set H homogeneous for f, $H \oplus C$ computes A. By strong cone avoidance of EM, seeing f as a tournament T such that $T(a, b)$ holds iff $f(a, b) = 1$, there exists an infinite subtournament U such that $U \oplus C$ does not compute A. Seeing U as an ordered set $\{a_0 < a_1 < \dots\}$, we can define a stable linear order L over ω such that $x <_L y$ iff $T(a_x, a_y)$ holds. For every infinite ascending or descending sequence S of L, $S \oplus U$ computes an infinite set H homogeneous for f, hence $S \oplus U \oplus C$ computes A. □

Wang gave a direct proof of Corollary 4.28 by constructing a stable linear order such that every infinite ascending or descending sequence computes the halting set.

Direct proof of Corollary 4.28 by Wang. Let $m : \omega \to \omega$ denote the modulus function of the halting set, i.e. $m(n) = \mu s(K_s \upharpoonright n = K \upharpoonright n)$. We can assume that m is strictly increasing. Define the $\emptyset'$-computable linear order L such that for each $x <_\omega y$, $x <_L y$ iff there is an n such that $x <_\omega m(n) \leq_\omega y$.

We claim that L is transitive and of order type ω. Suppose $x <_L y <_L z$.

- If $x <_\omega y$, then there is an n such that $x <_\omega m(n) \leq_\omega y$. If $y <_\omega z$ then n witnesses $x <_\omega m(n) \leq_\omega z$ so $x <_L z$. If $z <_\omega y$ then there is no n' such that $z <_\omega m(n') \leq_\omega y$, hence $m(n) \leq_\omega z$. So $x <_\omega m(n) \leq_\omega z$ and $x <_L z$.
- If $x >_\omega y$, then there is no n such that $y <_\omega m(n) \leq_\omega x$. If $y <_\omega z$ then there is an n such that $y <_\omega m(n) \leq_\omega z$, hence $x <_\omega m(n) \leq_\omega z$ and so $x <_L z$. If $z <_\omega y$ then $z <_\omega x$ and there is no n' such that $z <_\omega m(n') \leq_\omega y$ hence there is no n such that $z <_\omega m(n) \leq_\omega x$ and by definition $x <_L z$.

As m is strictly increasing, for each x, there is an n such that $m(n) >_\omega x$, so for each $y >_\omega m(n)$, $y >_L x$. Hence L is of order type ω. Let $A = \{a_0 < a_1 < \dots\}$ be an ascending sequence. Its principal function p_A defined by $p_A(i) = a_i$ majorizes m, hence computes the halting set. □

4.4. **The rainbow Ramsey theorem for pairs**

All the principles considered until now are consequences of RT^2_2. However strong 1-enum avoidance also holds for consequences of RT^n_2 which are not consequences of RT^{n-1}_2 for arbitrary n.

Definition 4.29 (Rainbow Ramsey theorem) A function $f : [\omega]^n \to \omega$ is k-bounded if $\left|f^{-1}(c)\right| \leq k$ for each $c \in \omega$. Let $f : [\omega]^n \to \omega$ be a k-bounded coloring function. A set R is a *rainbow for f* if f is injective on $[R]^n$. RRT^n_k is the statement "every k-bounded function $f : [\omega]^n \to \omega$ has an infinite rainbow for f". RRT is the statement $(\forall n)\,\mathsf{RRT}^n_2$.

RRT^n_2 is a strict consequence of RT^n_2, i.e. RRT^n_2 does not imply RT^n_2 [43, 9]. Although RRT^n_2 satisfies the same bounds as those proven by Jockush for RT^n_2 in [22], RRT^n_2 is combinatorially weak. The author proved in [33] that $\mathsf{RCA}_0 \vdash \mathsf{RRT}^3_2 \to \mathsf{STS}(2)$. Therefore RRT^3_2 does not admit 2 c.b-enum avoidance and *a fortiori* does not admit 1-enum avoidance.

4.4.1. *Strong 1-enum avoidance of* RRT^1_2. The rainbow Ramsey theorem for singletons is a principle combinatorially equivalent to DNR, a basic statement of computability theory.

Definition 4.30 (Diagonally non-computable) A function f is *diagonally non-computable* (d.n.c.) relative to a set X if $(\forall e) f(e) \neq \Phi^X_e(e)$. DNR is the statement "For every X, there exists a function d.n.c. relative to X". $\mathsf{DNR}[\emptyset']$ is the statement "For every X, there exists a function d.n.c. relative to X'".

DNR is a very weak principle, proven to be a consequence of SRT^2_2 [18], WWKL_0 [1], SEM [33], RWKL [13] and STS(2) [35] over RCA_0. Hirschfeldt et al. constructed in [18] an ω-model of COH not model of DNR, Ambos & al. in [1] an ω-model of DNR not model of WWKL_0. Finally Bienvenu & al. proved in [4] the existence of an ω-model of DNR model of neither RWKL nor AMT. We start by proving strong 1-enum avoidance of RRT^1_2. Strong 1-enum avoidance of DNR follows from its strong computable reduction to RRT^1_2.

Theorem 4.31 RRT^1_2 admits strong 1-enum avoidance.

Definition 4.32 A *2-bounded $\oplus_k$-function* is a set $\vec{f} = f_0 \oplus \dots \oplus f_{k-1}$ such that each f_i is a coding of a 2-bounded coloring over integers. One might think of an 2-bounded $\oplus_k$-function as a conjunction of 2-bounded functions. Thus notions over functions can be naturally extended to 2-bounded $\oplus_k$-functions: – e.g. A set F is a *rainbow* for a 2-bounded $\oplus_k$-function $\vec{f}$ if it is an f_i-rainbow for each $i < k$ –.

Proof of Theorem 4.31. Let C be a set computing no 1-enum of $\mathscr{C}$ for some class $\mathscr{C} \subseteq 2^\omega$ and $f : \omega \to \omega$ be a 2-bounded coloring. Our forcing conditions are tuples $(k, F, X, \vec{g})$ such that

(a) $\vec{g}$ is a normal 2-bounded $\oplus_k$-function
(b) X is an infinite set such that $F < X$ and $X \oplus C$ computes no 1-enum of $\mathscr{C}$
(c) F is a finite $\vec{g}$-rainbow.

A set G *satisfies* a condition $(k, F, X, \vec{g})$ if it satisfies the Mathias condition (F, X) and G if g_i-free for each $i < k$. Our initial condition is $(1, \emptyset, \omega, f)$. A condition $(m, F', X', \vec{g}')$ *extends* another condition $(k, F, X, \vec{g})$ if (F', X') Mathias extends (F, X), $m \geq k$ and $(\forall i < k) g_i = g'_i$.

Lemma 4.33 For every condition $(k, F, X, \vec{g})$ there exists an extension $(k, \tilde{F}, \tilde{X}, \vec{g})$ such that $|H| > |F|$.

Proof. Take $x \in X \setminus \bigcup_i g_i(F)$. $F \cup \{x\}$ is a $\vec{g}$-rainbow, hence $(k, F \cup \{x\}, X \setminus [0, x], \vec{g})$ is the desired extension. □

Lemma 4.34 For every condition $(k, F, X, \vec{g})$ and every $e \in \omega$, there exists an extension $(m, \tilde{F}, \tilde{X}, \vec{h})$ forcing $\Phi^{G \oplus C}_e$ not to be a 1-enum of $\mathscr{C}$, where G is the forcing variable.

Proof. Suppose there exists a $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C} = \emptyset$ and a finite set $F' \subseteq X$ such that $F \cup F'$ is g_i-free for each $i < k$ and $\Phi^{(F_0 \cup F') \oplus C}_e(|\sigma|) \downarrow = \sigma$. $(k, F \cup F', X \setminus [0, max(F')], \vec{g})$ is a condition forcing $\Phi^{G \oplus C}_e$ not to be a 1-enum of $\mathscr{C}$.

Suppose there is no such finite set $F' \subset X$. For each $\sigma \in 2^{<\omega}$, let $\mathscr{F}_\sigma$ denote the collection of 2-bounded $\oplus_k$-functions $\vec{h}$ such that F is $\vec{h}$-free and for each finite set $F' \subset X$ such that $F \cup F'$ is h_j-free for each $j < k$, either $\Phi_e^{(F\cup F')\oplus C}(|\sigma|) \uparrow$ or $\Phi_e^{(F\cup F')\oplus C}(|\sigma|) \neq \sigma$. Note that $\mathscr{F}_\sigma$ are uniformly $\Pi_1^{0,X\oplus C}$ classes. Because above case does not hold, $\vec{g} \in \mathscr{F}_\sigma$ for each σ such that $\mathscr{C} \cap [\sigma] = \emptyset$. The set $\{\sigma : \mathscr{F}_\sigma = \emptyset\}$ is $X \oplus C$-c.e. If for each $u \in \omega$ there exists a $\sigma \in 2^u$ such that $\mathscr{F}_\sigma = \emptyset$ then $X \oplus C$ computes a 1-enum of $\mathscr{C}$, contradicting our hypothesis. So there must be an $u \in \omega$ such that $\mathscr{F}_\sigma \neq \emptyset$ for each $\sigma \in 2^u$.

For each $\sigma \in 2^u$, let $\vec{h}_\sigma \in \mathscr{F}_\sigma$. $((2^u+1)k, F, X, \vec{g} \bigoplus_{\sigma\in 2^u} \vec{h}_\sigma)$ is a condition forcing $\Phi_e^{G\oplus C}(u) \uparrow$ or $\Phi_e^{G\oplus C}(u) \downarrow \notin 2^u$. □

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(1, \emptyset, \omega, f)$, where $c_s = (k_s, F_s, X_s, \vec{g}_s)$. The filter $\mathscr{F}$ yields a unique set $G = \bigcup_s F_s$. By Lemma 4.33, the set G is infinite. By definition of a forcing condition, G is an f-rainbow, and by Lemma 4.34, $G \oplus C$ computes no 1-enum of $\mathscr{C}$. □

Lemma 4.35 $\mathsf{DNR} \leq_{sc} \mathsf{RRT}^1_2[\emptyset']$

Proof. Fix a set X and a canonical enumeration of all finite sets $(D_i : i \in \omega)$. We construct a 2-bounded coloring $f : \omega \to \omega$ such that for every $e \in \omega$, if $\Phi_e^X(e)\downarrow$ and $D_{\Phi_e^X(e)}$ has at least $2(e+1)$ elements, then either $D_{\Phi_e^X(e)} \cap [0,e] \neq \emptyset$ or it is not an f-rainbow. We first show how, given an infinite f-rainbow H, we compute a function g d.n.c. relative to X. For every $e \in \omega$, $g(e) = i$ where D_i are the first $2(e+1)$ elements of H greater than e. Suppose for the sake of absurd that $g(e) = \Phi_e^X(e)$ for some e. Then $D_{\Phi_e^X(e)}$ is not an f-rainbow and therefore $D_{\Phi_e^X(e)} \neq D_{g(e)}$. Contradiction.

We now detail the construction of f by stages. At stage 0, $dom(f_0) = \emptyset$. Suppose that at stage s, $[0,s) \subseteq dom(f_s)$ and $|dom(f_s)| \leq 3s$. If $\Phi_s^X(s)\downarrow$ and $|D_{\Phi_s^X(s)}| \geq 2(s+1)$ and has no element before s, then by cardinality, there exists $u, v \in D_{\Phi_s^X(s)} \setminus dom(f_s)$. Set $f(u) = f(v)$ and give a fresh color to $f(s)$ if $s \notin dom(f_s)$. Then go to stage $s+1$. $f = \bigcup_s f_s$ is the desired coloring. Note that f is X'-computable. □

Corollary 4.36 DNR admits strong 1-enum avoidance.

Proof. By Theorem 4.31, Lemma 4.35 and Lemma 2.15. □

4.4.2. *Enum avoidance of* RRT^2_2. Miller proved [31] that RRT^2_2 and $\mathsf{DNR}[\emptyset']$ are computably equivalent. By Corollary 4.36, $\mathsf{DNR}[\emptyset']$ admits strong 1-enum avoidance, so *a fortiori* 1-enum avoidance, and we deduce directly by Lemma 2.15 that RRT^2_2 admits 1-enum avoidance. We provide another proof of 1-enum avoidance of RRT^2_2 without using the characterization by Miller.

Theorem 4.37 RRT^2_2 admits 1-enum avoidance.

The proof of 1-enum avoidance of RRT^2_2 follows a very standard technique in coloring principles: we use avoidance of COH to reduce the problem of avoidance of a coloring of n-tuples into strong avoidance of a coloring of $(n+1)$-tuples. This is where the notion of strong avoidance begins to have a direct practical application. Before proving Theorem 4.37, we need to focus on a canonical class of coloring over tuples.

Definition 4.38 A coloring $f : [\omega]^{n+1} \to \omega$ is *normal* if $f(\sigma, a) \neq f(\tau, b)$ for each $\sigma, \tau \in [\omega]^n$, whenever $a \neq b$.

Wang proved in [41] that for every 2-bounded coloring $f : [\omega]^n \to \omega$, every f-random computes an infinite set X on which f is normal. The author refined in [33] this result by proving that every function d.n.c. relative to f computes such a set.

Lemma 4.39 Let $f : [\omega]^2 \to \omega$ be a 2-bounded coloring. Every function d.n.c. relative to f computes a set H such that f is normal on H.

Proof. By [26], every function d.n.c. relative to f computes a function g such that if $|W_e^f| \leq n$ then $g(e,n) \notin W_e^f$. Given a finite f-normal set F, there exists at most $\binom{|F|}{r}$ elements x such that $F \cup \{x\}$ is not f-normal. We can define an infinite f-normal set H by stages. $H_0 = \emptyset$. Given a finite f-normal set H_s of cardinal s, set $H_{s+1} = H_s \cup \{g(e, \binom{s}{r})\}$ where e is a Turing index such that $W_e^f = \{x : H_s \cup \{x\} \text{ is not } f\text{-normal}\}$. □

Corollary 4.40 If RRT^n_2 for normal colorings admits (strong) 1-enum avoidance for some countable collection of sets $\mathscr{C}_0, \mathscr{C}_1, \dots$, then so does RRT^n_2.

Proof. We prove it in the case of strong 1-enum avoidance for $\vec{\mathscr{C}}$. The non-strong case is similar. Fix a set C computing no 1-enum of $\vec{\mathscr{C}}$. Let $f : [\omega]^n \to \omega$ be a 2-bounded coloring. By Lemma 4.39, every function d.n.c. relative to f computes a set H such that f is normal on H. By Corollary 4.36 and Lemma 3.4, DNR admits strong 1-enum avoidance for $\vec{\mathscr{C}}$, so we can assume that $H \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$. By strong 1-enum avoidance of RRT^n_2 for $\vec{\mathscr{C}}$ and normal colorings, there exists an infinite set $G \subseteq H$ such that $G \oplus H \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$. □

The following lemma has been proven by Wang [41] for $n = 2$ in the context of PA avoidance. The same proof holds for every notion of avoidance.

Lemma 4.41 If RRT^n_2 admits strong $\mathscr{C}$ avoidance and COH admits $\mathscr{C}$ avoidance, then RRT^{n+1}_2 admits $\mathscr{C}$ avoidance.

Proof. Fix a set C computing no member of $\mathscr{C}$ and $f : [\omega]^{n+1} \to \omega$ a 2-bounded coloring. By Corollary 4.40, we can assume w.l.o.g. that f is normal. Consider the following sets:

$$R_{\sigma,\tau} = \{s : f(\sigma,s) = f(\tau,s)\}$$

By $\mathscr{C}$-avoidance of COH applied to $\vec{R}$, there exists an infinite $\vec{R}$-cohesive set G such that $G \oplus C$ computes no member of $\mathscr{C}$. Define $\tilde{f} : [\omega]^n \to \omega$ as

$$\tilde{f}(\sigma) = min(\{\rho <_{lex} \sigma : \lim_{s \in G} f(\sigma,s) = f(\tau,s)\})$$

$\tilde{f}$ is 2-bounded, and by strong $\mathscr{C}$ avoidance of RRT^n_2, there exists an infinite $\tilde{f}$-rainbow H such that $H \oplus G \oplus C$ computes no member of $\mathscr{C}$. $H \oplus G$ computes an infinite f-rainbow. □

Proof of Theorem 4.37. Immediate by Corollary 4.40, Lemma 4.41 and Theorem 4.31. □

5. The weakness of the thin set and free set theorems

Ramsey's hierarchy is known to collapse at level 3 in reverse mathematics. Indeed, RT^n_2 is equivalent to ACA_0 over RCA_0 for every $n \geq 3$ (see Simpson [40]). In this section, we study some combinatorial consequences of Ramsey's theorem which happen to be computationally weak, in that the whole hierarchy is strictly below ACA_0 (see Wang [43]).

5.1. **The thin set theorem**

The thin set theorem is a natural weakening of Ramsey's theorem, in which we allow more than one color in the solution. It turns out that allowing sufficiently many colors in the output changes radically the computational power of the resulting principle, which does not even imply stable Ramsey's theorem for pairs over RCA_0 (see Patey [34]).

Definition 5.1 (Thin set theorem) Given a coloring $f : [\omega]^n \to k$ (resp. $f : [\omega]^n \to \omega$), an infinite set H is *f-thin* if $|f([H]^n)| \leq k-1$ (resp. $f([H]^n) \neq \omega$). For every $n \geq 1$ and $k \geq 2$, TS^n_k is the statement "Every coloring $f : [\omega]^n \to k$ has an f-thin set" and TS^n is the statement "Every coloring $f : [\omega]^n \to \omega$ has an f-thin set". STS^n_k is the restriction of TS^n_k to stable colorings. TS is the statement $(\forall n)\, \mathsf{TS}^n$.

The thin set theorem for unbounded colorings has been introduced by Friedman [14, 16], together with the free set theorem. Dorais, Dzhafarov, Hirst, Mileti and Shafer [10] proved that $\mathsf{TS}^{n+2}_{2^n}$ for

$n > 0$ is equivalent to ACA_0, hence does not admit strong c.b-enum avoidance. On the other hand, Wang proved in [43] that TS^n_d admits strong cone avoidance for sufficiently large d's. For every $d \geq 2$, TS^2_d implies TS^2 over RCA_0 and therefore does not admit 1-enum avoidance. We now develop a framework to prove that TS^n_d admits strong c.b-enum avoidance for every n and sufficiently large d's.

5.1.1. *A framework of pseudo partitions.* We need to extend the notion of k-partition of the integers to colorings over arbitrary tuples. The forcing in Liu's theorem involved Π^0_1 classes of ordered k-partitions of ω. Those partitions correspond to the sets which are simultaneously homogeneous for a finite number of 2-colorings of the integers. For example, three functions $g_0, g_1, g_2 : \omega \to 2$ induce the 6-partition

$$X^{g_0}_0 \cap X^{g_1}_0, X^{g_0}_0 \cap X^{g_2}_0, X^{g_1}_0 \cap X^{g_2}_0, X^{g_0}_1 \cap X^{g_1}_1, X^{g_0}_1 \cap X^{g_2}_1, X^{g_1}_1 \cap X^{g_2}_1$$

where X^g_i is the set of the integers x such that $g(x) = i$.

In our case, we are not manipulating colorings of integers but of tuples of integers. The sets homogeneous for a function $g : [\omega]^n \to k$ do not form a partition of the integers. This is why have to make explicit the formulas expressing the homogeneity constraints.

Definition 5.2 (Coloring formula) Fix some $d \geq 1$ and some finite set S.

1. A *coloring d-atom* over S is a pair (g, J) (written $g[J]$) where g is a function symbol and $J \subset S$ is a set of size d. A *coloring d-formula* over S is a (possibly empty) conjunction of coloring d-atoms over S. We denote by ε the empty conjunction of coloring d-atoms.
2. A *valuation* of a set of coloring d-formulas over S with function symbols $g_0, \dots, g_{t-1}$ is a function π with $\operatorname{dom}(\pi) \supseteq \{g_0, \dots, g_{t-1}\}$ and such that for every $g \in \operatorname{dom}(\pi)$, $\pi(g)$ is a finite set $J \subset S$ of size d.
3. A valuation π *satisfies* a coloring d-formula $\varphi = g_0[J_0] \wedge \dots \wedge g_{t-1}[J_{t-1}]$ (written $\pi \models \varphi$) if $\pi(g_i) = J_j$ for each $i < t$.
4. A *pseudo k-partition* of coloring d-formulas is an ordered k-set of coloring d-formulas $(\varphi_\nu : \nu < k)$ such that for every valuation π, $\pi \models \varphi_\nu$ for some $\nu < k$.

In particular, the singleton $\{\varepsilon\}$ is trivially a pseudo 1-partition. Given a coloring formula $\varphi = g_0[J_0] \wedge \dots \wedge g_k[J_k]$, we write $\operatorname{dom}(\varphi)$ for the set $\{g_0, \dots, g_k\}$. The domain of a pseudo k-partition is the union of the domain of its coloring d-formulas. We now prove some closure properties.

Lemma 5.3 For every pseudo k-partition of coloring d-formulas $\vec{\varphi} = (\varphi_\nu : \nu < k)$ over a finite set S, every $\mu < k$ and every function symbol g, the set $\vec{\psi} = (\varphi_\nu : \nu \neq \mu) \cup (\varphi_\mu \wedge g[I] : I \subseteq S \wedge |I| = d)$ is a pseudo $(k + \binom{|S|}{d} - 1)$-partition of coloring d-formulas.

Proof. Fix some valuation π with $\operatorname{dom}(\pi) \supseteq \operatorname{dom}(\vec{\varphi}) \cup \{g\}$. As $(\varphi_\nu : \nu < k)$ is a pseudo k-partition, there exists a $\nu < k$ such that $\pi \models \varphi_\nu$. If $\mu \neq \nu$, then we are done since $\varphi_\nu \in \vec{\psi}$. If $\mu = \nu$, then $\pi \models \varphi_\mu \wedge g[\pi(g)]$ and we are also done since $\varphi_\mu \wedge g[\pi(g)] \in \vec{\psi}$. □

We now need to redefine a few notions introduced by Liu in [30]. In the following, a *k-cover* of a set X is a k-tuple of sets $X_0, \dots, X_{k-1}$ such that $X_0 \cup \dots \cup X_{k-1} = X$. We do not require the sets X_i to be pairwise disjoint.

Definition 5.4 Fix some integers k and q.

1. A *k-supporter* $\vec{\mathcal{K}}$ of $\{1, \dots, q\}$ is k-tuple $(\mathcal{K}_\nu : \nu < k)$ where $\mathcal{K}_\nu = \{K_{\nu,i} : i < q_\nu\}$ such that each $K_{\nu,i}$ is a subset of $\{1, \dots q\}$ and for every ordered k-cover $(P_\nu : \nu < k)$ of $\{1, \dots, q\}$, there exists some $\mathcal{K}_\nu$ and some $K_{\nu,i} \in \mathcal{K}_\nu$ such that $\mathcal{K}_{\nu,i} \subseteq P_\nu$.
2. A sequence of q clopen sets $V^{(1)}, \dots, V^{(q)}$ is *k-disperse* if for every ordered k-cover $(P_\nu : \nu < k)$ of $\{1, \dots, q\}$, there exists a $\nu < k$ such that $\bigcap_{i \in P_\nu} V^{(i)} = \emptyset$.

3. Given q pseudo k-partitions of coloring d-formulas $\vec{\varphi}^1 = (\varphi^1_\nu : \nu < k), \dots, \vec{\varphi}^q = (\varphi^q_\nu : \nu < k)$ and a k-supporter $\vec{\mathscr{K}} = (\mathscr{K}_\nu : \nu < k)$ of $\{1,\dots,q\}$, let

$$Cross(\vec{\varphi}^1, \dots, \vec{\varphi}^q, \vec{\mathscr{K}}) = \left\{ \bigwedge_{i \in K_{\nu,j}} \varphi^i_\nu : K_{\nu,j} \in \mathscr{K}_\nu, \nu < k \right\}$$

Lemma 5.5 Let $\vec{\varphi}^1 = (\varphi^1_\nu : \nu < k), \dots, \vec{\varphi}^q = (\varphi^q_\nu : \nu < k)$ be q pseudo k-partitions of coloring d-formulas, let $\vec{\mathscr{K}} = (\mathscr{K}_\nu : \nu < k)$ be a k-supporter of $\{1,\dots,q\}$ and let $K' = \sum_{\nu<k} |\mathscr{K}_\nu|$. Then $Cross(\vec{\varphi}^1, \dots, \vec{\varphi}^q, \vec{\mathscr{K}})$ is a pseudo K'-partition of coloring d-formulas.

Proof. Fix a valuation π with $\text{dom}(\pi) \supseteq \bigcup_i \text{dom}(\vec{\varphi}^i)$. For every $i \in (0,q]$, since $(\varphi^i_\nu : \nu < k)$ is a pseudo k-partition of coloring d-formulas, there is some $\nu_i < k$ such that $\pi \models \varphi^i_{\nu_i}$. This induces an ordered k-partition $(P_\nu : \nu < k)$ of $\{1,\dots,q\}$ where $P_\nu = \{i \in \{1,\dots,q\} : \nu_i = \nu\}$. By definition of being a k-supporter of $\{1,\dots,q\}$, there exists some $\mathscr{K}_\nu$ and some $\mathscr{K}_{\nu,j} \in \mathscr{K}_\nu$ such that $\mathscr{K}_{\nu,j} \subseteq P_\nu$. By definition of P_ν and of the cross operator,

$$\bigwedge_{i \in \mathscr{K}_{\nu,j}} \varphi^i_{\nu_i} \text{ is the same as } \bigwedge_{i \in \mathscr{K}_{\nu,j}} \varphi^i_\nu \text{ which is in } Cross(\vec{\varphi}^1, \dots, \vec{\varphi}^q, \vec{\mathscr{K}})$$

and $\pi \models \bigwedge_{i \in \mathscr{K}_{\nu,j}} \varphi^i_{\nu_i}$. Hence $\pi \models \psi$ for some $\psi \in Cross(\vec{\varphi}^1, \dots, \vec{\varphi}^q, \vec{\mathscr{K}})$. □

A particular way of constructing k-supporters consists of using a k-disperse sequence of clopen sets.

Lemma 5.6 Let $(e_\nu : \nu < k)$ be k natural numbers and let $k' = \sum_{\nu<k} e_\nu$. If $V^{(1)}, \dots, V^{(q)}$ is a k'-disperse sequence of clopen sets, then $\vec{\mathscr{K}} = \{\mathscr{K}_\nu : \nu < k\}$ where

$$\mathscr{K}_\nu = \{K \subseteq \{1,\dots,q\} : \{V^{(i)}\}_{i \in K} \text{ is an } e_\nu\text{-disperse sequence}\}$$

is a k-supporter of $\{1,\dots,q\}$.

Proof. Suppose for the sake of absurd that there exists a k-cover $(P_\nu : \nu < k)$ of $\{1,\dots,q\}$ such that for all $\nu < k$, $P_\nu \notin K_\nu$, i.e., $\{V^{(i)}\}_{i \in P_\nu}$ is not an e_ν-disperse sequence of clopen sets. Then for each $\nu < k$, there exists an e_ν-cover $(P_{\nu,j} : j < e_\nu)$ of P_ν such that $(\forall j < e_\nu)(\bigcap_{i \in P_{\nu,j}} V^{(i)} \neq \emptyset)$. However then $(P_{\nu,j} : j < e_\nu, \nu < k)$ is a k'-cover of $\{1,\dots,q\}$ that contradicts the assumption that $V^{(1)}, \dots, V^{(q)}$ is a k'-disperse sequence of clopen sets. □

The following lemma is a pure application of the pigeonhole principle.

Lemma 5.7 Let $\vec{\psi}^0 = (\varphi^0_\nu : \nu < k), \dots, \vec{\psi}^k = (\varphi^k_\nu : \nu < k)$ be $k+1$ pseudo k-partitions of coloring d-formulas. The set $\vec{\psi} = \{\varphi^i_\nu \wedge \varphi^j_\nu : i < j \leq k, \nu < k\}$ is a pseudo $(k\binom{k+1}{2})$-partition of coloring d-formulas.

Proof. First, notice that $\vec{\psi} = Cross(\vec{\varphi}^0, \dots, \vec{\varphi}^k, \vec{\mathscr{K}})$, where $\vec{\mathscr{K}} = \{\mathscr{K}_\nu : \nu < k\}$ is defined by

$$\mathscr{K}_\nu = \{\{i,j\} : i < j \leq k\}$$

Thanks to Lemma 5.5, it suffices to prove that $\vec{\mathscr{K}}$ is a k-supporter of $\{0,\dots,k\}$. Fix some k-cover $(P_\nu : \nu < k)$ of $\{0,\dots,k\}$. For each $i \leq k$, let $\nu_i < k$ be such that $i \in P_{\nu_i}$. By the pigeonhole principle, there are some $i < j \leq k$ such that $\nu_i = \nu_j$. Hence $\{i,j\} \subseteq P_{\nu_i}$. Since $\{i,j\} \in \mathscr{K}_{\nu_i}$, we conclude. □

Given a set $\mathscr{C} \subseteq 2^\omega$ and some $n \in \omega$, define

$$C_n = \{\rho \in 2^n : [\rho] \cap \mathscr{C} \neq \emptyset\}$$

Lemma 5.8 . For every set D computing no c.b-enum of $\mathscr{C}$ and every $\Sigma^{0,D}_1$ formula $\varphi(V)$ where V is a clopen variable, one of the following must hold.

1. $\varphi(C_n)$ holds for some $n \in \omega$.

2. For every $k \in \omega$, there exists a k-disperse sequence of clopen sets $V^{(1)}, \dots, V^{(m)}$ such that for every $i = 1, \dots, m$, $\varphi(V^{(i)})$ does not hold.

Proof. Define the following D-c.e. set.

$$E = \{W \subseteq 2^{<\omega} : (\forall \rho, \sigma \in W)|\rho| = |\sigma| \wedge \varphi(W)\}$$

Suppose case 1 does not hold. In other words $C_n \notin E$ for every $n \in \omega$. We prove that for every $k \in \omega$ and almost every $n \in \omega$, the following is a k-disperse:

$$\mathscr{W}_n = \{W \subseteq 2^{<\omega} : (\forall \rho \in W)|\rho| = n \wedge W \notin E\}$$

Note that $\mathscr{W}_n$ is co-D-c.e. uniformly in n. Fix some $k \in \omega$. Let $\mathscr{W}_{n,t}$ denote $\mathscr{W}_n$ at stage t. We have $\mathscr{W}_{n,t+1} \subseteq \mathscr{W}_{n,t}$. Therefore if there exists a k-cover $(P_\nu : \nu < k)$ of $\mathscr{W}_n$ such that $(\forall \nu < k)\bigcap_{W \in P_\nu} W \neq \emptyset$, then this cover can be found in a finite amount of time. Furthermore $C_n \in P_\nu$ for some $\nu < k$, so

$$(\forall \rho \in \bigcap_{W \in P_\nu} W)[\rho] \cap \mathscr{C} \neq \emptyset$$

It follows that if there exists infinitely many n such that such a k-cover exists, we can D-computably find infinitely of them and define the D-computable enumeration h which on input n returns $(\rho_\nu : \nu < k)$ such that there exists some $t, m \geq n$ and a k-cover $(P_\nu : \nu < k)$ of $\mathscr{W}_{m,t}$ such that $(\forall \nu < k)\bigcap_{W \in P_\nu} W \neq \emptyset$ and ρ_ν is the leftmost string in $\bigcap_{W \in P_\nu} W$. This contradicts the fact that D computes no c.b-enum of $\mathscr{C}$. □

5.1.2. *Strong PA avoidance of* TS^2_3. Before proving strong c.b-enum avoidance of TS^n_d for every n and sufficiently large d's, we prove strong PA avoidance of TS^2_3 as a warm-up. It already contains the core tools used for the general case. In this section, we shall consider only coloring 1-formulas over $\{0, 1, 2\}$. We therefore omit the parameters and simply say "coloring formula".

Theorem 5.9 TS^2_3 admits strong PA avoidance.

Proof. Fix a set C of non-PA degree and let $f : [\omega]^2 \to 3$ be a coloring. By strong PA avoidance of COH (Wang [41, Theorem 3.1]), there exists an infinite set X_0 such that $X_0 \oplus C$ is not of PA degree and

$$(\forall x)(\exists i < 3)(\forall^\infty s \in X_0) f(x, s) = i$$

By strong PA avoidance of RT^1_3 (Liu [29]) there exists an infinite set $X_1 \subseteq X_0$ and a color $i_f < 3$ such that $X_1 \oplus C$ is not of PA degree and

$$(\forall x \in X_1)(\forall^\infty s \in X_1) f(x, s) = i_f$$

We will construct simultaneously three infinite sets G_0, G_1, G_2 such that one of the $G_i \oplus C$'s is not of PA degree, and for each $i < n$, $f([G_i]^2) \subseteq \{i, i_f\}$. Thus the G's are all f-thin. The requirements to ensure that all the G_i's are infinite are

$$Q_m : (\forall i < 3)(\exists w > m)(w \in G_i)$$

whereas the requirements to ensure that $G_i \oplus C$ is not of PA degree for some $i < 3$ are

$$R_{e_0,e_1,e_2} : (\exists i < 3)(\Phi^{G_i \oplus C}_{e_i} \text{ total} \to (\exists w)\Phi^{G_i \oplus C}_{e_i}(w) \neq \Phi_w(w)\downarrow)$$

Before defining our notion of forcing, we need to provide some semantics to the notion of partition of coloring formulas. The way we understand the notion of partition of coloring formulas strongly depends on the notion of forcing we consider. In our case, the function symbols are interpreted by functions of type $[\omega]^2 \to 3$.

Definition 5.10 (Assignment) An *assignment* of a coloring formula φ is a function κ such that $\operatorname{dom}(\kappa) \supseteq \operatorname{dom}(\varphi)$ and for every $g \in \operatorname{dom}(\kappa)$, $\kappa(g)$ is a function of type $[\omega]^2 \to 3$. Given a coloring formula $\varphi = g_0[J_0] \wedge \cdots \wedge g_{t-1}[J_{t-1}]$ over $\{0, 1, 2\}$ and an assignment κ, a set $F \subseteq \omega$ *satisfies* φ (written $(F, \kappa) \models \varphi$) if $\kappa(g_j)([F]^2) \in J_j \cup \{i_f\}$ for each $j < t$.

In other words, $(F,\kappa) \models \varphi$ iff there is a valuation $\pi \models \varphi$ such that $\kappa(g_j)([F]^2) \subseteq \pi(g_j) \cup \{i_f\}$ for each $j < t$. Note that this definition of satisfaction is parameterized by the color i_f. Given an assignment κ, we let κ^+ be the assignment of domain $\operatorname{dom}(\kappa) \sqcup \{f\}$ extending κ and such that $\kappa^+(f) = f$. Here, we identify the function symbol f and the actual function f of type $[\omega]^2 \to 3$. Now we have defined the suitable interpretation of the notion of pseudo k-partition of coloring formula, we can define our notion of forcing to build an infinite f-thin set for some coloring $f : [\omega]^2 \to 3$.

Definition 5.11 (Single condition)

1. A *single condition* is a tuple $(F_0, F_1, F_2, X, \varphi, \kappa)$ where (F_i, X) is a Mathias condition, φ is a coloring formula and κ is an assignment such that for each $i < 3$,
$$(\forall x \in F_i)(\forall t \in F_i \cup X)(\{x,t\}, \kappa^+) \models \varphi \wedge f[\{i\}]$$
2. A single condition $d = (\vec{H}, Y, \psi, \gamma)$ *extends* $c = (\vec{F}, X, \varphi, \kappa)$ if for each $i < 3$, (H_i, Y) extends the Mathias condition (F_i, X), $\kappa \subseteq \gamma$ and there exists a coloring formula θ such that $\psi = \varphi \wedge \theta$.
3. An 3-tuple of sets $\vec{G}$ *satisfies* a single condition $(\vec{F}, X, \varphi, \kappa)$ if G_i satisfies the Mathias condition (F_i, X) and $(G_i, \kappa^+) \models \varphi \wedge f[\{i\}]$ for each $i < 3$.

Definition 5.12 (Condition)

1. A *condition* is a tuple $(k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ where $k > 0$, $\vec{F}$ is a $3k$-tuple of finite sets $(F_i^\nu : \nu < k, i < 3)$, D is not of PA degree, $X \oplus C \leq_T D$, $\vec{\varphi} = (\varphi_\nu : \nu < k)$ is a pseudo k-partition of coloring formulas, $\mathscr{P}$ is a non-empty $\Pi^{0,D}_1$ class of assignments and for each $\kappa \in \mathscr{P}$ and each $\nu < k$, $(F_0^\nu, F_1^\nu, F_2^\nu, X, \varphi_\nu, \kappa)$ is a single condition.
2. A condition $d = (m, \vec{H}, Y, E, \vec{\psi}, \mathscr{Q})$ *extends* $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ if $D \leq_T E$ and there is a function $f : m \to k$ with the following property: for each $\gamma \in \mathscr{Q}$, there is some $\kappa \in \mathscr{P}$ such that the single condition $(H_0^\nu, H_1^\nu, H_2^\nu, Y, \psi_\nu, \gamma)$ extends $(F_0^{f(\nu)}, F_1^{f(\nu)}, F_2^{f(\nu)}, X, \varphi_{f(\nu)}, \kappa)$. In this case, the function f *witnesses* the extension and *part ν of d refines part $f(\nu)$ of c*.
3. A 3-tuple of sets $\vec{G}$ *satisfies* some condition $(k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ on part ν if there is some $\kappa \in \mathscr{P}$ such that $\vec{G}$ satisfies the single condition $(F_0^\nu, F_1^\nu, F_2^\nu, X, \varphi_\nu, \kappa)$. $\vec{G}$ satisfies d if it satisfies d on some of its parts.
4. A condition $(k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ *forces Q_m on part ν* if for every $i < 3$, there exists $w > m$ such that $w \in F_i^\nu$.
5. A condition *d forces R_{e_0,e_1,e_2} on part ν* if every 3-tuple of sets $\vec{G}$ satisfying d on part ν satisfies R_{e_0,e_1,e_2}.
6. *Part ν of $(k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ is acceptable* if there is an infinite set $Y \subseteq X$ such that $Y \oplus D$ is not of PA degree and there is some $\kappa \in \mathscr{P}$ such that
$$(\forall x \in Y)(\forall^\infty t \in Y)(\{x,t\}, \kappa) \models \varphi_\nu$$

The three following lemma are sufficient to build the desired infinite f-thin set.

Lemma 5.13 Every condition has an acceptable part.

Lemma 5.14 For every condition c and every m, there is a condition d extending c such that d forces Q_m on each of its acceptable parts.

Lemma 5.15 For every condition c and every $e_0, e_1, e_2 \in \omega$ there exists an extension d forcing R_{e_0,e_1,e_2} on each of its acceptable parts.

The construction of the sets G_0, G_1 and G_2 given the three lemmas above is strictly the same as in [17, Lagniappe]: We build an infinite, decreasing sequence of conditions $c_0 \geq c_1 \geq \dots$ starting with $c_0 = (1, \emptyset, \emptyset, \emptyset, \omega, C, \{\varepsilon\}, \emptyset)$ where ε is the empty conjunction, with the following properties assuming that $c_s = (k_s, \vec{F}_s, X_s, D_s, \vec{\varphi}_s \mathscr{P}_s)$:

1. Each condition c_s has an acceptable part.
2. If part ν of c_s is acceptable, then c_s forces R_{e_0,e_1,e_2}, where $s = \langle e_0, e_1, e_2 \rangle$.

3. If part ν of c_s is acceptable, then c_s forces Q_s on part ν.

If part ν of c_{s+1} is acceptable and refines part μ of c_s, then so is part μ of c_s. Hence the acceptable parts of the sequence of conditions form an infinite, finitely branching tree. By König's lemma, there exists an infinite sequence $\nu_0, \nu_1, \dots$ where part ν_{s+1} of c_{s+1} refines part ν_s of condition c_s. One easily checks that $G_i = \bigcup_s F_{s,i}^{\nu_s}$ is the desired set.

Proof of Lemma 5.13. Let $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ be a condition. As $\mathscr{P}$ is non-empty, there exists an assignment $\kappa \in \mathscr{P}$. Suppose that $\text{dom}(\kappa) = \{g_0, \dots, g_{t-1}\}$. Thanks to strong PA avoidance of COH (Wang [41, Theorem 3.1]) and of RT^1_3 (Liu [29]), define a finite decreasing sequence $X \supseteq Y_0 \supseteq \dots \supseteq Y_{t-1}$ such that for each $i < t$

1. $Y_i \oplus D$ is not of PA degree
2. there is some $c_i < 3$ such that,

$$(\forall x \in Y_i)(\forall^\infty s \in Y_i)\kappa(g_i)(x,s) = c_i$$

Indeed, at step i, first apply strong PA avoidance of COH to the sequence of sets $\vec{R}$ defined by $R_{x,c} = \{s : \kappa(g_i)(x,s) = c\}$ to obtain a set $\tilde{Y}_i$ such that

$$(\forall x \in \tilde{Y}_i)(\exists c < 3)(\forall^\infty s \in \tilde{Y}_i)\kappa(g_i)(x,s) = c_i$$

Then apply strong PA avoidance of RT^1_3 to the function $\tilde{h} : \tilde{Y}_i \to 3$ defined by $\tilde{h}(x) = \lim_{s \in \tilde{Y}_i} \kappa(g_i)(x,s)$ to obtain a color $c_i < 3$ and the set Y_i.

Let π be the valuation defined by $\pi(g_i) = \{c_i\}$ for each $i < t$. Since $\vec{\varphi} = (\varphi_\nu : \nu < k)$ is a pseudo k-partition, there is some $\nu < k$ such that $\pi \models \varphi_\nu$. We claim that ν and Y_{t-1} satisfy the desired properties. For each $i < t$, by definition of π,

$$(\forall x \in Y_{t-1})(\forall^\infty s \in Y_{t-1})\kappa(g_i)(x,s) \in \pi(g_i)$$

Therefore

$$(\forall x \in Y_{t-1})(\forall^\infty s \in Y_{t-1})(\forall i < t)\kappa(g_i)(x,s) \in \pi(g_i)$$

Since $(\{x,s\}, \kappa) \models \varphi$ iff $(\forall i < t)\kappa(g_i)(x,s) \in \pi(g_i)$ for some valuation $\pi \models \varphi$,

$$(\forall x \in Y_{t-1})(\forall^\infty s \in Y_{t-1})(\{x,s\}, \kappa) \models \varphi_\nu$$

Therefore part ν of c is acceptable. □

Proof of Lemma 5.14. Fix some $m \geq 0$. It suffices to prove that given some condition $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$, if part μ is acceptable, then there exists an extension $d = (k, \vec{H}, Y, Y \oplus D, \vec{\varphi}, \mathscr{Q})$ which forces Q_m on part μ and whose extension is witnessed by the identity map. By iterating the process at most k times, we obtain an extension satisfying the statement of the lemma. Fix an acceptable part μ. By definition of being acceptable, there is some $\kappa \in \mathscr{P}$ and an infinite subset $Y_0 \subseteq X$ such that $Y_0 \oplus D$ is not of PA degree and

$$(\forall x \in Y_0)(\forall^\infty s \in Y_0)(\{x,s\}, \kappa) \models \varphi_\mu$$

Recall that given a set F, the statement $(F, \kappa) \models g[I]$ is defined by $(\forall \{x,y\} \in [F]^2)\kappa(g)(x,y) \in I \cup \{i_f\}$. By the choice of our initial condition, $\forall x \in Y_0$ and $\forall^\infty s \in Y_0$, $f(x,s) = i_f$, therefore for every $i < 3$

$$(\forall x \in Y_0)(\forall^\infty s \in Y_0)(\{x,s\}, \kappa^+) \models f[\{i\}]$$

By choosing some $y \in Y_0 \cap (m, +\infty)$ and removing finitely many elements from Y_0, we obtain a set $Y \subseteq Y_0$ such that $Y \oplus D$ is not of PA degree and

$$(\forall i < 3)(\forall s \in Y)(\{y,s\}, \kappa^+) \models \varphi_\mu \wedge f[\{i\}]$$

By the fact that $(F_0^\mu, F_1^\mu, F_2^\mu, Y, \varphi_\mu, \kappa)$ is a single condition,

$$(\forall i < 3)(\forall x \in F_i \cup \{y\})(\forall s \in F_i \cup \{y\} \cup Y_0)(\{x,s\}, \kappa^+) \models \varphi_\mu \wedge f[\{i\}]$$

For each $i < 3$, let $H_i^\nu = F_i^\nu$ if $\nu \neq \mu$ and $F_i^\mu \cup \{y\}$ otherwise. Let $\mathscr{Q}$ be the $\Pi_1^{0, Y \oplus D}$ class of assignments $\kappa \in \mathscr{P}$ such that $(\forall i < 3)(\forall s \in Y)(\{y,s\}, \kappa^+) \models \varphi_\mu \wedge f[\{i\}]$. The condition $(k, \vec{H}, Y, Y \oplus D, \vec{\varphi}, \mathscr{Q})$ is an extension forcing Q_m on part μ. □

It remains to prove Lemma 5.15. Given a condition c, and any $e_0, e_1, e_2 \in \omega$, let $U_{e_0,e_1,e_2}(c)$ be the set of the acceptable parts ν such that c does not force R_{e_0,e_1,e_2} on part ν. If $U_{e_0,e_1,e_2}(c) = \emptyset$, we are already done as condition c already forces R_{e_0,e_1,e_2} of each of its acceptable parts. In order to prove Lemma 5.15, it suffices to prove and iterate the following lemma.

Lemma 5.16 For every condition c and every $e_0, e_1, e_2 \in \omega$ such that $U_{e_0,e_1,e_2}(c) \neq \emptyset$, there exists an extension d of c such that $\left|U_{e_0,e_1,e_2}(d)\right| < \left|U_{e_0,e_1,e_2}(c)\right|$.

We need first to redefine a few notions introduced in [17, Lagniappe].

Definition 5.17

1. A *valuation* is a finite partial function of type $\omega \to 2$. A valuation α is *correct* if $\alpha(n) = \Phi_n(n)\downarrow$ for all $n \in dom(\alpha)$. Two valuations α and β are *incompatible* if there is an n such that $\alpha(n) \neq \beta(n)$.
2. Let $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ be a condition and α be a valuation. We say that part μ of c *disagrees* with α if for every $\kappa \in \mathscr{P}$ and every function $h : [\omega]^2 \to 3$, there is an $i < 3$, a $w \in dom(\alpha)$ and a finite set $F' \subset X$ such that $(F', \kappa) \models \varphi_\mu$, $h([F']^2) \subseteq \{i, i_f\}$ and $\Phi_{e_i}^{(F_i^\mu \cup F') \oplus C}(w) \downarrow \neq \alpha(w)$.

Note that the set E of all valuations α such that part μ of c disagrees with α is C-c.e. The following lemma has exactly the same proof as Lemma L.33 in [17, Lagniappe].

Lemma 5.18 . For every condition c and $e_0, e_1, e_2 \in \omega$, one of the following must hold.

1. There is a correct valuation α and a $\mu \in U_{e_0,e_1,e_2}(c)$ such that α disagrees with part μ of c.
2. There are infinitely many pairwise incompatible valuations $\alpha_0, \alpha_1, \dots$ such that for every $\mu \in U_{e_0,e_1,e_2}(c)$ and every $i < \omega$, α_i does not disagree with part μ of c.

We need one last definition before proving Lemma 5.16. The acceptation of a part ν of a condition $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ intuitively means that we can find an infinite set $Y \subseteq X$ such that $(k, \vec{F}, Y, Y \oplus D, \vec{\varphi}, \mathscr{P})$ is a valid extension and there exists an assignment $\kappa \in \mathscr{P}$ such that

$$(\forall x \in Y)(\forall^\infty s \in Y)(\{x, s\}, \kappa) \models \varphi_\nu$$

The condition $(k, \vec{F}, Y, Y \oplus D, \vec{\varphi}, \mathscr{P})$ has the same number of parts and its part ν can take Y as a witness of being acceptable. This process can be iterated so that we obtain a condition $d = (k, \vec{F}, Z, E, \vec{\varphi}, \mathscr{P})$ such that for every acceptable part ν of d, there exists an assignment $\kappa \in \mathscr{P}$ such that Z is a witness of acceptation of part ν. Such a condition is said to *witness its acceptable parts*. Every condition can be extended to a condition witnessing its acceptable parts. We are now ready to prove Lemma 5.16.

Proof of Lemma 5.16. Fix an extension $d = (k, \vec{F}, Y, D, \vec{\varphi}, \mathscr{P})$ of c witnessing its acceptable parts. The proof is done by a case analysis of Lemma 5.18 applied to d. In the first case, we will construct an extension d' whose witnessing function is the identity function and forcing R_{e_0,e_1,e_2} on an part of d' refining a part in $U_{e_0,e_1,e_2}(d)$. Therefore $|U_{e_0,e_1,e_2}(d')| < |U_{e_0,e_1,e_2}(d)|$. In the other case, we will construct an extension d' forcing R_{e_0,e_1,e_2} simultaneously on every part of d' refining any part in $U_{e_0,e_1,e_2}(d)$. In this case $U_{e_0,e_1,e_2}(d') = \emptyset$ and we are done.

Suppose that case 1 of Lemma 5.18 holds for d. Fix the correct valuation α and the accepting part μ of d. By choice of d, there exists an assignment $\kappa \in \mathscr{P}$ such that

$$(\forall x \in Y)(\forall^\infty s \in Y)(\{x, s\}, \kappa) \models \varphi_\mu$$

Take f as the function h is the definition of disagreeing with α. There exists a finite set $F' \subseteq X$, an $i < 3$ and a $w \in \omega$ such that $(F', \kappa) \models \varphi_\mu$, $f([F']^2) \subseteq \{i, i_f\}$ and $\Phi_{e_i}^{(F_i^\mu \cup F') \oplus C}(w) \downarrow \neq \alpha(w)$. Set $H_j^\nu = F_i^\mu \cup F'$ if $\mu = \nu$ and $i = j$. Otherwise set $H_j^\nu = F_j^\nu$. By removing finitely many elements from Y, we obtain a set $Z \subseteq Y$ such that

$$(\forall x \in F_i^\mu \cup F')(\forall s \in F_i^\mu \cup F' \cup Z)(\{x, s\}, \kappa^+) \models \varphi_\mu \wedge f[\{i\}]$$

Let $\mathscr{Q}$ be the $\Pi^{0,D}_1$ class of all the assignments $\kappa \in \mathscr{P}$ satisfying the above property. The condition $(k, \vec{H}, Z, D, \vec{\varphi}, \mathscr{Q})$ is a valid extension forcing R_{e_0,e_1,e_2} on part μ.

Suppose now that case 2 of Lemma 5.18 holds for d. Let $\alpha_0, \dots, \alpha_{3k}$ be pairwise incompatible valuations such that for all $\ell < 3k+1$ and all $\nu \in U_{e_0,e_1,e_2}(d)$, part ν of d does not disagree with α_ℓ. For each $\ell < 3k+1$, let $\vec{\varphi}^\ell = (\varphi_\nu \wedge g_\nu[i] : \nu < k, i < 3)$ be the set of coloring formulas obtained by adding k new function symbols $(g_\nu : \nu < k)$. By Lemma 5.3, $\vec{\varphi}^\ell$ is a pseudo $3k$-partition of coloring formulas. Let $\mathscr{Q}_\ell$ be the $\Pi^{0,Y\oplus D}_1$ class of all assignments κ such that $\mathrm{dom}(\kappa) \supseteq \mathrm{dom}(\vec{\varphi}) \cup \{g_\nu : \nu < k\}$ and such that for every $w \in dom(\alpha_\ell)$, $\nu \in U_{e_0,e_1,e_2}(d)$, every $i < 3$, every finite set $F' \subset Y$ such that $(F', \kappa) \models \varphi_\nu \wedge g_\nu[\{i\}]$,

$$\Phi_{e_i}^{(F_i^\nu \cup F')\oplus C}(w)\uparrow \text{ or } \Phi_{e_i}^{(F_i^\nu \cup F')\oplus C}(w) = \alpha_\ell(w)$$

$\mathscr{Q}_\ell$ is non-empty since for every $\nu \in U_{e_0,e_1,e_2}(d)$, part ν of d does not disagree with any of the α's. By renaming the constant symbols, we now suppose that the pseudo $3k$-partitions $\mathrm{dom}(\vec{\varphi}^0), \dots, \vec{\varphi}^{3k}$ have pairwise disjoint domains.

Let $\vec{\psi} = \{\varphi^i_\nu \wedge \varphi^j_\nu : i < j < 3k+1, \nu < 3k\}$. By Lemma 5.7, $\vec{\psi}$ is a pseudo $(3k\binom{3k+1}{2})$-partition of coloring formulas. Let $\mathscr{Q}$ be the $\Pi^{0,Y\oplus D}_1$ class of all assignements κ such that $\kappa = \kappa^0 \sqcup \dots \sqcup \kappa^{3k+1}$ for some $\kappa^\ell \in \mathscr{Q}_\ell$. Let $\vec{H}$ be obtained from $\vec{F}$ by duplicating the sets $3\binom{3k+1}{2}$ times. The condition $(3k\binom{3k+1}{2}), \vec{H}, Y, Y \oplus D, \vec{\psi}, \mathscr{Q})$ is an extension forcing R_{e_0,e_1,e_2} on each part refining any part of $U_{e_0,e_1,e_2}(d)$. ☐

This last lemma finishes the proof. ☐

5.1.3. *Strong 1-enum avoidance of generalized cohesiveness.* Before proving strong c.b-enum avoidance of $\mathsf{TS}^n_{d_n+1}$, we need to prove strong c.b-enum avoidance of a generalized notion of cohesiveness. Cohesiveness can be seen as taking a coloring function over pairs $f : [\omega]^2 \to k$, fixing the first parameter to obtain an infinite sequence of colorings of the integers $f_x : \omega \to k$ for each $x \in \omega$ and creating a set which will be eventually homogeneous for each coloring f_x.

Going further in the reasoning, we can consider that cohesiveness is a degenerate case of taking a coloring function over pairs $f : [\omega]^2 \to \omega$, fixing the first parameter to obtain an infinite sequence of colorings of integers $f_x : \omega \to \omega$ and creating a set such that for each color i, either eventually the color will be avoided, by f_x or f_x will be eventually homogeneous with color i.

We can also generalize the notion to colorings over tuples $f : [\omega]^n \to \omega$, seeing them as an infinite sequence of colorings over k-uples $f_\sigma : [\omega]^k \to \omega$ for each $\sigma \in [\omega]^{n-k}$. We will create a set such that at most d_k colors will appear arbitrarily far for each function f_σ by applying $\mathsf{TS}^k_{d_k+1}$ to f_σ.

Theorem 5.19 Fix a coloring $f : [\omega]^n \to \omega$, some $t \le n$ and a closed set $\mathscr{C} \subseteq \omega^\omega$ for which $\mathsf{TS}^s_{d_s+1}$ admits strong path avoidance for each $s \in (0, t]$. For every set C computing no member of $\mathscr{C}$, there exists an infinite set G such that $G \oplus C$ computes no member of $\mathscr{C}$ and for every $\sigma \in [\omega]^{<\omega}$ such that $n - t \le |\sigma| < n$,

$$\left|\left\{x : (\forall b)(\exists \tau \in [G \cap (b, +\infty)]^{n-|\sigma|}) f(\sigma, \tau) = x\right\}\right| \le d_{n-|\sigma|}$$

Proof. Our forcing conditions are Mathias conditions (F,X) where $X \oplus C$ computes no member of $\mathscr{C}$. By Lemma 4.6, it suffices to prove the following lemma.

Lemma 5.20 For every condition (F,X) and $\sigma \in [\omega]^{<\omega}$ such that $n - t \le |\sigma| < n$, for every finite set I such that $|I| = d_{n-|\sigma|}$, there exists an extension $(F,\tilde{X})$ such that

$$\{f(\sigma,\tau) : \tau \in [\tilde{X}]^{n-|\sigma|}\} \subseteq I \quad \text{or} \quad I \not\subseteq \{f(\sigma,\tau) : \tau \in [\tilde{X}]^{n-|\sigma|}\}$$

Proof. Define the function $g : [X]^{n-|\sigma|} \to I \cup \{\bot\}$ by $g(\tau) = f(\sigma, \tau)$ if $f(\sigma, \tau) \in I$ and $g(\tau) = \bot$ otherwise. Since $n - |\sigma| \in (0, t]$, then by strong path avoidance of $\mathsf{TS}^{n-|\sigma|}_{d_{n-|\sigma|}+1}$ for $\mathscr{C}$, there exists an infinite subset $\tilde{X} \subseteq X$ such that $\tilde{X} \oplus C$ computes no member of $\mathscr{C}$ and $\left|\{g(\tau) : \tau \in [\tilde{X}]^{n-|\sigma|}\}\right| \le d_{n-|\sigma|}$. The condition $(F,\tilde{X})$ is the desired extension. ☐

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \omega)$, where $c_s = (F_s, X_s)$. The filter $\mathscr{F}$ yields a unique infinite set $G = \bigcup_s F_s$. By Lemma 4.6, $G \oplus C$ computes no member of $\mathscr{C}$. We claim that G satisfies our hypothesis. Fix a $\sigma \in [\omega]^{<\omega}$ such that $n - t \leq |\sigma| < n$. Suppose there exists $d_{n-|\sigma|} + 1$ elements $x_0, \dots, x_{d_{n-|\sigma|}}$ such that $(\forall b)(\exists \tau \in [G \cap (b, +\infty)]^{n-|\sigma|}) f(\sigma, \tau) = x_i$ for each $i \leq d_{n-|\sigma|}$. Let $I = \{x_0, \dots, x_{d_{n-|\sigma|}-1}\}$. By Lemma 5.20, the set G satisfies some condition $c_s \in \mathscr{F}$ such that $\{f(\sigma, \tau) : \tau \in [X_s]^{n-|\sigma|}\} \subseteq I$ or $I \not\subseteq \{f(\sigma, \tau) : \tau \in [X_s]^{n-|\sigma|}\}$. In the first case it contradicts the choice of $x_{d_{n-|\sigma|}}$ and in the second case it contradicts the choice of an element of I. □

When considering a function $f : [\omega]^n \to k$ and taking $t = n - 1$, we obtain a set similar to the one constructed in section 3.1 in [43].

Theorem 5.21 Fix a coloring $f : [\omega]^n \to k$ and a closed set $\mathscr{C} \subseteq \omega^\omega$ for which $\mathsf{TS}^s_{d_s+1}$ admits strong path avoidance for each $s \in (0, n)$. For every set C which does not compute a member of $\mathscr{C}$, there exists an infinite set G such that $G \oplus C$ computes no member of $\mathscr{C}$ and a sequence $(I_\sigma : 0 < |\sigma| < n)$ such that for each $\ell \in (0, n)$ and each $\sigma \in [\omega]^\ell$

(a) I_σ is a subset of $\{0, \dots, k-1\}$ with at most $d_{n-\ell}$ elements
(b) $(\exists b)(\forall \tau \in [G \cap (b, +\infty)]^{n-\ell}) f(\sigma, \tau) \in I_\sigma$

Proof. Let G be the set constructed by Theorem 5.19 for $t = n - 1$. For each $\sigma \in [\omega]^{<\omega}$ such that $0 < |\sigma| < n$, let

$$I_\sigma = \{x < k : (\forall b)(\exists \tau \in [G \cap (b, +\infty)]^{n-|\sigma|}) f(\sigma, \tau) = x$$

By choice of G, $|I_\sigma|$ has at most $d_{n-|\sigma|}$ many elements. Moreover, for each $y < k$ such that $y \notin I_\sigma$, there exists a bound b_y such that $(\forall \tau \in [G \cap (b_y, +\infty)]^{n-|\sigma|}) f(\sigma, \tau) \neq y$. So taking $b = max(b_y : y < k \wedge y \notin I_\sigma$

$$(\forall \tau \in [G \cap (b, +\infty)]^{n-|\sigma|}) f(\sigma, \tau) \in I_\sigma$$

□

In the proof of strong PA avoidance of TS^2_3, cohesiveness is always followed by an application of $\mathsf{RT}^1_{<\infty}$ in order to obtain an infinite subset on which the coloring $f : [\omega]^2 \to 3$ eventually uses one fixed color. This is also the case for proving strong c.b-enum avoidance of $\mathsf{TS}^n_{d_n+1}$. Therefore we need to prove the following theorem.

Theorem 5.22 Fix a coloring $f : [\omega]^n \to k$ and a closed set $\mathscr{C} \subseteq \omega^\omega$ for which $\mathsf{TS}^s_{d_s+1}$ admits strong path avoidance for each $s \in (0, n)$. For every set C with does not compute a member of $\mathscr{C}$, there exists an infinite set G such that $G \oplus C$ computes no member of $\mathscr{C}$ and a finite set $(\mathscr{I}_s : 0 < s < n)$ such that for each $s \in (0, n)$

(a) $\mathscr{I}_s$ is a finite set of at most d_s sets of colors, and $|I| \leq d_{n-s}$ for each $I \in \mathscr{I}_s$.
(b) $(\forall \sigma \in [G]^s)(\exists b)(\exists I \in \mathscr{I}_s)(\forall \tau \in [G \cap (b, +\infty)]^{n-s}) f(\sigma, \tau) \in I$

Proof. Let X be the infinite set and $(I_\sigma : 0 < |\sigma| < n)$ be the infinite sequence constructed in Theorem 5.21. For each $s \in (0, n)$ and $\sigma \in [G]^s$, let $F_s(\sigma) = I_\sigma$. Using strong path avoidance of $\mathsf{TS}^s_{d_s+1}$ for $\mathscr{C}$, we build a finite sequence $X \supseteq X_1 \supseteq \cdots \supseteq X_{n-1}$ such that for each $s \in (0, n)$

1. $X_s \oplus C$ computes no member of $\mathscr{C}$
2. $|F_s([X_s]^s)| \leq d_s$

Let $G = X_{n-1}$ and $\mathscr{I}_s = F_s([G]^s)$ for each $s \in (0, n)$. We now check that property (b) is satisfied. Fix a $\sigma \in [G]^s$. Because $G \subseteq X$, $(\exists b)(\forall \tau \in [G \cap (b, +\infty)]^{n-s}) f(\sigma, \tau) \in I_\sigma$. So $F_s(\sigma) = I_\sigma$, but $\sigma \in [G]^s$, hence $I_\sigma \in \mathscr{I}_s$. □

In particular, in our ongoing forcing, we will use the following corollary.

Corollary 5.23 Fix a coloring $f : [\omega]^n \to k$ and some set $\mathscr{C} \subseteq 2^\omega$ for which $\mathsf{TS}^s_{d_s+1}$ admits strong c.b-enum avoidance for each $s \in (0, n)$. For every set C which does not compute a c.b-enum of $\mathscr{C}$, there exists an infinite set G such that $G \oplus C$ computes no c.b-enum of $\mathscr{C}$ and a finite set $I \subseteq \{0,\dots,k-1\}$ such that

$$|I| \leq \sum_{0<s<n} d_s d_{n-s}$$

and for each $s \in (0,n)$

$$(\forall \sigma \in [G]^s)(\exists b)(\forall \tau \in [G \cap (b,+\infty)]^{n-s}) f(\sigma,\tau) \in I$$

Proof. Apply Theorem 5.22 taking $I = \bigcup \mathscr{I}$ and Lemma 3.20 using strong c.b-enum avoidance of $\mathsf{TS}^s_{d_s+1}$ for $\mathscr{C}$. □

5.1.4. *Strong c.b-enum avoidance of* TS. The thin set theorem is, together with the Ramsey-type weak König's lemma, the only statement studied in the paper whose strong c.b-enum avoidance is not proved using a relative simultaneous 1-enum avoidance. It is unknown whether such a relative proof exists.

Theorem 5.24 TS^n_d admits strong c.b-enum avoidance for every $n \geq 1$ and sufficiently large d's.

Corollary 5.25 TS admits strong c.b-enum avoidance.

Proof. Wang proved in [43] that $\mathsf{TS} \leq_{sc} (\forall n)(\exists d_n)\mathsf{TS}^n_{d_n+1}$. Apply Lemma 2.15 and Theorem 5.24. □

The proof of Theorem 5.24 is done by induction over $n \geq 1$, assuming that $\mathsf{TS}^s_{d_s+1}$ admits strong c.b-enum avoidance for every $s < n$. Define d_n inductively as follows:

$$d_1 = 1 \qquad d_n = 2\sum_{0<s<n} d_s d_{n-s} \quad \text{for } n > 1$$

The case of $n = 1$ is nothing but Liu's theorem since $\mathsf{TS}^1_{d_1+1}$ is RT^1_2.

Proof of Theorem 5.24. Fix a set C computing no c.b-enum of $\mathscr{C}$ for some set $\mathscr{C} \subseteq 2^\omega$ and let $f : [\omega]^n \to d_n + 1$ be a coloring. Let $d = \sum_{0<s<n} d_s d_{n-s}$. By Corollary 5.23, there exists a finite set I_f of cardinality d and an infinite set X_0 such that $X_0 \oplus C$ computes no c.b-enum of $\mathscr{C}$ and for every $s \in (0,n)$,

$$(\forall \sigma \in [G]^s)(\exists b)(\forall \tau \in [G \cap (b,+\infty)]^{n-s}) f(\sigma,\tau) \in I_f$$

Let $p = \binom{d_n+1}{d}$ and let $I_0,\dots,I_{p-1}$ be the sequence of all finite d-subsets of $\{0,\dots,d_n\}$. We will construct simultaneously p sets $G_0,\dots,G_{p-1}$ such that $G_i \oplus C$ computes no c.b-enum of $\mathscr{C}$ for some $i < p$. We furthermore ensure that for each $i < p$,

$$f([G_i]^n) \subseteq I_i \cup I_f$$

therefore G_i will be f-thin, as $|I_i \cup I_f| \leq 2d = d_n$. The requirements to ensure that all G_i's are infinite are

$$Q_s : (\forall i < p)(\exists w > s)(w \in G_i)$$

The requirements to ensure that $G_i \oplus C$ computes no c.b-enum of $\mathscr{C}$ for some $i < p$ are

$$\mathcal{R}_{e_0,\dots,e_{p-1}} : \mathcal{R}_{e_0} \vee \dots \vee \mathcal{R}_{e_{p-1}}$$

where

$$\mathcal{R}_{e_i} : (\Phi^{G_i \oplus C}_{e_i} \text{ total} \to (\exists w)|\Phi^{G_i \oplus C}_{e_i}(w) \neq \Phi_w(w)| > e_i \vee [\Phi^{G_i \oplus C}_{e_i}(w)] \cap \mathscr{C} = \emptyset$$

Once again, we shall define some suitable semantics to the notion of coloring formula, as we did for the thin set theorem for pairs. We shall only consider coloring d-formulas over $\{0,\dots,d_n\}$. In the following, we will omit the parameters d and $\{0,\dots,d_n\}$.

Definition 5.26 (Assignment) An *assignment* of a coloring formula φ is a function κ such that $\operatorname{dom}(\kappa) \supseteq \operatorname{dom}(\varphi)$ and for every $g \in \operatorname{dom}(\kappa)$, $\kappa(g)$ is a function of type $[\omega]^n \to d_n + 1$. Given a coloring formula $\varphi = g_0[J_0] \wedge \dots \wedge g_{t-1}[J_{t-1}]$ and an assignment κ, a set $F \subseteq \omega$ *satisfies* φ (written $(F,\kappa) \models \varphi$) if $\kappa(g_j)([F]^n) \subseteq J_j \cup I_f$ for each $j < t$.

In other words, $(F,\kappa) \models \varphi$ iff there is a valuation $\pi \models \varphi$ such that $\kappa(g_j)([F]^n) \subseteq \pi(g_j) \cup I_f$ for each $j < t$. Again, given an assignment κ, we let κ^+ be the assignment of domain $\mathrm{dom}(\kappa) \sqcup \{f\}$ extending κ and such that $\kappa^+(f) = f$.

Definition 5.27 Given a Turing functional Φ_e, a finite set F, a clopen V, a coloring formula φ, and assignment κ and a set X, we say that $\Phi_e^{F\oplus C}$ *abandons* V on φ, κ and X if there is a $w \in \omega$ and a finite set $F' \subset X$ such that $(F',\kappa) \models \varphi$ and

$$|\Phi_e^{(F\cup F')\oplus C}(w)| > e \vee [\Phi_e^{(F\cup F')\oplus C}(w)] \cap V = \emptyset$$

The following lemma tells us that computing an e-enum and not abandoning an e-disperse sequence of clopen sets is incompatible.

Lemma 5.28 Let $V^{(1)},\dots,V^{(q)}$ be an e-disperse sequence of clopen sets. Suppose $\Phi_e^{F\oplus C}$ does not abandon $V^{(j)}$ on φ, κ and X for every $j = 1,\dots,q$. Then for every set G satisfying the Mathias condition (F,X) such that $(G,\kappa) \models \varphi$, $\Phi_e^{G\oplus C}$ is not total or is not an e-enum.

Proof. Fix such a set G and a $j \in \{1,\dots,q\}$. Because $\Phi_e^{F\oplus C}$ does not abandon $V^{(j)}$ on φ, κ and X, for every $w \in \omega$ such that $\Phi_e^{G\oplus C}(w)\downarrow$, the following holds

$$|\Phi_e^{G\oplus C}(w)| \leq e \wedge [\Phi_e^{G\oplus C}(w)] \cap V^{(j)} \neq \emptyset$$

By convention, if $\rho \in \Phi_e^{G\oplus C}(w)$ then $|\rho| = w$. Taking w large enough, we have for every $j \in \{1,\dots,q\}$ and every $\rho \in \Phi_e^{G\oplus C}(w)$

$$[\rho] \cap V^{(j)} \neq \emptyset \to [\rho] \subseteq V^{(j)}$$

For $i < e$, let ρ_i be the ith string in $\Phi_e^{G\oplus C}(w)$. The string ρ_i induces an e-cover $(P_i : i < e)$ of the clopen sets defined by

$$P_i = \{V^{(j)} : [\rho_i] \subseteq V^{(j)}\}$$

But then for each $i < e$, $[\rho_i] \subseteq \bigcap_{j\in P_i} V^{(j)} \neq \emptyset$ contradicting the assumption that $V^{(1)},\dots,V^{(q)}$ is e-disperse. □

We are now ready to define the actual notion of forcing and prove that every sufficiently generic filter yields the desired p-tuple of sets.

Definition 5.29 (Single condition)

1. A *single condition* is a tuple $(F_0,\dots,F_{p-1},X,\varphi,\kappa)$ where (F_i,X) is a Mathias condition, φ is a coloring formula and κ is an assignment such that for each $i < p$ and each $s \in (0,n)$,

$$(\forall \sigma \in [F_i]^s)(\forall \tau \in [F_i \cup X]^{n-s})(\sigma\tau,\kappa^+) \models \varphi \wedge f[I_i]$$

2. A single condition $d = (\vec{H},Y,\psi,\gamma)$ *extends* $c = (\vec{F},X,\varphi,\kappa)$ if for each $i < p$, (H_i,Y) extends the Mathias condition (F_i,X), $\kappa \subseteq \gamma$ and there exists a coloring formula θ such that $\psi = \varphi \wedge \theta$.
3. A p-tuple of sets $\vec{G}$ *satisfies* a single condition $(\vec{F},X,\varphi,\kappa)$ if G_i satisfies the Mathias condition (F_i,X) and $(G_i,\kappa^+) \models \varphi \wedge f[I_i]$ for each $i < p$.

Definition 5.30 (Condition)

1. A *condition* is a tuple $(k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ where $k > 0$, $\vec{F}$ is a kp-tuple of finite sets $(F_i^\nu : \nu < k, i < p)$, D computes no c.b-enum of $\mathscr{C}$, $X \oplus C \leq_T D$, $\vec{\varphi} = (\varphi_\nu : \nu < k)$ is a pseudo k-partition of coloring formulas, $\mathscr{P}$ is a non-empty $\Pi_1^{0,D}$ class of assignments and for each $\kappa \in \mathscr{P}$, each $\nu < k$, $(F_0^\nu,\dots,F_{p-1}^\nu,X,\varphi_\nu,\kappa)$ is a single condition.
2. A condition $d = (m,\vec{H},Y,E,\vec{\psi},\mathscr{Q})$ *extends* $c = (k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ if $D \leq_T E$ and there is a function $f : m \to k$ with the following property: for each $\gamma \in \mathscr{Q}$, there is some $\kappa \in \mathscr{P}$ such that the single condition $(H_0^\nu,\dots,H_{p-1}^\nu,Y,\psi_\nu,\gamma)$ extends $(F_0^{f(\nu)},\dots,F_{p-1}^{f(\nu)},X,\varphi_{f(\nu)},\kappa)$. In this case, the function f *witnesses* the extension and *part s of d refines part $f(s)$ of c*.

3. A p-tuple of sets $\vec{G}$ *satisfies* some condition $(k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ on part ν if there is some $\kappa \in \mathscr{P}$ such that $\vec{G}$ satisfies the single condition $(F_0^\nu,\dots,F_{p-1}^\nu,X,\varphi_\nu,\kappa)$. $\vec{G}$ satisfies d if it satisfies d on some of its parts.
4. A condition $(k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ *forces* Q_u *on part* ν if for every $i < p$, there exists $w > u$ such that $w \in F_i^\nu$.
5. A condition d *forces* $R_{e_0,\dots,e_{p-1}}$ *on part* ν if every tuple of sets $\vec{G}$ satisfying d on part ν satisfies $R_{e_0,\dots,e_{p-1}}$.
6. *Part* ν *of* $(k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ *is acceptable* if there is an infinite set $Y \subseteq X$ such that $Y \oplus D$ computes no c.b-enum of $\mathscr{C}$ and there is a $\kappa \in \mathscr{P}$ such that for each $s \in (0,n)$,

$$(\forall \sigma \in [Y]^s)(\exists b)(\forall \tau \in [Y \cap (b,+\infty)]^{n-s})(\sigma\tau,\kappa) \models \varphi_\nu$$

Lemma 5.31 Every condition has an acceptable part.

Lemma 5.32 For every condition c and every $u \in \omega$, there is a condition d extending c such that d forces Q_u on each of its acceptable parts.

Lemma 5.33 For every condition c and every $e_0,\dots,e_{p-1} \in \omega$ there exists an extension d forcing $R_{e_0,\dots,e_{p-1}}$ on each of its acceptable parts.

The construction of $G_0,\dots,G_{p-1}$ given the three lemmas above is strictly the same as in [17, Lagniappe]: We build an infinite, decreasing sequence of conditions $c_0 \geq c_1 \geq \dots$ starting with $c_0 = (1,\emptyset,\dots,\emptyset,\omega,C,\{\varepsilon\},\emptyset)$ where ε is the empty conjunction, with the following properties assuming that $c_s = (k_s,\vec{F}_s,X_s,D_s,\vec{\varphi}_s,\mathscr{P}_s)$:

1. Each c_s has an acceptable part.
2. If part ν of c_s is acceptable, then c_s forces $R_{e_0,\dots,e_{p-1}}$ if $s = \langle e_0,\dots,e_{p-1}\rangle$.
3. If part ν of c_s is acceptable, then c_s forces Q_s on part ν.

If part ν of c_{s+1} is acceptable and refines part μ of c_s, then part μ of c_s is also acceptable. Hence the acceptable parts of the conditions form an infinite finitely branching tree. By König's lemma, there exists an infinite sequence $\nu_0,\nu_1,\dots$ where part ν_{s+1} of c_{s+1} refines part ν_s of condition c_s. One easily checks that $G_i = \bigcup_s F_{s,i}^{\nu_s}$ is the desired set,.

Proof of Lemma 5.31. Let $c = (k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ be a condition. As $\mathscr{P}$ is non-empty, there exists an assignment $\kappa \in \mathscr{P}$. Thanks to Corollary 5.23, define a finite decreasing sequence $X \supseteq Y_0 \supseteq \dots \supseteq Y_{t-1}$ such that for each $i < t$

1. $Y_i \oplus D$ computes no c.b-enum of $\mathscr{C}$
2. there is a set J_i of size d such that for each $s \in (0,n)$,

$$(\forall \sigma \in [Y_i]^s)(\exists b)(\forall \tau \in [Y_i \cap (b,+\infty)]^{n-s})\kappa(g_i)(\sigma,\tau) \in J_i$$

Let π be the valuation defined by $\pi(g_i) = J_i$ for each $i < t$. Since $\vec{\varphi} = (\varphi_\nu : \nu < k)$ is a pseudo k-partition, there is some $\nu < k$ such that $\pi \models \varphi_\nu$. We claim that ν and Y_{t-1} satisfy the desired properties. For each $s \in (0,n)$ and $i < t$, by definition of π,

$$(\forall \sigma \in [Y_{t-1}]^s)(\exists b)(\forall \tau \in [Y_{t-1} \cap (b,+\infty)]^{n-s})\kappa(g_i)(\sigma,\tau) \in \pi(g_i)$$

Therefore, for each $s \in (0,n)$,

$$(\forall \sigma \in [Y_{t-1}]^s)(\exists b)(\forall \tau \in [Y_{t-1} \cap (b,+\infty)]^{n-s})(\forall i < t)\kappa(g_i)(\sigma,\tau) \in \pi(g_i)$$

Since $(\sigma\tau,\kappa) \models \varphi$ iff $(\forall i < t)\kappa(g_i)(\sigma,\tau) \in \pi(g_i)$ for some valuation $\pi \models \varphi$,

$$(\forall \sigma \in [Y_{t-1}]^s)(\exists b)(\forall \tau \in [Y_{t-1} \cap (b,+\infty)]^{n-s})(\sigma\tau,\kappa) \models \varphi_\nu$$

Therefore part ν of c is acceptable. □

Proof of Lemma 5.32. Fix some $u \in \omega$. It suffices to prove that given a condition $c = (k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$, if part μ is acceptable, then there exists an extension $d = (k,\vec{H},Y,Y \oplus D,\vec{\varphi},\mathscr{Q})$ which forces Q_u on part μ and whose extension is witnessed by the identity map. By iterating the process, we obtain an extension satisfying the statement of the lemma.

Fix an acceptable part μ. By definition, there exists an assignment $\kappa \in \mathscr{P}$ and an infinite subset $Y_0 \subseteq X$ such that $Y_0 \oplus D$ computes no c.b-enum of $\mathscr{C}$ and for each $s \in (0,n)$,

$$(\forall \sigma \in [Y_0]^s)(\exists b)(\forall \tau \in [Y_0 \cap (b,+\infty)]^{n-s})(\sigma\tau,\kappa) \models \varphi_\mu$$

By the choice of our initial condition and since the statement $(F,\kappa) \models g[J]$ is defined by $\kappa(g)([F]^n) \subseteq J \cup I_f$, for every $i < p$ and each $s \in (0,n)$,

$$(\forall \sigma \in [Y_0]^s)(\exists b)(\forall \tau \in [Y_0 \cap (b,+\infty)]^{n-s})(\sigma\tau,\kappa^+) \models f[I_i]$$

By the fact that $(F_0^\mu,\dots,F_{p-1}^\mu,Y_0,\varphi_\mu)$ is a single condition, for each $s \in (0,n)$,

$$(\forall i < p)(\forall \sigma \in [F_i]^s)(\forall \tau \in [F_i \cup Y_0]^{n-s})(\sigma\tau,\kappa^+) \models \varphi_\mu \wedge f[I_i]$$

therefore by taking $y \in Y_0 \cap (u,+\infty)$ and removing finitely many elements from Y_0, we obtain a set Y such that for each $s \in (0,n)$,

$$(\forall i < p)(\forall \sigma \in [F_i \cup \{y\}]^s)(\forall \tau \in [F_i \cup Y]^{n-s})(\sigma\tau,\kappa^+) \models \varphi_\mu \wedge f[I_i]$$

For each $i < p$, let $H_i^\nu = F_i^\nu$ if $\nu \neq \mu$ and $F_i^\mu \cup \{y\}$ otherwise. Let $\mathscr{Q}$ be the $\Pi_1^{0,Y\oplus D}$ collection of all the assignments $\kappa \in \mathscr{P}$ such that the above formula holds. The condition $(k,\vec{H},Y,Y\oplus D,\vec{\varphi},\mathscr{Q})$ is an extension forcing Q_u on part μ. □

It remains to prove Lemma 5.33. Given a condition c, and any $e_0,\dots,e_{p-1} \in \omega$, let $U_{e_0,\dots,e_{p-1}}(c)$ be the set of all acceptable parts ν such that c does not force $R_{e_0,\dots,e_{p-1}}$ on part ν. If $U_{e_0,\dots,e_{p-1}}(c) = \emptyset$, we are already done as condition c already forces $R_{e_0,\dots,e_{p-1}}$ of each of its acceptable parts. In order to prove Lemma 5.33, it suffices to prove and iterate the following lemma.

Lemma 5.34 For every condition c and every $e_0,\dots,e_{p-1} \in \omega$ such that $U_{e_0,\dots,e_{p-1}}(c) \neq \emptyset$, there exists an extension d such that $\left|U_{e_0,\dots,e_{p-1}}(d)\right| < \left|U_{e_0,\dots,e_{p-1}}(c)\right|$.

The proof of Lemma 5.34 is divided into three main lemmas: Lemma 5.8 asserts that only two cases can happen: a case where we can find a piece of oracle in a part of $U_{e_0,\dots,e_{p-1}}(d)$, forcing the Turing functional we consider to halt on a "wrong" input, i.e., on a clopen set which does not intersect $\mathscr{C}$. In the other case, it states the existence of many clopen sets which are intersected by the Turing functional whenever it halts. The second lemma states the existence of a finite extension forcing the Turing functional to halt on a wrong input on a part of $U_{e_0,\dots,e_{p-1}}(d)$ when the first case of the previous lemma holds. The third lemma states the existence of an extension forcing the Turing functionals to diverge or not to be an e_i-enum on each of the parts of $U_{e_0,\dots,e_{p-1}}(d)$ when the second case of the first lemma holds. Before stating and proving the three lemmas, we need to extend the abandoning terminology to a condition.

Definition 5.35 Let $c = (k,\vec{F},X,D,\vec{\psi},\mathscr{P})$ be a condition and V be a clopen set.

1. We say that part μ of c *abandons* V on some assignment κ if for every function $h : [\omega]^n \to d_n + 1$, there is an $i < p$, such that $\Phi_{e_i}^{F_i^\mu \oplus C}$ abandons V on $\varphi_\mu \wedge g[I_i]$, $\kappa + \{g \mapsto h\}$ and X.
2. We say that part μ of c *abandons* V if for every assignment $\kappa \in \mathscr{P}$, part μ of c abandons V on κ. The condition c *abandons* V if it abandons V on some part $\mu \in U_{e_0,\dots,e_{p-1}}(c)$.

Given a condition $c = (k,\vec{F},X,D,\vec{\kappa},\mathscr{P})$, k new function symbols $(g_\nu : \nu < k)$ and a clopen set V, let $\vec{\varphi}^V = \{\varphi_\nu \wedge g_\nu[I_i] : i < p, \nu < k\}$. By Lemma 5.3, $\vec{\varphi}^V$ is a pseudo pk-partition of coloring formulas. Moreover, define the following $\Pi_1^{0,D}$ class of assignments for $\vec{\varphi}^V$:

$$\mathscr{P}_V = \{\kappa + \{g_\nu \mapsto h_\nu : \nu < k\} : \kappa \in \mathscr{P} \wedge (\forall \nu \in U_{e_0,\dots,e_{p-1}}(c))(\forall i < p) \\ \Phi_{e_i}^{F_i \oplus C} \text{ does not abandon } V \text{ on } \varphi_\nu \wedge g_\nu[I_i], \kappa + \{g_\nu \mapsto h_\nu\} \text{ and } X\}$$

Notice that c abandons V iff $\mathscr{P}_V = \emptyset$, hence the set E of all clopens V such that $(k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$ abandons V is $X \oplus C$-c.e. as it can be written $\{V : \mathscr{P}_V = \emptyset\}$. We are now about to prove the second lemma, but need one last definition. The acceptation of a part ν of a condition $c = (k,\vec{F},X,D,\vec{\varphi},\mathscr{P})$

intuitively means that we can find an infinite set $Y \subseteq X$ such that $(k, \vec{F}, Y, Y \oplus D, \vec{\varphi}, \mathscr{P})$ is a valid extension and there exists an assignment $\kappa \in \mathscr{P}$ such that for each $s \in (0, n)$,

$$(\forall \sigma \in [Y]^s)(\exists b)(\forall \tau \in [Y \cap (b, +\infty)]^{n-s})(\sigma\tau, \kappa) \models \varphi_\nu$$

The condition $(k, \vec{F}, Y, Y \oplus D, \vec{\varphi}, \mathscr{P})$ has the same number of parts and its part ν can take Y as a witness of being acceptable. This process can be iterated so that we obtain a condition $d = (k, \vec{F}, Z, E, \vec{\varphi}, \mathscr{P})$ such that for every acceptable part ν of d, there exists an assignment $\kappa \in \mathscr{P}$ such that Z is a witness of acceptation of part ν. Such a condition is said to *witness its acceptable parts*. Every condition can be extended to a condition witnessing its acceptable parts. Recall that for each $n \in \omega$,

$$C_n = \{\rho \in 2^n : [\rho] \cap \mathscr{C} \neq \emptyset\}$$

Lemma 5.36 Let $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ be a condition witnessing its acceptable parts, and let $\mu \in U_{e_0, \dots, e_{p-1}}(c)$ such that part μ of c abandons C_n for some $n \in \omega$. There exists an extension d with the same parts as c, such that for every p-tuple of sets $G_0, \dots, G_{p-1}$ satisfying d on part μ and every $i < p$, $\Phi_{e_i}^{G_i \oplus C}$ is not an e_i-enum of $\mathscr{C}$.

Proof. By definition of witnessing its acceptable parts, there exists an assignment $\kappa \in \mathscr{P}$ such that for each $s \in (0, n)$,

$$(\forall \sigma \in [X]^s)(\exists b)(\forall \tau \in [X \cap (b, +\infty)]^{n-s})(\sigma\tau, \kappa) \models \varphi_\mu$$

As part μ of c abandons C_n, then for every function $h : [\omega]^n \to d_n + 1$, there is an $i < p$, such that $\Phi_{e_i}^{F_i^\mu \oplus C}$ abandons C_n on $\varphi_\mu \wedge g[I_i]$, $\kappa + \{g \mapsto h\}$ and X. In particular, for $h = f$, there is an $i < p$ such that $\Phi_{e_i}^{F_i^\mu \oplus C}$ abandons C_n on $\varphi_\mu \wedge f[I_i]$, κ^+ and X. Unfolding the definition, there exists a $w \in \omega$ and finite set $F' \subseteq X$, such that $(F', \kappa^+) \models \varphi_\mu \wedge f[I_i]$ and

$$|\Phi_{e_i}^{(F_i^\mu \cup F') \oplus C}(w)| > e_i \vee [\Phi_{e_i}^{(F_i^\mu \cup F') \oplus C}(w)] \cap C_n (\supseteq \mathscr{C}) = \emptyset$$

Set $H_j^\nu = F_i^\mu \cup F'$ if $\mu = \nu$ and $i = j$. Otherwise set $H_j^\nu = F_j^\nu$. By removing finitely many elements from X, we obtain a set $Y \subseteq X$ such that for each $s \in (0, n)$,

$$(\forall \sigma \in [F_i^\mu \cup F']^s)(\forall \tau \in [F_i^\mu \cup F' \cup Y]^{n-s})\sigma\tau \models \varphi_\mu \wedge f[I_i]$$

Let $\mathscr{Q}$ be the $\Pi_1^{0,D}$ class of all the assignments $\kappa \in \mathscr{P}$ satisfying the above property. The condition $(k, \vec{H}, Y, D, \vec{\varphi}, \mathscr{P})$ is a valid extension forcing $R_{e_0, \dots, e_{p-1}}$ on part μ. □

We now prove the third lemma stating the existence of an extension forcing $\Phi_{e_i}^{G_i \oplus C}$ to diverge or not to be a e_i-enum on each of the parts refining a part in $U_{e_0, \dots, e_{p-1}}(c)$.

Lemma 5.37 Let $V^{(1)}, \dots, V^{(q)}$ be an e-disperse sequence of clopen sets for $e = k \sum_{i<p} e_i$ and let $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$ be a condition which does not abandon $V^{(j)}$ for every $j = 1, \dots, q$. There exists an extension d such that for every p-tuple of sets $G_0, \dots, G_{p-1}$ satisfying d and every $i < p$, $\Phi_{e_i}^{G_i \oplus C}$ is either partial or is not an e_i-enum of $\mathscr{C}$.

Proof. By the hypothesis that c does not abandon $V^{(j)}$ for every $j \in \{1, \dots, q\}$, the $\Pi_1^{0,D}$ class $\mathscr{P}_{V^{(j)}}$ is non-empty. Let $\mathscr{K}$ be the e-supporter of $\{1, \dots, q\}$ constructed in Lemma 5.6 and let $K' = \sum_{\mathscr{K}_i \in \mathscr{K}} |\mathscr{K}_i|$. By renaming the function symbols, we can suppose that $\vec{\varphi}^{V^{(1)}}, \dots, \vec{\varphi}^{V^{(q)}}$ have a pairwise disjoint domain. By Lemma 5.5, $\vec{\psi} = Cross(\vec{\varphi}^{V^{(1)}}, \dots, \vec{\varphi}^{V^{(q)}}, \mathscr{K})$ is a pseudo K'-partition of coloring formulas. Let $\mathscr{Q}$ be the $\Pi_1^{0,D}$ class of all assignments κ such that $\kappa = \kappa^1 \sqcup \dots \sqcup \kappa^q$ for some $\kappa^j \in \mathscr{P}_{V^{(j)}}$. For each $\mu < K'$, let $H^\mu = F^\nu$ if part μ refines part ν of c. Then the condition $d = (K', \vec{H}, X, D, \vec{\psi}, \mathscr{Q})$ is a valid extension of c. We claim that d forces $\Phi_{e_i}^{G_i \oplus C}$ to be either partial or not to be an e_i-enum of $\mathscr{C}$ for each p-tuple of sets $\vec{G}$ satisfying d.

Fix such a p-tuple of sets $G_0, \dots, G_{p-1}$ satisfying c on some part $\mu < K'$ of d. If part μ of d refines a part of c which is not in $U_{e_0,\dots,e_{p-1}}(c)$ then by definition part μ of d already forces $\Phi_{e_i}^{G_i \oplus C}$ not to be an e_i-enum of $\mathscr{C}$. So suppose that part μ of d refines a part ν of c such that $\nu \in U_{e_0,\dots,e_{p-1}}(c)$.

By definition of satisfaction and the definition of the cross operator, there exists a $K \in \mathscr{K}_\nu$ and for each $j \in \{1, \dots, q\}$ an assignment $\kappa^j \in \mathscr{P}_{V(j)}$ such that $\vec{G}$ satisfies the single condition

$$(F_0^\nu, \dots, F_{p-1}^\nu, X, \bigwedge_{j \in K} \varphi_\nu^j \wedge g_\nu^j[I_{i_j}], \kappa^j)$$

In particular, G_i satisfies the Mathias condition (F_i^ν, X) and

$$(G_i, \kappa^j) \models \bigwedge_{j \in K} \varphi_\nu^j \wedge g_\nu^j[I_{i_j}]$$

Fix some $i < p$. By construction of $\vec{\mathscr{K}}$, $\{V^{(j)} : j \in K\}$ is an e_i-disperse sequence of clopen sets. By definition of $\mathscr{P}_{V(j)}$, $\Phi_{e_i}^{F_i^\nu \oplus C}$ does not abandon $V^{(j)}$ on $\varphi_\nu^j \wedge g_\nu^j[I_{i_j}]$, κ^j and X. So in particular $\Phi_{e_i}^{F_i^\nu \oplus C}$ does not abandon $V^{(j)}$ on $\bigwedge_{j \in K} \varphi_\nu^j \wedge g_\nu^j[I_{i_j}]$, $\kappa^1 \sqcup \dots \sqcup \kappa^q$ and X. Applying Lemma 5.28, we deduce that G_i is not total or does not compute an e_i-enum. □

Proof of Lemma 5.34. Fix a condition $c = (k, \vec{F}, X, D, \vec{\varphi}, \mathscr{P})$. We can furthermore assume without loss of generality that c witnesses its acceptable parts. If $U_{e_0,\dots,e_{p-1}}(c) = \emptyset$, we are done. So suppose $U_{e_0,\dots,e_{p-1}}(c) \neq \emptyset$. By Lemma 5.8 applied to the $\Sigma_1^{0,D}$ formula "c abandons V", we have two cases:

1. There exists an $n \in \omega$ and a part $\mu \in U_{e_0,\dots,e_{p-1}}(c)$ such that part μ of c abandons C_n. In this case, by Lemma 5.36 there exists an extension d having the same parts as c, and such that for every p-tuple of sets $G_0, \dots, G_{p-1}$ satisfying d on part μ and every $i < p$, $\Phi_{e_i}^{G_i \oplus C}$ is not an e_i-enum of $\mathscr{C}$. Therefore, $U_{e_0,\dots,e_{p-1}}(d) = U_{e_0,\dots,e_{p-1}}(c) \setminus \{\mu\}$ and we are done.
2. For $e = k \sum_{i<p} e_i$, there exists an e-disperse sequence of clopen sets $V^{(1)}, \dots, V^{(m)}$ such that for every $i = 1, \dots, m$, c does not abandon $V^{(i)}$. By Lemma 5.37, there exists an extension d such that for every p-tuple of sets $G_0, \dots, G_{p-1}$ satisfying d and every $i < p$, $\Phi_{e_i}^{G_i \oplus C}$ is either partial or is not an e_i-enum of $\mathscr{C}$. In this case we have $U_{e_0,\dots,e_{p-1}}(d) = \emptyset$.

□

This last lemma finishes the proof. □

5.1.5. Π_1^0 *classes of coloring of tuples.* As well as there exists a $\emptyset'$-computable coloring of pairs $f : [\omega]^2 \to 2$ such that every infinite set homogeneous for f computes the halting set, one may wonder whether there exists a Π_1^0 class of colorings of pairs $\mathscr{P}$ such that for every $f \in \mathscr{P}$, every infinite f-homogeneous set is of PA degree.

Definition 5.38 For every principle P, $\Pi_1^0(\mathsf{P})$ is the statement "For every Π_1^0 class $\mathscr{P}$ of instances of P, there exists a solution to one of the instances of $\mathscr{P}$."

In particular, $\Pi_1^0(\mathsf{RT}_2^1)$ is RWKL.

Theorem 5.39 Suppose that $\mathsf{TS}_{d_s+1}^s$ admits strong c.b-enum avoidance for each $s < n$. Then $\Pi_1^0(\mathsf{TS}_{d+1}^n)$ admits c.b-enum avoidance with

$$d = \sum_{0<s<n} d_s d_{n-s}$$

Proof. The proof is almost exactly the same as the one of Theorem 5.24. The main difference comes from the satisfaction of a coloring formula. The set of colors I_f does not any more exist, so given an assignment κ, a set F *satisfies* a coloring formula $\varphi = g_0[J_0] \wedge \dots \wedge g_{t-1}[J_{t-1}]$ (written $(F, \kappa) \models \varphi$) if $g_j([F]^n) \subseteq J_j$ for each $j < t$. Therefore $g_j([F]^n)$ will use at most $|J_j| \leq d$ colors and not $|J_j \cup I_f| \leq 2d$ colors. Given a $\Pi_1^{0,C}$ class $\mathscr{P}$ of colorings of n-tuples into $d+1$ colors, the initial condition is $(n, F^0, \dots, F^{n-1}, C, \vec{\varphi}, \mathscr{P})$ where $\vec{\varphi} = (g[J_i] : i < p)$. □

Corollary 5.40 $\Pi^0_1(\mathsf{RT}^2_2)$ admits c.b-enum avoidance.

Proof. RT^1_1 admits strong c.b-enum avoidance, so we can apply Theorem 5.39 taking $d_1 = 1$. □

5.2. **The free set theorem**

The free set theorem is, together with the thin set theorem, another consequence of full Ramsey's theorem which is sufficiently weak to admit strong c.b-enum avoidance. Altough it is not known to be a consequence of TS, its proof of strong c.b-enum avoidance of FS seems to deeply rely on strong c.b-enum avoidance of TS.

Definition 5.41 (Free set theorem) Let $f : [\omega]^n \to \omega$ be a coloring function. A set A is *free for* f if for every $x_1 < \dots < x_n \in A$, if $f(x_1, \dots, x_n) \in A$ then $f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$. FS^n is the statement "every function $f : [\omega]^n \to \omega$ has an infinite set free for f".

FS^n lies in between RT^n_2 and RRT^n_2 [43, 7]. Wang proved in [43] that FS^n does not imply RT^n_2 for $n \neq 3$. Lerman & al. [28] and Wang [42] proved independently that EM does not imply FS^2 over ω-models. FS^2 implies TS^2, therefore FS^n does not admit 1-enum avoidance. However, for any fixed class $\mathscr{C}$ for which the thin set theorem admits 1-enum avoidance, so does FS. We first restrict our study of FS to particular functions – trapped functions –. This notion will be useful for talking about Π^0_1 classes of functions.

The thin set theorems are useful in particular for proving avoidance theorems for principles like the free set theorem.

Theorem 5.42 If $\mathsf{TS}^s_{d_s+1}$ admits simultaneous strong 1-enum avoidance for some countable collection of classes $\mathscr{C}_0, \mathscr{C}_1, \dots$ and each $0 < s \leq n$, then so does FS^n.

Corollary 5.43 FS admits strong c.b-enum avoidance.

Proof. By strong c.b-enum avoidance of $\mathsf{TS}^s_{d_s+1}$ for every s and sufficiently large d_s, Theorem 5.42 and Lemma 3.21. □

Lemma 5.44 (Wang [43]) For each $n \geq 1$, if FS^n and COH admit strong $\mathscr{C}$ avoidance, then FS^{n+1} admits $\mathscr{C}$ avoidance.

Before proving Theorem 5.42, we deduce strong c.b-enum avoidance for the rainbow Ramsey theorem.

Corollary 5.45 RRT admits strong c.b-enum avoidance.

Proof. Wang proved in [43] that $\mathsf{RRT} \leq_{sc} \mathsf{FS}$. Apply Lemma 2.15 and Corollary 5.43. □

The proof of strong 1-enum avoidance of FS relative to strong 1-enum avoidance of TS uses a case analysis only on two kinds of functions: left trapped and right trapped functions.

Definition 5.46 A function $f : [\omega]^n \to \omega$ is *left (resp. right) trapped* if for every $\sigma \in [\omega]^n$, $f(\sigma) \leq \sigma(n-1)$ (resp. $f(\sigma) > \sigma(n-1)$).

Lemma 5.47 (Wang in [43]) For each $n \geq 1$, if FS^n for trapped functions admits (strong) $\mathscr{C}$ avoidance for some set $\mathscr{C} \subseteq \omega^\omega$, then so does FS^n.

Proof. We prove it in the case of strong $\mathscr{C}$ avoidance. The proof of $\mathscr{C}$ avoidance is similar. Let $f : [\omega]^n \to \omega$ be a coloring and C be set computing no member of $\mathscr{C}$. For each $\sigma \in [\omega]^n$ and $i \leq n$, let

$$f_0(\sigma) = min(f(\sigma), max(\sigma)) \qquad f_1(\sigma) = max(f(\sigma), max(\sigma) + 1)$$

By strong $\mathscr{C}$ avoidance of FS^n for trapped functions, we can define a finite sequence $\omega \supseteq H_0 \supseteq H_1$ such that for each $i \leq n$

1. H_i is an infinite f_i-free set

2. $H_i \oplus C$ computes no member of $\mathscr{C}$.

We claim that H_1 is f-free. Let $\sigma \in [H_n]^n$. $f(\sigma) = f_i(\sigma)$ for some $i \in \{0,1\}$. As H_1 is free for f_i, $f(\sigma) \notin H_1 \setminus \sigma$. □

5.2.1. *Case of right trapped functions.*

Lemma 5.48 Let $f : [\omega]^n \to \omega$ be a right trapped function. Every function d.n.c. relative to f computes an infinite set free for f.

Proof. By [26], every function d.n.c. relative to f computes a function g such that if $|W_e^f| \leq m$ then $g(e,m) \notin W_e^f$. Given a finite f-free set F, there exists at most $\binom{|F|}{n}$ elements x such that $F \cup \{x\}$ is not f-free. We can define an infinite f-free set H by stages. $H_0 = \emptyset$. Given a finite f-free set H_s of cardinal s, set $H_{s+1} = H_s \cup \{g(e, \binom{s}{n})\}$ where e is a Turing index such that $W_e^f = \{x : F \cup \{x\}$ is not $f-$free$\}$. □

Lemma 5.49 For each $n \geq 1$, if FS^n for left trapped functions admits (strong) $\mathscr{C}$ avoidance for some set $\mathscr{C} \subseteq \omega^\omega$, then so does FS^n.

Proof. Again, we prove it in the case of strong $\mathscr{C}$ avoidance. By Lemma 5.47, it suffices to prove that FS^n for right trapped functions admits strong $\mathscr{C}$ avoidance. Let $f : [\omega]^n \to \omega$ be a right trapped function and C be a set computing no member of $\mathscr{C}$. By Rice [35], there exists an f-computable stable left trapped function g such that every infinite set thin for g computes a function d.n.c. relative to f. By [7, Theorem 3.2], every infinite set free for g is, up to finite variation, a set thin for g. By strong $\mathscr{C}$ avoidance of FS^n for left trapped functions, there is an infinite g-free set H such that $H \oplus C$ computes no member of $\mathscr{C}$. By Lemma 5.48, H computes an infinite f-free set. □

By Lemma 5.49 it remains to prove strong 1-enum avoidance of FS^n for left trapped functions, assuming strong 1-enum avoidance of TS^m_d for each $m \leq n$ and sufficiently large d's.

5.2.2. *Case of left trapped functions.*

Theorem 5.50 If $\mathsf{TS}^s_{d_s+1}$ admits simultaneous strong 1-enum avoidance for some countable collection of classes $\vec{\mathscr{C}}$ and each $0 < s \leq n$, then so does FS^n for left trapped functions.

The proof will be by induction over n. Base case is easy and follows directy from strong 1-enum avoidance of TS^1_2 for $\vec{\mathscr{C}}$.

Lemma 5.51 If RT^1_2 admits strong 1-enum avoidance for $\vec{\mathscr{C}}$ then so does FS^1.

Proof. Cholak et al. proved in [7] that FS^1 for left trapped functions is strongly computably reducible to RT^1_4. Apply Lemma 2.15 to deduce strong 1-enum avoidance of FS^1 for left trapped functions for $\vec{\mathscr{C}}$. Conclude with Lemma 5.49. □

The two following lemmas will ensure that promise sets of our forcing conditions will have good properties, so that conditions will be extensible.

Lemma 5.52 Suppose FS^s admits strong $\mathscr{C}$ avoidance for each $s < n$ for some class $\mathscr{C}$. Fix a set C computing no member of $\mathscr{C}$, a finite set F and an infinite set X computable in C. For every function $f : [X]^n \to \omega$ there exists an infinite set $Y \subseteq X$ such that $Y \oplus C$ computes no member of $\mathscr{C}$ and $(\forall \sigma \in [F]^t)(\forall \tau \in [Y]^{n-t}) f(\sigma,\tau) \notin Y \setminus \tau$ for each $0 < t < n$.

Proof. Fix the finite enumeration $\sigma_1, \dots, \sigma_k$ for all $\sigma \in [F]^t$ for some $0 < t < n$. Start with $Y_0 = X$. Suppose that $Y_{m-1} \oplus C$ computes no member of $\mathscr{C}$ and for all $i < m$, $\forall \tau \in [Y_{m-1}]^{n-|\sigma_i|} f(\sigma_i, \tau) \notin Y_{m-1} \setminus \tau$. Define the function $f_{\sigma_m} : [Y_{m-1}]^{n-|\sigma_m|} \to \omega$ by $f_{\sigma_m}(\tau) = f(\sigma_m, \tau)$. By strong $\mathscr{C}$ avoidance of $\mathsf{FS}^{n-|\sigma_m|}$, there exists an infinite set $Y_m \subseteq Y_{m-1}$ such that $Y_m \oplus C$ computes no member of $\mathscr{C}$ and $(\forall \tau \in [Y_m]^{n-|\sigma_m|}) f(\sigma_m, \tau) \notin Y_m \setminus \tau$. Y_k is the desired set. □

Lemma 5.53 Suppose that $\mathsf{TS}^s_{d_s+1}$ admits strong 1-enum avoidance for $\vec{\mathscr{C}}$ and for each $0 < s \leq n$ and FS^s admits strong 1-enum avoidance for $\vec{\mathscr{C}}$ and for each $0 < s < n$. For every function $f : [\omega]^n \to \omega$ and every set C computing no 1-enum of $\vec{\mathscr{C}}$, there exists an infinite set X such that $X \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$ and for every $\sigma \in [G]^{<\omega}$ such that $0 \leq |\sigma| < n$,

$$(\forall x \in G \setminus \sigma)(\exists b)(\forall \tau \in [G \cap (b, +\infty)]^{n-|\sigma|}) f(\sigma, \tau) \neq x$$

Proof. Let X be an infinite set satisfying property of Theorem 5.19 with $t = n$. For each $s < n$ and $i < d_{n-s}$, let $f_{s,i} : [X]^s \to \omega$ be the function such that $f_{s,i}(\sigma)$ is the ith element of the set

$$\{x : (\forall b)(\exists \tau \in [X \cap (b, +\infty)]^{n-s}) f(\sigma, \tau) = x\}$$

if it exists, and 0 otherwise. Define a finite sequence $X \supseteq X_0 \supseteq \cdots \supseteq X_{n-1}$ such that for each $s < n$

1. X_s is $f_{s,i}$-free for each $i < d_{n-s}$
2. $X_s \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$

We claim that X_{n-1} is the desired set. Fix $s < n$ and take any $\sigma \in [X_{n-1}]^s$ and any $x \in X_{n-1} \setminus \sigma$. If $(\forall b)(\exists \tau \in [G \cap (b, +\infty)]^{n-s}) f(\sigma, \tau) = x$, then by choice of X, there exists an $i < d_{n-s}$ such that $f_{s,i}(\sigma) = x$, contradicting $f_{s,i}$-freeness of X_{n-1}. So $(\exists b)(\forall \tau \in [G \cap (b, +\infty)]^{n-s}) f(\sigma, \tau) \neq x$. □

Proof of Theorem 5.50. Fix a countable collection of classes $\mathscr{C}_0, \mathscr{C}_1, \dots$ for which $\mathsf{TS}^s_{d_s+1}$ admits strong 1-enum avoidance for each $0 < s \leq n$. Let $f : [\omega]^n \to \omega$ be a left trapped function and C be a set computing no 1-enum of $\vec{\mathscr{C}}$. Our forcing conditions are tuples $(k, \vec{F}, X, \vec{g})$ such that

(a) $\vec{g}$ is a left trapped $\oplus_k$-function, $\vec{F}$ is a finite $\oplus_k$-set
(b) X is an infinite set such that $F_0 < X$ and $X \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$
(c) $(\forall \sigma \in [F_i \cup X]^n) g_i(\sigma) \notin F_i \setminus \sigma$ for each $i < k$
(d) $(\forall \sigma \in [F_i \cup X]^t)(\forall x \in (F_i \cup X) \setminus \sigma)(\exists b)(\forall \tau \in [(F_i \cup X) \cap (b, +\infty)]^{n-t})$ $g_i(\sigma, \tau) \neq x$ for each $i < k$ and $0 \leq t < n$.
(e) $(\forall \sigma \in [F_i]^t)(\forall \tau \in [X]^{n-t}) g_i(\sigma, \tau) \notin X \setminus \tau$ for each $i < k$ and $0 < t < n$

Properties (d) and (e) will be obtained by Lemma 5.53 and Lemma 5.52 and are present to ensure to have extensions such that (c) holds. A set G *satisfies* a condition $(k, \vec{F}, X, \vec{g})$ if it satisfies the Mathias condition (F_0, X) and $G \setminus (F_0 \setminus F_i)$ if g_i-free for each $i < k$. Our initial condition is $(1, \emptyset, Y, f)$ where Y is obtained by Lemma 5.53. A condition $(m, \vec{F}', X', \vec{g}')$ *extends* another condition $(k, \vec{F}, X, \vec{g})$ if $X' \subseteq X$, $m \geq k$, $(\forall i < k) g_i = g'_i$ and there is a finite $E \subset X$ such that

(i) for every $i < k$, $F_i \subseteq F'_i$ and $F'_i \setminus F_i = E$
(ii) for every $k \leq i < m$, $F'_i = E$

Lemma 5.54 For every condition $(k, \vec{F}, X, \vec{g})$ there exists an extension $(k, \vec{H}, \tilde{X}, \vec{g})$ such that $|H_i| > |F_i|$ for each $i < k$.

Proof. Choose an $x \in X$ such that $(\forall j < k)(\forall \sigma \in [F_j]^n) g_j(\sigma) \neq x$ and set $H_i = F_i \cup \{x\}$ for each $i < k$. By property (d) of $(k, \vec{F}, X, \vec{g})$, there exists a b such that $(\forall i < k)(\forall \sigma \in [F_i]^t)(\forall \tau \in [X \cap (b, +\infty)]^{n-t}) g_i(\sigma, \tau) \neq \{x\} \setminus \sigma$ for each $0 \leq t \leq n$. By k applications of Lemma 5.52, there exists a $\tilde{X} \subseteq X \setminus [0, b]$ such that $\tilde{X} \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$ and property (e) is satisfied for $(k, \vec{H}, \tilde{X}, \vec{g})$. We claim that $(k, \vec{H}, \tilde{X}, \vec{g})$ is a valid condition. Properties (a), (b) and (d) trivially hold. It remains to check property (c). By property (c) of $(k, \vec{F}, X, \vec{g})$, we only need to check that $(\forall \sigma \in [F_i \cup \tilde{X}]^n) g_i(\sigma) \neq x$ for each $i < k$. This follows from our choice of b. □

Lemma 5.55 For every condition $(k, \vec{F}, X, \vec{g})$ and every $e, i \in \omega$, there exists an extension $(m, \vec{H}, \tilde{X}, \vec{h})$ forcing $\Phi_e^{G \oplus C}$ not to be a 1-enum of $\mathscr{C}_i$, where G is the forcing variable.

Proof. By removing finitely many elements to X, we can suppose w.l.o.g. that $(\forall j < k)(\forall \sigma \in [F_j]^n) g_j(\sigma) \notin X$. Suppose there exists a $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathscr{C}_i = \emptyset$ and a finite set $F' \subseteq X$ which is g_j-free for each $j < k$ and $\Phi_e^{(F_0 \cup F') \oplus C}(|\sigma|) \downarrow = \sigma$. Set $H_j = F_j \cup F'$ for each $j < k$. By property (d) of $(k, \vec{F}, X, \vec{g})$, there exists a b such that $(\forall \sigma \in [H_i]^t)(\forall x \in H_i)(\forall \tau \in [X \cap (b, +\infty)]^{n-t}) g_i(\sigma, \tau) \neq \{x\} \setminus \sigma$ for each $i < k$ and $0 \leq t < n$. By k applications of Lemma 5.52, there exists a $\tilde{X} \subseteq$

$X \cap (b, +\infty)$ such that $\tilde{X} \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$ and property (e) is satisfied for $(k, \vec{H}, \tilde{X}, \vec{g})$. We claim that $(k, \vec{H}, \tilde{X}, \vec{g})$ is a valid condition.

Properties (a), (b), (d) and (e) trivially hold. It remains to check property (c). By our choice of b, we need only to check that $(\forall \sigma \in [H_i]^n)(\forall x \in H_i) g_i(\sigma) \neq \{x\} \setminus \sigma$ for each $i < k$. By property (c) of $(k, \vec{F}, X, \vec{g})$, it suffices to check that $(\forall \sigma \in [H_i]^n) g_i(\sigma) \not\in F' \setminus \sigma$ for each $i < k$. By property (e) of $(k, \vec{F}, X, \vec{g})$, it remains the case $(\forall \sigma \in [F']^n) g_i(\sigma) \not\in F' \setminus \sigma$ for each $i < k$, which is exactly $\vec{g}$-freeness of F'.

Suppose there is no such finite set $F' \subset X$. For each $\sigma \in 2^{<\omega}$, let $\mathscr{F}_\sigma$ denote the collection of left trapped $\oplus_k$-functions $\vec{g}$ such that for each finite set $F' \subset X$ which is g_j-free for each $j < k$, either $\Phi_e^{(F_0 \cup F') \oplus C}(|\sigma|) \uparrow$ or $\Phi_e^{(F_0 \cup F') \oplus C}(|\sigma|) \neq \sigma$. Note that $\mathscr{F}_\sigma$ are uniformly $\Pi_1^{0, X \oplus C}$ classes. Because above case does not hold, $\vec{g} \in \mathscr{F}_\sigma$ for each σ such that $\mathscr{C}_i \cap [\sigma] = \emptyset$. The set $\{\sigma : \mathscr{F}_\sigma = \emptyset\}$ is $X \oplus C$-c.e. If for each $u \in \omega$ there exists a $\sigma \in 2^u$ such that $\mathscr{F}_\sigma = \emptyset$ then $X \oplus C$ computes a 1-enum of $\mathscr{C}_i$, contradicting our hypothesis. So there must be an $u \in \omega$ such that $\mathscr{F}_\sigma \neq \emptyset$ for each $\sigma \in 2^u$.

For each $\sigma \in 2^u$, let $\vec{h}_\sigma \in \mathscr{F}_\sigma$. Set $H_j = F_j$ for each $j < k$ and $H_j = \emptyset$ for each $k \leq j < (2^u + 1)k$. By 2^u applications of Lemma 5.53, there exists an infinite set $\tilde{X} \subseteq X$ such that $\tilde{X} \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$ and property (d) of $((2^u + 1)k, \vec{H}, \tilde{X}, \vec{g} \bigoplus_{\sigma \in 2^u} \vec{h}_\sigma)$ holds. As conditions (a-c) and (e) trivially hold, $((2^u + 1)k, \vec{H}, \tilde{X}, \vec{g} \bigoplus_{\sigma \in 2^u} \vec{h}_\sigma)$ is a valid condition. Moreover it forces $\Phi_e^{G \oplus C}(u) \uparrow$ or $\Phi_e^{G \oplus C}(u) \downarrow \not\in 2^u$. □

Let $\mathscr{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(1, \emptyset, Y, f)$, where $c_s = (k_s, \vec{F}_s, \vec{g}_s)$. The filter $\mathscr{F}$ yields a unique real $G = \bigcup_s F_{s,0}$. By definition of a forcing condition, G is an f-free set. By Lemma 5.54, G is infinite, and by Lemma 5.55, $G \oplus C$ computes no 1-enum of $\vec{\mathscr{C}}$. □

6. Remarks and applications

We now prove corollaries stated in introduction and study some equivalent formulations of strong c.b-enum avoidance.

6.1. **Avoiding countably many cones**

Original Seetapun's theorem states strong cone avoidance of $\mathsf{RT}^1_{<\infty}$ for countably many cones simultaneously. We prove now that this stronger statement is still subsumed by the notion of strong c.b-enum avoidance. Of course, given a countable collection of non-computable reals $A_0, A_1, \dots \subseteq \omega$, the set of reals $\mathscr{C} = \{A_i : i \in \omega\}$ has no computable member but may have a computable 1-enum. For example, fix any non-computable set A and set A_i to be A prefixed by i zeros. No A_i is computable, but corresponding set of reals $\mathscr{C}$ will have a trivial 1-enum consisting of all finite sequences of zero's. However, even for such a collection of sets, we can construct an increasing sequence of sets of reals $\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \dots \subseteq 2^\omega$ such that computing a c.b-enum of $\vec{\mathscr{C}}$ is equivalent to compute one of the reals A_i.

Lemma 6.1 Let $A_0, A_1, \dots \subseteq \omega$ be a countable collection non-computable reals. There exists a countable collection of closed sets of reals $\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \dots$ such that there is no computable c.b-enum of $\mathscr{C}_n$ and for every n, A_n computes a c.b-enum of $\mathscr{C}_n$.

Proof. By induction over n. Case $n = 0$ is satisfied by defining $\mathscr{C}_0 = \{A_0\}$. As by Corollary 3.23, every c.b-enum of $\{A_0\}$ computes A_0, there exists no computable c.b-enum of $\mathscr{C}_0$. Suppose we have defined $\mathscr{C}_n$ and consider A_{n+1}. If A_{n+1} computes a c.b-enum of $\mathscr{C}_n$, then set $\mathscr{C}_{n+1} = \mathscr{C}_n$.

Suppose now that A_{n+1} computes no c.b-enum of $\mathscr{C}_n$. Set $\mathscr{C}_{n+1} = \mathscr{C}_n \cup \{A_{n+1}\}$. If there exists a computable c.b-enum of $\mathscr{C}_{n+1}$ $(D_i : i < \omega)$, then $A_{n+1} \not\in [D_i]$ for infinitely many i, otherwise it would be a computable c.b-enum of $\{A_{n+1}\}$ and would compute A_{n+1} by Corollary 3.23. So A_{n+1} computes a c.b-enum of $\mathscr{C}_n$ by looking on input i to the least $j \geq i$ such that $A_{n+1} \not\in D_j$ and returning $D_j \upharpoonright i$. This contradicts our hypothesis. □

Corollary 6.2 (Seetapun [37]) Let $A_0, A_1, \dots$ be a countable collection non C-computable reals. Every function $f : [\omega]^2 \to 2$ has an infinite set of integerss H homogeneous for f such that $H \oplus C$ computes no A_n for every $n \in \omega$.

Proof. Apply c.b-enum avoidance of RT^2_2 with $\vec{\mathscr{C}}$ defined as in Lemma 6.1 to obtain an infinite set of integers H homogeneous for f such that $H \oplus C$ computes no c.b-enum of $\vec{\mathscr{C}}$. $H \oplus C$ computes no real A_n for every $n \in \omega$ as otherwise it would compute a c.b-enum of $\vec{\mathscr{C}}$. ☐

6.2. **From the Cantor space to the Baire space**

The proofs of strong enum avoidance can be easily adapted to consider compacts sets in the Baire space. However, there is no need to go into the forcing arguments to extend the results to the Baire space. We will now prove that c.b-enum avoidance for compact sets in the Baire space is equivalent to c.b-enum avoidance for closed sets in the Cantor space.

For each $n \in \omega$, let $\sigma_n \in 2^{<\omega}$ be the binary representation of n. In particular σ_0 is the empty string. Given $n \in \omega$, let $g(n)$ be the unique $\tau \in 2^{2|\sigma_n|+2}$ such that $\tau(2i) = 1$, $\tau(2i+1) = \sigma_n(i)$ for each $i < |\sigma_n|$ and $\tau(2|\sigma_n|) = \tau(2|\sigma_n|+1) = 0$. The function g is a computable bijection. Given a sequence $X \in \omega^\omega$, let Ψ^X be the sequence $g(X(0))^\frown g(X(1))^\frown \dots$. For example, the sequence $\langle 3, 4, \dots \rangle$ is transformed into $g(3)^\frown g(4)^\frown \dots = 111100^\frown 11101000^\frown \dots$.

Lemma 6.3 Every set of sequences $\mathscr{C} \subseteq \omega^\omega$ is Medvedev equivalent to the set of reals $\{\Psi^X : X \in \mathscr{C}\}$.

Proof. Ψ is a functional witnessing the Medvedev reduction of $\{\Psi^X : X \in \mathscr{C}\}$ to $\mathscr{C}$. Conversely, given a real $Y = \Psi^X$ for some $X \in \mathscr{C}$, we can compute $X \restriction n$ by looking for a string $\rho \prec Y$ such that $|\{j : \rho(2j) = 0\}| = n$, and decode the string $\sigma \in \omega^n$ using the bijection g. By definition of Ψ, $\sigma \prec X$. ☐

Corollary 6.4 The degrees bounding a c.b-enum of a set of sequences $\mathscr{C} \subseteq \omega^\omega$ and those bounding a c.b-enum of $\{\Psi^X : X \in \mathscr{C}\}$ coincide.

Proof. Since $\mathscr{C}$ is compact, Ψ is continuous and Cantor space is separated, $\{\Psi^X : X \in \mathscr{C}\}$ is compact. Conclude by Lemma 6.3 and Lemma 3.2. ☐

Corollary 6.5 For every (arbitrary) function h, there exists an ω-model of $\mathsf{RT}^2_2 + \mathsf{RRT}$ containing no h-bounded d.n.c. function.

Proof. Apply c.b-enum avoidance of $\mathsf{RT}^2_2 + \mathsf{FS}$ to the effectively closed set $\mathscr{C} = \{f \in h^\omega : (\forall x) f(x) \neq \Phi_x(x)\}$. Note that $\mathscr{C}$ is homogeneous, so by Lemma 5.6 of [30], every c.b-enum of $\mathscr{C}$ computes a member of $\mathscr{C}$. ☐

Corollary 6.6 For every set X and every (non-necessarily computable) function h, there exists a function d.n.c. relative to X computing no h-bounded d.n.c. function.

Proof. Consider the effectively closed set $\mathscr{C}$ of Corollary 6.5. By applying strong c.b-enum avoidance of RT^1_2 to any Martin-Löf random R relative to X, we obtain an infinite subset Y in R or $\overline{R}$ computing no c.b-enum of $\mathscr{C}$. By [26], Y computes a function d.n.c. relative to X but computes no h-bounded d.n.c. function. ☐

6.3. **Restricting the scheme**

Here, we explain that c.b-enum is really a scheme of avoidance, and so it can even be used to separate statements which do not admit c.b-enum avoidance in its full generality. Algorithmic randomness is very useful for studying how a typical set behaves with a computability theoretic notion.

Definition 6.7 (Martin-Löf randomness) A set R is *Martin-Löf random* if there exists a constant $c \in \omega$ such that $(\forall n)[K(R \restriction n) \geq n - c]$ where K is the prefix-free Kolmogorov complexity.

Martin-Löf randomness is a very robust notion admitting various characterizations in terms of Martin-Löf tests, martingales, etc... In a reverse mathematical point of view, the existence for every set X of a Martin-Löf random relative to X is equivalent to the principle WWKL_0 over RCA_0 [2]. The initial purpose of c.b-enum avoidance has been the separation of RT^2_2 from WWKL_0 over ω-models.

Theorem 6.8 (Liu [30]) WWKL_0 does not admit c.b-enum avoidance.

Proof. Consider the effectively closed set of positive measure $\mathscr{C} = \{Z : (\forall n)K(Z \upharpoonright n) \geq n\}$. By Lemma 3.19, every c.b-enum of $\mathscr{C}$ computes a 1-enum of $\mathscr{C}$. By [26], every 1-enum of $\mathscr{C}$ computes a d.n.c. function, therefore $\mathscr{C}$ has no computable 1-enum. □

It is currently unknown whether every 1-enum of $\mathscr{C}$ computes a member of $\mathscr{C}$ for the class $\mathscr{C} = \{Z : (\forall n)K(Z \upharpoonright n) \geq n\}$.

Theorem 6.9 (Jockusch & al. [23]) Fix a set C computing no 1-enum of some homogeneous closed set $\mathscr{C} \subseteq 2^\omega$.

$$\mu(\{Z : Z \oplus C \text{ computes a member of } \mathscr{C}\}) = 0$$

Corollary 6.10 WWKL_0 admits strong $\mathscr{C}$-avoidance for every homogeneous closed set $\mathscr{C}$.

Proof. Fix a set C computing no 1-enum of $\mathscr{C}$ and consider a tree T of positive measure. By Theorem 6.9, $\mu(\{Z : Z \oplus C$ computes a member of $\mathscr{C}\}) = 0$, so there exists an infinite path P in T such that $ZP \oplus C$ computes no member of $\mathscr{C}$. By homogeneity of $\mathscr{C}$, every 1-enum of $\mathscr{C}$ computes a member of $\mathscr{C}$, so $P \oplus C$ computes no 1-enum of $\mathscr{C}$. □

Corollary 6.11 There exists an ω-model of $\mathsf{RT}^2_2 \wedge \mathsf{TS} \wedge \mathsf{FS} \wedge \mathsf{WWKL}_0$ which is not a model of WKL_0.

Proof. Consider the effectively closed set $\mathscr{C}$ of all completions of Peano arithmetic. By definition, WKL_0 does not admit $\mathscr{C}$-avoidance. By Corollary 4.11, Theorem 5.24 and Corollary 5.43, RT^2_2, TS and FS admit c.b-enum avoidance of $\mathscr{C}$. By Lemma 5.6 of [30], the degrees of members of $\mathscr{C}$ and of c.b-enum of $\mathscr{C}$ coincide. By Corollary 6.10, WWKL_0 admits $\mathscr{C}$-avoidance. Therefore $\mathsf{RT}^2_2 \wedge \mathsf{TS} \wedge \mathsf{FS}$ admit $\mathscr{C}$-avoidance and we conclude. □

6.4. **Open questions**

Question 6.12 Does RWKL admit 1-enum avoidance ?

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