The complexity of satisfaction problems in Reverse Mathematics

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Abstract. Satisfiability problems play a central role in computer science and engineering as a general framework for studying the complexity of various problems. Schaefer proved in 1978 that truth satisfaction of propositional formulas given a language of relations is either NP-complete or tractable. We classify the corresponding satisfying assignment construction problems in the framework of Reverse Mathematics and show that the principles are either provable over RCA_0 or equivalent to WKL_0. We formulate also a Ramseyan version of the problems and state a different dichotomy theorem. However, the different classes arising from this classification are not known to be distinct.

1 Introduction

A common way to solve a constrained problem in industry consists of reducing it to a satisfaction problem over propositional logic and using a SAT solver. The generality of the framework and its multiple applications make it a natural subject of interest for the scientific community and constraint satisfaction problems remains an active field of research.

In 1978, Schaefer [9] gave a great insight in the understanding of the complexity of satisfiability problems by studying a parameterized class of problems and showing they admit a dichotomy between NP-completeness and tractability. Many other dichotomy theorems have been proven since, about refinements to AC^0 reductions [1], variants about counting, optimization, 3-valued domains and many others [4,7,3]. The existence of dichotomies for n-valued domains with $n > 3$ remains open.

Reverse Mathematics is a vast program of classification of the strength of mathematical theorems by emphasizing on their computational content. This study has led to the main observation that many theorems are computationally equivalent to one of four axioms. On particular axiom is Weak König's lemma (WKL_0) which allows formalization of many compactness arguments and the solution of many satisfiability problems. We believe that studying constraint satisfaction problems within this framework can lead to insights in both fields: in Reverse Mathematics, we can exploit the generality of constraint satisfaction problems to compare existing principles by reducing them to satisfaction problems. In CSP, Reverse Mathematics can yield a better understanding of the computational strength of satisfiability problems for particular classes of formulas. In particular we answer to the question of Marek & Remmel [8] whether there exists dichotomy theorems for infinite recursive versions of constraint satisfaction problems.

Definition 1. *As set of Boolean formulas C is* satisfiable *if every conjunction of a finite set of formulas in C is satisfiable.* SAT *is the statement "for every satisfiable set C of Boolean formulas over an infinite set of variables V there is an infinite assignment $v : V \to \{\mathtt{T}, \mathtt{F}\}$ satisfying C." The pair (V, C) forms an* instance *of* SAT.

The weak system on which relations are based is called RCA_0, standing for Recursive Comprehension Axiom. It consists of basic Peano axioms together with a comprehension scheme restricted to Δ^0_1 formulas and an the induction restricted to Σ^0_1 formulas.

Theorem 2 (Simpson [10]). $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \longleftrightarrow \mathsf{SAT}$

RWKL, a weakening of WKL_0, has been recently introduced by Flood in [5]. Given an infinite binary tree, the principle does not assert the existence of a path, but rather of an infinite subset of a path in the tree. Initially called RKL, it has been renamed to RWKL in [2] to give a consistent R prefix to Ramseyan principles. This principle has been shown to be strictly weaker than SRT^2_2 and WKL_0 by Flood, and stricly stronger than DNR by Bienvenu & al. in [2]. By analogy with RWKL, we formulate Ramsey-type versions of satisfiability problems.

Definition 3. *Let C be a set of Boolean formulas over an infinite set of variables V. A set H is homogeneous for C if there is a $c \in \{\mathtt{T}, \mathtt{F}\}$ such that every conjunction of a finite set of formulas in C is satisfiable by a truth assignment v such that $(\forall a \in H)(v(a) = c)$.*

Definition 4. LRSAT *is the statement " Let C be a satisfiable set of Boolean formulas over an infinite set of variables V For every infinite set $L \subseteq V$ there exists an infinite set $H \subseteq L$ homogeneous for C." The corresponding* instance of LRSAT *is the tuple (V, C, L).* RSAT *is obtained by restricting* LRSAT *to $L = V$. Then an* instance of RSAT *is an ordered pair (V, C).*

The equivalence between WKL_0 and SAT over RCA_0 extends to their Ramseyan version. The proof is relatively easy and directly adaptable from proof of Theorem 2.

Theorem 5 (Bienvenu & al. [2]). $\mathsf{RCA}_0 \vdash \mathsf{RWKL} \longleftrightarrow \mathsf{RSAT} \longleftrightarrow \mathsf{LRSAT}$

1.1 Definitions and notations

Some classes of Boolean formulas – bijunctive, affine, horn, ... – have been extensively studied in Complexity Theory, leading to the well-known dichotomy theorem due to Schaefer. We give a precise definition of those classes in order to state our dichotomy theorems.

Definition 6. *A* literal *is either a Boolean variable (positive literal), or its negation (negative literal). A* clause *is a disjunction of literals. A clause is* horn *if it has at most one positive literal,* co-horn *if it has at most one negative literal and* bijunctive *if it has at most 2 literals. If we number Boolean variables, we can associate to each Boolean formula φ with Boolean variables $x_1, \dots, x_n$ a relation $[\varphi] \subseteq \{\mathtt{F}, \mathtt{T}\}^n$ such that $\boldsymbol{a} \in [\varphi]$ iff $\varphi(\boldsymbol{a})$. If S is a set of relations, an S-*formula *over a set of variables V is a formula of the form $R(y_1, \dots, y_n)$ for some $R \in S$ and $y_1, \dots, y_n \in V$.*

Example 7. Let $S = \{\rightarrow\}$. $(x \rightarrow y)$ is an S-formula but $(x \rightarrow \neg y)$ is not. Neither is $(x \rightarrow y) \wedge (y \rightarrow z)$. The formula $(x \rightarrow y)$ is equivalent to the horn clause $(\neg x \vee y)$ where the literals are $\neg x$ and y.

Definition 8. *A formula φ is* i-valid *for $i = 0, 1$ if $\varphi(i, \dots, i)$ is true. It is* horn *(resp.* co-horn*,* bijunctive*) if it is a conjunction of horn (resp. co-horn, bijunctive) clauses. A formula is* affine *if it is a conjunction of formulas of the form $x_1 \oplus \dots \oplus x_n = i$ for $i \in \{0, 1\}$ where $\oplus$ is the exclusive or.*

A relation $R \subseteq \{0, 1\}^n$ is *bijunctive* (resp. *horn*, *co-horn*, *affine*, *i-valid*) if there is bijunctive (resp. horn, co-horn, affine, i-valid) formula φ such that $R = [\varphi]$. A relation R is *i-default* for $i = 0, 1$ if for every finite set $I \subseteq \mathbb{N}$, if $\boldsymbol{r} \in R$ with $\boldsymbol{r}(k) = i$ for every $k \in I$ then $\boldsymbol{s}$, defined by $\boldsymbol{s}(k) = 1-i$ for every $k \in I$ and $\boldsymbol{s}(k) = i$ otherwise, is also in R. In particular every i-default relation is i-valid, as witnessed by taking $I = \emptyset$. We denote by $\mathsf{ISAT}(S)$ the class of satisfiable conjunctions of S-formulas.

1.2 Dichotomies

Theorem 9 (Schaefer's dichotomy [9]). *Let S be a finite set of Boolean relations. If S satisfies one of the conditions $(a) - (f)$ below, then $\mathsf{ISAT}(S)$ is polynomial-time decidable. Otherwise, $\mathsf{ISAT}(S)$ is log-complete in NP.*

(a) Every relation in S is 0-valid.
(b) Every relation in S is 1-valid.
(c) Every relation in S is horn
(d) Every relation in S is co-horn
(e) Every relation in S is affine.
(f) Every relation in S is bijunctive.

In the remainder of this paper, S will be a – possibly infinite – class of Boolean relations. Note that there is no effectiveness requirement on S.

Definition 10. $\mathsf{SAT}(S)$ *is the following statement: for every set C of S-formulas over an infinite set of variables V such that every finite set $C_0 \subseteq C$ is satisfiable there is an infinite assignment $v : V \rightarrow \{\mathtt{T}, \mathtt{F}\}$ satisfying C.*

We will prove the following dichotomy theorem based on Schaefer's theorem.

Theorem 11. *If S satisfies one of the conditions $(a) - (d)$ below, then $\mathsf{SAT}(S)$ is provable over RCA_0. Otherwise $\mathsf{SAT}(S)$ is equivalent to WKL_0 over RCA_0.*

(a) Every relation in S is 0-valid.
(b) Every relation in S is 1-valid.
(c) If $R \in S$ is not 0-default then $R = [x]$.

(d) If $R \in S$ is not 1-default then $R = [\neg x]$.

$\mathsf{SAT}(S)$ principles are not fully satisfactory as these are not robust notions: if we define $\mathsf{SAT}(S)$ in terms of satisfiable sets of *conjunctions* of S-formulas, this yields a different dichotomy theorems. In particular, $\mathsf{RCA}_0 \vdash \mathsf{SAT}([x],[\neg y])$ whereas $\mathsf{RCA}_0 \vdash \mathsf{SAT}([x \wedge \neg y]) \longleftrightarrow \mathsf{WKL}_0$. Ramseyan versions of satisfaction problems have better properties.

Definition 12. $\mathsf{RSAT}(S)$ *is the following statement: for every satisfiable set C of S-formulas over an infinite set of variables V, there is an infinite set $H \subseteq V$ homogeneous for C.*

Usual reductions between satisfiability problems involve fresh variable introductions. This is why it is natural to define a *localized* version of those principles, i.e. where the homogeneous set has to lie within a pre-specified set.

Definition 13. $\mathsf{LRSAT}(S)$ *is the following statement: for every satisfiable set C of S-formulas over an infinite set of variables V and every infinite set $X \subseteq V$, there is an infinite set $H \subseteq X$ homogeneous for C.*

In particular, we define LRSAT(0-valid) (resp. LRSAT(1-valid), LRSAT(Horn), LRSAT(CoHorn), LRSAT(Bijunctive) or LRSAT(Affine)) to denote $\mathsf{LRSAT}(S)$ where S is the set of all 0-valid (resp. 1-valid, horn, co-horn, bijunctive or affine) relations. We will prove the following dichotomy theorem.

Theorem 14. *Either $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S)$ or $\mathsf{LRSAT}(S)$ is equivalent to one of the following principles over RCA_0:*

1. LRSAT
2. $\mathsf{LRSAT}([x \neq y])$
3. LRSAT(*Affine*)
4. LRSAT(*Bijunctive*)

As we will see in Theorem 37, each of those principles are equivalent to their non localized version. As well, $\mathsf{LRSAT}([x \neq y])$ coincides with an already existing principle about bipartite graphs called RCOLOR_2 and LRSAT is equivalent to RWKL over RCA_0. Hence $\mathsf{LRSAT}(S)$ is either provable over RCA_0, or equivalent to one of RCOLOR_2, RSAT(Affine), RSAT(Bijunctive) and RWKL over RCA_0.

2 Schaefer's dichotomy theorem

Definition 15. *Let S be a class of Boolean relations and V be a set of variables. Let φ be an S-formula over V. We denote by $\mathsf{Var}(\varphi)$ the set variables occurring in φ. An* assignment *for φ is a function $v : \mathsf{Var}(\varphi) \to \{\mathtt{T}, \mathtt{F}\}$. An assignment can be naturally extended to a function over formulas by the natural interpretation rules for logical connectives. Then an assignment v* satisfies *φ if $v(\varphi) = \mathtt{T}$. The set of assignments of φ is written* $\mathsf{Assign}(\varphi)$. Variable substitution *is defined in the usual way and is written $\varphi[y/x]$, meaning that all occurrences of x in φ are replaced by y. We will also write $\varphi[y/X]$ where X is a set of variables to denote substitution of all occurrences of a variable of X in φ by y. A* constant *is either* 0 *or* 1.

Definition 16. *Let S be a class of relations over Booleans. The class of existentially quantified S-formulas with constants – i.e. of the form $(\exists \boldsymbol{x})\varphi[\boldsymbol{x},\boldsymbol{y},\mathtt{T},\mathtt{F}]$ with $\varphi \in S$ – is denoted by $Gen(S)$. We also define $Rep(S) = \{[R] : R \in Gen(S)\}$, ie. the relations represented by existentially quantified S-formula with constants. By abuse of notation, we may use $Rep(R)$ when R is a relation to denote $Rep(\{R\})$. We can also define similar relations without constants, denoted by Gen_{NC} and Rep_{NC}.*

Lemma 17 (Schaefer in [9, 4.3]). *At least one of the following holds:*

(a) *Every relation in S is 0-valid.*
(b) *Every relation in S is 1-valid.*
(c) *$[x]$ and $[\neg x]$ are contained in $Rep_{NC}(S)$.*
(d) *$[x \neq y] \in Rep_{NC}(S)$.*

One easily sees that if every relation in S is 0-valid (resp. 1-valid) then $\mathsf{RCA}_0 \vdash \mathsf{SAT}(S)$ as the assignment always equal to $\mathtt{F}$ (resp. $\mathtt{T}$) is a valid assignment and is computable. We will now see that problems parameterized by relations either 0-default or $[x]$ (resp. 1-default or $[\neg x]$) are also solvable.

Lemma 18. *If the only relation in S which is not 0-default is $[x]$ or the only relation which is not 1-default is $[\neg x]$ then $\mathsf{RCA}_0 \vdash \mathsf{SAT}(S)$.*

The strategy for solving such an instance (V,C) of $\mathsf{SAT}(S)$ consists in defining an assignment which given a variable x will give it the default value $\mathtt{F}$ unless it finds the clause $(x \vee x) \in C$.

Lemma 19. *If $[x \neq y] \in Rep_{NC}(S)$ then $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow \mathsf{SAT}(S)$.*

Lemma 19 holds because $\mathsf{SAT}([x \neq y])$ can be seen as a reformulation of COLOR_2 which is equivalent to WKL_0 over RCA_0 [6].

Theorem 11 is proven by a case analysis using Lemma 17, by noticing that when we are not in cases already handled by Lemma 18 and Lemma 19, we can find n-ary formulas encoding $[x]$ and $[\neg x]$ with $n \geq 2$. Thus diagonalizing against some values becomes a Σ^0_1 event.

3 Ramsey-type Schaefer's dichotomy theorem

Proof of Theorem 14 can be split into four steps, each of them being dichotomies themselves. The first one, Theorem 22, states the existence of a gap between provability in RCA_0 and implying RCOLOR_2 over RCA_0. Then we focus successively on two classes of boolean formulas: bijunctive formulas (Theorem 29) and affine formulas (Theorem 33) whose corresponding principles happen to be either a consequence of RCOLOR_2 or equivalent to the full class of bijunctive (resp. affine) formulas. Remaining cases are handled by Theorem 34. We first state a trivial relation between a satisfaction principle and its Ramseyan version.

Lemma 20. $\mathsf{RCA}_0 \vdash \mathsf{SAT}(S) \rightarrow \mathsf{LRSAT}(S)$

Lemma 21. *Let T be a c.e. set of Boolean relations such that $[x \neq y] \in Rep_{NC}(T)$. If $S \subseteq Rep_{NC}(T \cup \{[x],[\neg x]\})$ then $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(T) \rightarrow \mathsf{LRSAT}(S)$.*

3.1 From provability to $\mathsf{LRSAT}([x \neq y])$

Our first dichotomy for Ramseyan principles is between RCA_0 and $\mathsf{LRSAT}([x \neq y])$.

Theorem 22. *If S satisfies one of the conditions (a)-(d) below then* $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S)$. *Otherwise* $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \to \mathsf{LRSAT}([x \neq y])$.

(a) Every relation in S is 0-valid. *(c) Every relation in S is horn.*
(b) Every relation in S is 1-valid. *(d) Every relation in S is co-horn.*

Lemma 23 (Schaefer in [9, 3.2.1]). *If S contains some relation which is not horn and some relation which is not co-horn, then $[x \neq y] \in Rep(S)$.*

Lemma 24. *At least one of the following holds:*

(a) Every relation in S is 0-valid. *(d) Every relation in S is co-horn.*
(b) Every relation in S is 1-valid. *(e) $[x \neq y] \in Rep_{NC}(S)$.*
(c) Every relation in S is horn.

Proof. Assume none of cases (a), (b) and (e) holds. Then by Lemma 17, $[x]$ and $[\neg x]$ are contained in $Rep_{NC}(S)$, hence $Rep_{NC}(S) = Rep(S)$. So by Lemma 23, either every relation in S is horn, or every relation in S is co-horn. □

It is easy to see that LRSAT(0-valid) and LRSAT(1-valid) both hold over RCA_0. We will now prove that so do LRSAT(Horn) and LRSAT(CoHorn), but first we must introduce the powerful tool of *closure under functions*.

Definition 25. *We say that a relation $R \subseteq \{0,1\}^n$ is* closed *or* invariant *under an m-ary function f and that f is a* polymorphism *of R if for every m-tuple $\langle v_1, \dots, v_m \rangle$ of vectors of R, $\boldsymbol{f}(v_1, \dots, v_m) \in R$ where $\boldsymbol{f}$ is the coordinate-wise application of the function f.*

We denote the set of all polymorphisms of R by $\mathsf{Pol}(R)$, and for a set Γ of Boolean relations we define $\mathsf{Pol}(\Gamma) = \{f : f \in \mathsf{Pol}(R) \text{ for every } R \in \Gamma\}$. Similarly for a set B of Boolean functions, $\mathsf{Inv}(B) = \{R : B \subseteq \mathsf{Pol}(R)\}$ is the set of *invariants* of B. For any set S of Boolean relations, $\mathsf{Pol}(R)$ is in Post's lattice.

Definition 26. *The conjunction function* $\mathsf{conj} : \{0,1\}^2 \to \{0,1\}$ *is defined by* $\mathsf{conj}(a,b) = a \wedge b$*, the disjunction function* $\mathsf{disj} : \{0,1\}^2 \to \{0,1\}$ *by* $\mathsf{disj}(a,b) = a \vee b$*, the affine function* $\mathsf{aff} : \{0,1\}^3 \to \{0,1\}$ *by* $\mathsf{aff}(a,b,c) = a \oplus b \oplus c = 1$ *and the majority function* $\mathsf{maj} : \{0,1\}^3 \to \{0,1\}$ *by* $\mathsf{maj}(a,b,c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$.

The following theorem due to Schaefer characterizes relations in terms of closure under some functions. The proof involves finite objects and hence can be easily proven to hold over RCA_0.

Theorem 27 (Schaefer [9]). *A relation is*

1. *horn iff it is closed under conjunction function*
2. *co-horn iff it is closed under disjunction function*
3. *affine iff it is closed under affine function*

4. *bijunctive iff it is closed under majority function*

In other words, using Post's lattice, a relation R is horn iff $E_2 \subseteq \mathsf{Pol}(R)$, co-horn iff $V_2 \subseteq \mathsf{Pol}(R)$, affine iff $L_2 \subseteq \mathsf{Pol}(R)$ and bijunctive iff $D_2 \subseteq \mathsf{Pol}(R)$.

Theorem 27 is powerful because it does not only imply the closure of valid assignments under some functions. As we will see in Theorem 37, this can be interpreted as "the localized version of the principles parametrized by one of classes 1-4 is not stronger than their corresponding non-localized versions". The closure of valid assignments under some functions enables us to prove Theorem 28 below.

Theorem 28. *If every relation in S is horn (resp. co-horn) then* $\mathsf{RSAT} \vdash \mathsf{LRSAT}(S)$*.*

Proof. We will prove it over RCA_0 for the horn case. The proof for co-horn relations is similar. Let (V, C, L) be an instance of LRSAT(Horn) and $F \subseteq L$ be the collection of variables $x \in L$ such that there is a finite $C_{fin} \subseteq C$ for which every valid assignment ν for C_{fin} satisfies $\nu(x) = \mathsf{T}$.

Case 1: F is infinite. Because F is Σ^0_1, we can take a infinite Δ^0_1 subset of F as homogeneous set for C with color T.

Case 2: F is finite. We take $H = L \setminus F$ as infinite set homogeneous for C with color F. If H is not homogeneous for C, then there exists a finite $C_{fin} \subseteq C$ witnessing it. Let $H_{fin} = \mathsf{Var}(C_{fin}) \cap H$. For every valid assignment ν for C_{fin}, there is an $x \in H_{fin}$ such that $\nu(x) = \mathsf{T}$. By definition of H, for each $x \in H$ there is a valid assignment ν_x such that $\nu_x(x) = \mathsf{F}$. By Theorem 27, the class valid assignments of a finite horn formula is closed under conjunction. So $\nu = \bigwedge_{x \in H_{fin}} \nu_x$ is a valid assignment for C_{fin} such that $\nu(x) = \mathsf{F}$ for each $x \in H_{fin}$. Contradiction. □

Proof (of Theorem 22). If every relation in S is 0-valid (resp. 1-valid) then $\mathsf{LRSAT}(S)$ holds obviously over RCA_0. If every relation in S is horn (resp. co-horn) then by Theorem 28, $\mathsf{LRSAT}(S)$ holds also over RCA_0. By Lemma 24, it remains the case where $[x \neq y] \in Rep_{NC}(S)$. By Lemma 21, $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \rightarrow \mathsf{LRSAT}([x \neq y])$. □

3.2 Bijunctive satisfiability

Our second dichotomy theorem concerns bijunctive relations. Either the related principle is a consequence of $\mathsf{LRSAT}([x \neq y])$ over RCA_0, or it has full strength of LRSAT(Bijunctive). In the remaining of this subsection, we will assume that S contains only bijunctive relations and $[x \neq y] \in Rep_{NC}(S)$. In other words we suppose that $D_2 \subseteq \mathsf{Pol}(S) \subseteq D$.

Theorem 29. *If S contains only affine relations then* $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}([x \neq y]) \rightarrow \mathsf{LRSAT}(S)$*. Otherwise* $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \longleftrightarrow \mathsf{LRSAT}$(*Bijunctive*)*.*

Definition 30. *For any set S of relations, the* co-clone *of S is the closure of S by existential quantification, equality and conjunction. We denote it by $\langle S \rangle$.*

Remark that in general, $Rep_{NC}(S)$ may be different from $\langle S\rangle$ if $[x = y] \notin Rep_{NC}(S)$. However in our case, we assume that $[x \neq y] \in Rep_{NC}(S)$, hence $[x = y] \in Rep_{NC}(S)$ and $Rep_{NC}(S) = \langle S\rangle$. The following property will happen to be very useful for proving that a relation $R \in Rep_{NC}(S)$.

Lemma 31 (Folklore). $\mathsf{Inv}(\mathsf{Pol}(S)) = \langle S\rangle$

Lemma 32. *One of the following holds:*

(a) $Rep_{NC}(S)$ *contains all bijunctive relations.*
(b) $S \subseteq Rep_{NC}(\{[x], [x \neq y]\})$.

Proof. By hypothesis, $D_2 \subseteq \mathsf{Pol}(S) \subseteq D$. Either $D_1 \subseteq \mathsf{Pol}(S)$ – meaning that every relation in S is affine – in which case $S \subseteq \mathsf{Inv}(D_1) = Rep_{NC}(\{[x], [x \neq y]\})$. Or $\mathsf{Pol}(S) = D_2$. Then $Rep_{NC}(S) = \langle S\rangle = \mathsf{Inv}(\mathsf{Pol}(S)) = \mathsf{Inv}(D_2)$ which is the set of all bijunctive relations. □

Proof (of Theorem 29). By Lemma 32, either $Rep_{NC}(S)$ contains all bijunctive relations or $S \subseteq Rep_{NC}(\{[x], [x \neq y]\})$. In the latter case, by Lemma 21 $\mathsf{LRSAT}([x \neq y])$ implies $\mathsf{LRSAT}(S)$ over RCA_0. In the former case, there exists a finite basis $S_0 \subseteq S$ such that $Rep_{NC}(S_0)$ contains all bijunctive relations. In particular S_0 is a c.e. set, so $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S_0) \rightarrow \mathsf{LRSAT}(\text{Bijunctive})$. Any instance of $\mathsf{LRSAT}(S_0)$ being an instance of $\mathsf{LRSAT}(S)$, $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \rightarrow \mathsf{LRSAT}(\text{Bijunctive})$. The reverse implication follows directly from the assumption that every relation in S is bijunctive. So $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \leftrightarrow \mathsf{LRSAT}(\text{Bijunctive})$. □

3.3 Affine satisfiability

We now suppose that $L_2 \subset \mathsf{Pol}(S) \subsetneq D$, i.e. S contains only affine relations, $[x \neq y] \in Rep_{NC}(S)$ and S contains a relation which is not bijunctive.

Theorem 33. $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \leftrightarrow \mathsf{LRSAT}(\mathit{Affine})$

Proof. By assumption, every relation in S is affine. Hence $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(\text{Affine}) \rightarrow \mathsf{LRSAT}(S)$. As $L_2 \subseteq \mathsf{Pol}(S) \subsetneq D$, $\mathsf{Pol}(S)$ is either L_3 or L_2. In particular, $\mathsf{Pol}(S \cup \{[x], [\neg x]\}) = L_2$. Considering the corresponding invariants, $\mathsf{Inv}(L_2) \subseteq \mathsf{Inv}(\mathsf{Pol}(S \cup \{[x], [\neg x]\})) = \langle S \cup \{[x], [\neg x]\}\rangle = Rep_{NC}(S \cup \{[x], [\neg x]\})$. $\mathsf{Inv}(L_2)$ being the set of affine relations, by Lemma 21, $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \rightarrow \mathsf{LRSAT}(\text{Affine})$. □

3.4 Remaining cases

Based on Post's lattice, the only remaining cases are $\mathsf{Pol}(S) = N_2$ or $\mathsf{Pol}(S) = I_2$.

Theorem 34. *If* $\mathsf{Pol}(S) \subseteq N_2$ *then* $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \leftrightarrow \mathsf{LRSAT}$.

Proof. The direction $\mathsf{RCA}_0 \vdash \mathsf{LRSAT} \rightarrow \mathsf{LRSAT}(S)$ is obvious. We will prove the converse. Because $\mathsf{Pol}(S) \subseteq N_2$, $\mathsf{Pol}(S \cup \{[x]\}) = I_2$. $Rep_{NC}(S \cup \{[x]\}) = \langle S \cup \{[x]\}\rangle = \mathsf{Inv}(\mathsf{Pol}(S \cup \{[x]\})) \supseteq \mathsf{Inv}(I_2)$. But $\mathsf{Inv}(I_2)$ is the set of all Boolean relations. As $\mathsf{Inv}(I_2)$ has a finite basis, there exists a finite $S_0 \subseteq S$ such that $Rep_{NC}(S_0 \cup \{[x]\})$ contains all Boolean relations. By Lemma 21, $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S_0) \rightarrow \mathsf{LRSAT}$. Hence $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \leftrightarrow \mathsf{LRSAT}$. □

Proof (of Theorem 14). By case analysis over $\mathsf{Pol}(S)$. If I_1, I_0, V_2 and E_2 are included in $\mathsf{Pol}(S)$ then by Theorem 22, $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S)$. If $D_1 \subseteq \mathsf{Pol}(S) \subseteq D$ then $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \longleftrightarrow \mathsf{LRSAT}([x \neq y])$ by Theorem 29. By the same theorem, if $\mathsf{Pol}(S) = D_2$ then $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \longleftrightarrow \mathsf{LRSAT}(\text{Bijunctive})$. If $L_2 \subseteq \mathsf{Pol}(S) \subseteq L_3$ then by Theorem 33, $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \longleftrightarrow \mathsf{LRSAT}(\text{Affine})$. Otherwise, $I_2 \subseteq \mathsf{Pol}(S) \subseteq N_2$ in which case $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(S) \longleftrightarrow \mathsf{LRSAT}$. □

In fact, $\mathsf{LRSAT}([x \neq y])$ coincides with an already existing principle about bipartite graphs. For $k \in \mathbb{N}$, we say that a graph $G = (V, E)$ is *k-colorable* if there is a function $f : V \to k$ such that $(\forall (x, y) \in E)(f(x) \neq f(y))$, and we say that a graph is *finitely k-colorable* if every finite induced subgraph is k-colorable.

Definition 35. *Let $G = (V, E)$ be a graph. A set $H \subseteq V$ is* homogeneous for G *if every finite $V_0 \subseteq V$ induces a subgraph that is k-colorable by a coloring that colors every $v \in V_0 \cap H$ color* 0. $\mathsf{LRCOLOR}_k$ *is the following statement: for every infinite, finitely k-colorable graph $G = (V, E)$ and every infinite $L \subseteq V$ there is an infinite $H \subseteq L$ that is homogeneous for G.* RCOLOR_k *is the restriction of* $\mathsf{LRCOLOR}_k$ *with $L = V$. An instance of* $\mathsf{LRCOLOR}_k$ *is a pair (G, L). For* RCOLOR_k*, it is simply the graph G.*

Theorem 36. $\mathsf{RCA}_0 \vdash \mathsf{RCOLOR}_2 \longleftrightarrow \mathsf{LRSAT}([x \neq y])$

4 The strength of satisfiability

Localized principles are relatively easy to manipulate as they can express relations defined using existential quantifier by restricting the localized set L to the variables not captured by any quantifier. However we will see that when the set of relations has some good closure properties, the unlocalized version of the principle is as expressive as its localized one.

Theorem 37. *Let S be a c.e.* co-clone. $\mathsf{RCA}_0 \vdash \mathsf{RSAT}(S) \longleftrightarrow \mathsf{LRSAT}(S)$

Noticing that affine (resp. bijunctive) relations form a co-clone, we immediately deduce the following corollary.

Corollary 38. RSAT(*Affine*) *and* RSAT(*Bijunctive*) *are equivalent to their local version over* RCA_0.

A useful principle below WKL_0 for studying the strength of a statement is the notion of *diagonally non-computable function*.

Definition 39. *A total function f is* diagonally non-computable *if* $(\forall e) f(e) \neq \Phi_e(e)$. DNR *is the corresponding principle, i.e. for every X, there exists a function d.n.c. relative to X.*

DNR is known to coincide with the restriction of RWKL to trees of positive measure ([5,2]). On the other side, there exists an ω-model of DNR which is not a model of RCOLOR_2 ([2]). We will now prove that we can compute a diagonally non-computable function from any infinite set homogeneous for a particular set of affine formulas. As RSAT implies LRSAT(Affine) over RCA_0, it gives another proof of $\mathsf{RCA}_0 \vdash \mathsf{RWKL} \to \mathsf{DNR}$.

Theorem 40. *There exists a computable set C of affines formulas over a computable set V of variables such that every infinite set homogeneous for C computes a diagonally non-computable function.*

Corollary 41. $\mathsf{RCA}_0 \vdash \mathsf{RSAT}(\mathit{Affine}) \rightarrow \mathsf{DNR}$.

5 Conclusions

Satisfaction principles happen to collapse in the case of a full assignment existence statement. The definition is not robust and the conditions of the corresponding dichotomy theorem evolve if we make the slight modification of allowing conjunctions in our definition of formulas.

However, the proposed Ramseyan version leads to a much more robust dichotomy theorem with four main subsystems. The conditions of "tractability" – here provability over RCA_0 – differ from those of Schaefer dichotomy theorem but the considered classes of relations remain the same. We obtain the surprising result that infinite versions of Horn and co-Horn satisfaction problems are provable over RCA_0 and strictly weaker than bijunctive and affine corresponding principles, whereas the complexity classification of [1] has shown that Horn satisfiability was P-complete under AC^0 reduction, hence at least as strong as Bijunctive satisfiability which is NL-complete.

Question 42. Does RCOLOR_2 imply DNR over RCA_0 ? Does it imply RWKL ?

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