COLORING THE RATIONALS IN REVERSE MATHEMATICS

EMANUELE FRITTAION AND LUDOVIC PATEY

Abstract. Ramsey's theorem for pairs asserts that every 2-coloring of the pairs of integers has an infinite monochromatic subset. In this paper, we study a strengthening of Ramsey's theorem for pairs due to Erdős and Rado, which states that every 2-coloring of the pairs of rationals has either an infinite 0-homogeneous set or a 1-homogeneous set of order type η, where η is the order type of the rationals. This theorem is a natural candidate to lie strictly between the arithmetic comprehension axiom and Ramsey's theorem for pairs. This Erdős-Rado theorem, like the tree theorem for pairs, belongs to a family of Ramsey-type statements whose logical strength remains a challenge.

1. Introduction

In this paper, we investigate the reverse mathematics of a well-known theorem due to Erdős and Rado about 2-colorings of pairs of rationals. This theorem is a natural strengthening of Ramsey's theorem for pairs and two colors. We say that an order type α is *Ramsey*, and write $\alpha \to (\alpha)^2_2$, if for every coloring $f\colon [L]^2 \to 2$, where L is a linear order of order type α, there is a homogeneous set H such that $(H, \leq_L)$ has order type α. Ramsey's theorem for pairs and two colors asserts that ω is Ramsey. It turns out that ω and ω^* are the only countable Ramsey order types. In particular, $\eta \to (\eta)^2_2$ does not hold, where η is the order type of the rationals. A standard counterexample is as follows. Fix a one-to-one map $i\colon \mathbb{Q} \to \mathbb{N}$. Define $f\colon [\mathbb{Q}]^2 \to 2$ by letting

$$f(x, y) = \begin{cases} 0 & \text{if } x <_{\mathbb{Q}} y \wedge i(x) < i(y); \\ 1 & \text{if } x <_{\mathbb{Q}} y \wedge i(x) > i(y). \end{cases}$$

A homogeneous set of order type η would give an embedding of $\mathbb{Q}$ into ω (with color 0) or ω^* (with color 1), which is impossible. Even though Ramsey's theorem for rationals fails, Erdős and Rado [6, Theorem 4, p. 427] proved the following Ramsey-type theorem (see also Rosenstein [17, Theorem 11.7, p. 207]).

Theorem 1.1 (Erdős-Rado theorem) The partition relation $\eta \to (\aleph_0, \eta)^2$ holds.

The relation $\eta \to (\aleph_0, \eta)^2$ asserts that for every coloring $f\colon [L]^2 \to 2$, where L is a linear order of order type η, there is either an infinite 0-homogeneous set or a 1-homogeneous set H such that $(H, \leq_{\mathbb{Q}})$ has order type η.

We study Theorem 1.1 within the framework of reverse mathematics (see Simpson [21]). Reverse mathematics is a vast mathematical program whose goal is to study the logical strength of ordinary theorems in terms of set existence axioms. It uses the framework of subsystems of second-order arithmetic, with the base theory RCA_0 (recursive comprehension axiom). RCA_0 is composed of P^-, that is, the basic first-order Peano axioms for $0, 1, +, \times, <$, together with Δ^0_1-comprehension and Σ^0_1-induction with number and set parameters. RCA_0 is usually thought of as capturing *computable mathematics*. It turns out that the large majority of countable mathematics can be proven in ACA_0, where ACA_0 is RCA_0 together with arithmetic comprehension. See Hirschfeldt [8] for a gentle presentation of the reverse mathematics below ACA_0.

We formalize Theorem 1.1 in RCA_0 as follows.

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ER^2_2 For every coloring $f\colon [\mathbb{Q}]^2 \to 2$ there exists either an infinite 0-homogeneous set or a 1-homogeneous set H such that $(H, \leq_{\mathbb{Q}})$ is dense.

Here $\mathbb{Q}$ is any fixed primitive recursive presentation of the rationals. We may safely assume that the domain of $\mathbb{Q}$ is $\mathbb{N}$. Note that provably in RCA_0 every two (countable) linear orders of order type η are isomorphic and any dense linear order obviously contains a linear order of order type η. Therefore ER^2_2 is provably equivalent over RCA_0 to the statement of Theorem 1.1.

In order to study ER^2_2 we also consider a version of the infinite pigeonhole principle over the rationals, namely the statement:

ER^1 For every n and for every n-coloring $f\colon \mathbb{Q} \to n$ there exists a dense homogeneous set.

The early study of reverse mathematics has led to the observation that most of the theorems happen to be equivalent to five main subsystems of second-order arithmetic that Montalbán [13] called the "Big Five". However, Ramsey's theory provides many statements escaping this observation. Perhaps the most well-known example is Ramsey's theorem for pairs and two colors (RT^2_2). The effective analysis of Ramsey's theorem was started by Jockusch [10]. In the framework of reverse mathematics, Simpson (see [21]), building on Jockusch results, proved that whenever $n \geq 3$ and $k \geq 2$, RT^n_k is equivalent to ACA_0 over RCA_0. The case of RT^2_2 had been a long-standing open problem until Seetapun [19] proved that RT^2_2 is strictly weaker than ACA_0 over RCA_0. Cholak, Jockusch and Slaman [1] paved the way to the reverse mathematics analysis of Ramsey's theorem for pairs. Since then, many consequences of Ramsey's theorem for pairs have been studied, leading to a whole zoo of independent statements. However, no natural statement besides Ramsey's theorem for pairs (RT^2) is known to be strictly between ACA_0 and RT^2_2 over RCA_0. The only known candidate is the tree theorem for pairs (TT^2_2) studied in [2, 3, 5, 15]. We show that ER^2_2 also lies between ACA_0 and RT^2_2, and so represents another candidate, arguably more natural than TT^2_2.

Although no relation is known between them, TT^2_2 and ER^2_2 share some essential combinatorial features and put the emphasis on a new family of Ramsey-type theorems, characterized by what we call a *disjoint extension commitment*. See section 5 for a discussion on this notion. Some separations known for variants of TT^2_2 are essentially due to this common feature, which enables us to prove the same separations for variants of ER^2_2. In particular, we prove that ER^2_2 does not computably reduce to Ramsey's theorem for pairs *with an arbitrary number of colors* (RT^2). However, we cannot simply adapt this "one-step separation" to a separation over ω-models, and in particular over RCA_0, as in the case of TT^2_2 [15]. This is the first known example of such an inability. Indeed, a diagonalization against an RT^2_4-instance is similar to a diagonalization against two RT^2_2-instances. Therefore, diagonalizing against RT^2 has some common flavor with a separation over standard models.

Among the consequences of Ramsey's theorem for pairs, Ramsey's theorem for singletons (RT^1), also known as the infinite pigeonhole principle, is of particular interest. RT^1 happens to be equivalent to the Σ^0_2 bounding scheme (see Hirst [9]). The Σ^0_2 bounding scheme ($\mathsf{B}\Sigma^0_2$) is formally defined as

$$(\forall x < a)\exists y\varphi(x, y, a) \implies \exists b(\forall x < a)(\exists y < b)\varphi(x, y, n)$$

where φ is any Σ^0_2 formula. One may think of $\mathsf{B}\Sigma^0_2$ as asserting that the finite union of finite sets is finite (see for instance [7]). We show that ER^1, the corresponding pigeonhole principle for rationals, is strictly stronger than $\mathsf{B}\Sigma^0_2$, and hence has the same reverse mathematics status as the tree theorem for singletons (TT^1) [3].

For the purpose of separating ER^2_2 from RT^2 over computable reducibility, we also introduce the asymmetric version of ER^1 for two colors, namely $\mathsf{a}\text{-}\mathsf{ER}^1_2$, stating that for every partition $A_0 \cup A_1 = \mathbb{Q}$ of the rationals there exists either an infinite subset of A_0 or a dense subset of A_1. Indeed, we show the existence of a Δ^0_2-instance of $\mathsf{a}\text{-}\mathsf{ER}^1_2$, and hence of a computable instance of ER^2_2, which does not reduce to any computable instance of RT^2.

1.1. **Definitions and notation**

String. A *string* is an ordered tuple of bits $b_0, \dots, b_{n-1}$, that is, such that $b_i < 2$ for every $i < n$. The empty string is written $\langle\rangle$. A *real* is an infinite listing of bits $b_0, b_1, \dots$. Given $s \in \omega$, 2^s is the set of strings of length s and $2^{<s}$ is the set of strings of length $< s$. Similarly, $2^{<\omega}$ is the set of finite strings and 2^ω is the set of reals. Given a string $\sigma \in 2^{<\omega}$, we denote by $|\sigma|$ its length. Given two strings $\sigma, \tau \in 2^{<\omega}$, we write $\sigma^\frown\tau$ for the concatenation of σ and τ, and we say that σ is a *prefix* of τ (written $\sigma \preceq \tau$) if there exists a string $\rho \in 2^{<\omega}$ such that $\sigma^\frown\rho = \tau$. Given a real X, we write $\sigma \prec X$ if $\sigma = X{\upharpoonright}n$ for some $n \in \omega$, where $X{\upharpoonright}n$ denotes the restriction of X to its first n elements. We may identify a real with a set of integers by considering that the real is its characteristic function.

Tree, path. A *binary tree* $T \subseteq 2^{<\omega}$ is a set downward-closed under the prefix relation. A real P is a *path* though T if for every $\sigma \prec P$, $\sigma \in T$.

Sets, partitions. Given two sets A and B, we denote by $A < B$ the formula $(\forall x \in A)(\forall y \in B)[x < y]$ and by $A \subseteq^* B$ the formula $(\forall^\infty x \in A)[x \in B]$, meaning that A is included in B *up to finitely many elements.* Given a set X and some integer k, a *k-partition of X* is a k-uple of pairwise disjoint sets $A_0, \dots, A_{k-1}$ such that $A_0 \cup \dots \cup A_{k-1} = X$. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (F_1, X_1) *extends* (F, X) (written $(F_1, X_1) \le (F, X)$) if $F \subseteq F_1$, $X_1 \subseteq X$ and $F_1 \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

2. The Erdős-Rado theorem in reverse mathematics

We start off the analysis of the Erdős-Rado theorem by proving that the statement ER^2_2 lies between ACA_0 and RT^2_2. On the lower bound hand, ER^2_2 can be seen as an immediate strengthening of RT^2_2. The upper bound is an effectivization of the original proof of ER^2_2 by Erdős and Rado in [6].

Lemma 2.1 (RCA_0) $\mathsf{ER}^2_2 \to \mathsf{RT}^2_2$.

Proof. An instance of RT^2_2 can be regarded as an instance of ER^2_2. Moreover, provably in RCA_0, a dense set is infinite. ☐

The rest of this section is devoted to show that ER^2_2 is provable in ACA_0. For this purpose, we give the following definition.

Definition 2.2 (RCA_0) By *interval* we mean a set of the form $I = (x, y)_\mathbb{Q}$ for $x, y \in \mathbb{Q}$. We say that $A \subseteq \mathbb{Q}$ is *somewhere dense* if A is dense in some interval of $\mathbb{Q}$, i.e., there exists an interval I such that for all intervals $J \subseteq I$ we have that $A \cap J \neq \emptyset$. We call A *nowhere dense* otherwise.

Notice that the above notion of nowhere dense is the usual topological notion with respect to the order topology of $\mathbb{Q}$. In general, the nowhere dense sets of a topological space form an ideal. This is crucial in the proof by Erdős and Rado. For this reason, we also use the terminology *positive* and *small* for somewhere dense and nowhere dense respectively. In RCA_0 we can show that nowhere dense subsets of $\mathbb{Q}$ are small, meaning that:

(1) If $A \subseteq \mathbb{Q}$ is small and $B \subseteq A$, then B is small;
(2) If $A, B \subseteq \mathbb{Q}$ are small, then $A \cup B$ is small.

With enough induction, it is possible to generalize (2) to finitely many sets.

Lemma 2.3 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$) If A_i is a small subset of $\mathbb{Q}$ for all $i < n$, then $\bigcup_{i<n} A_i$ is small.

Proof. Suppose that A_i is small for every $i < n$. Fix an interval I. We aim to show that $A^n = \bigcup_{i<n} A_i$ is not dense in I. By Σ^0_2-induction we prove that for all $i \le n$ there exists an interval $J \subseteq I$ such that $A^i \cap J = \emptyset$, where $A^i = \bigcup_{j<i} A_j$. For $i = n$ we have the desired conclusion. The case $i = 0$ is trivial. Suppose $i + 1 \le n$. By induction there exists an interval

$J \subseteq I$ such that $A^i \cap J = \emptyset$. By the assumption A_i is small and so there exists an interval $K \subseteq J$ such that $A_i \cap K = \emptyset$. It follows that $A^{i+1} \cap K = (A^i \cup A_i) \cap K = \emptyset$. □

Theorem 2.4 ER^2_2 is provable in ACA_0.

Proof. Let $f\colon [\mathbb{Q}]^2 \to 2$ be given. For any $x \in \mathbb{Q}$, let $\mathrm{Red}(x) = \{y \in \mathbb{Q} \setminus \{x\} \colon f(x,y) = 0\}$. Define $\mathrm{Blue}(x)$ accordingly. We say that $A \subseteq \mathbb{Q}$ is *red-admissible* if there exists some $x \in A$ such that $A \cap \mathrm{Red}(x)$ is positive.

Case I. Every positive subset of $\mathbb{Q}$ is red-admissible. We aim to show that there exists an infinite 0-homogeneous set. We define by arithmetical recursion a sequence $(x_n)_{n\in\mathbb{N}}$ as follows. Suppose we have defined x_i for all $i < n$, and assume by arithmetical induction that $A_n = \bigcap_{i<n} \mathrm{Red}(x_i)$ is positive, and hence red-admissible (where $\bigcap_{i<0} \mathrm{Red}(x_i) = \mathbb{Q}$). Search for the ω-least $x_n \in A_n$ such that $A_n \cap \mathrm{Red}(x_n) = \bigcap_{i<n+1} \mathrm{Red}(x_i)$ is positive. By definition, the set $\{x_n \colon n \in \mathbb{N}\}$ is infinite and 0-homogeneous.

Case II. There is a positive subset A of $\mathbb{Q}$ which is not red-admissible. In this case, we show that there exists a dense 1-homogeneous set. Let I be a witness of A being positive. Fix an enumeration $(I_n)_{n\in\mathbb{N}}$ of all subintervals of I. Notice that by definition A intersects every I_n.

We define by arithmetical recursion a sequence $(x_n)_{n\in\mathbb{N}}$ as follows. Let $x_0 \in A \cap I_0$. Suppose we have defined $x_i \in A \cap I_i$ for all $i < n$. By Lemma 2.3, since every $A \cap \mathrm{Red}(x_i)$ with $i < n$ is small, it follows that $E = \bigcup_{i<n} \big(A \cap \mathrm{Red}(x_i)\big)$ is small. Let $J \subseteq I_n$ be such that $E \cap J = \emptyset$. We may safely assume that no x_i with $i < n$ belongs to J. Since A is dense in I and $J \subseteq I$, we can find $x_n \in A \cap J$. In particular, $x_n \in \bigcap_{i<n} \mathrm{Blue}(x_i)$. Therefore $\{x_n \colon n \in \mathbb{N}\}$ is dense and 1-homogeneous. □

Remark 2.5 A similar proof shows that RT^2_2 is provable in ACA_0. In fact, we can consider the ideal of finite sets of $\mathbb{N}$ so that a positive set is just an infinite set and a red-admissible set is a set $A \subseteq \mathbb{N}$ such that $A \cap \mathrm{Red}(x)$ is infinite for some $x \in A$.

3. Pigeonhole principle on $\mathbb{Q}$

We next consider the statement ER^1 asserting that every finite coloring of rationals has a dense homogeneous set. The main result is that ER^1 is stronger than $\mathsf{B}\Sigma^0_2$ over RCA_0. We achieve this by adapting the model-theoretic proof of Corduan, Groszek, and Mileti [3] that separates TT^1 from $\mathsf{B}\Sigma^0_2$. Basically, in a model of $\mathsf{RCA}_0 + \neg\,\mathsf{I}\Sigma^0_2$, there are a real X and an X-recursive instance of ER^1 with no X-recursive solutions. Before going into the details of this proof, we establish the following simple reverse mathematics facts.

Lemma 3.1 Over RCA_0,

1) $\mathsf{ER}^2_2 \vee \mathsf{I}\Sigma^0_2 \to \mathsf{ER}^1$
2) $\mathsf{ER}^1 \to \mathsf{RT}^1$.

Proof. 1) Let $f\colon \mathbb{Q} \to n$ be a given coloring. First assume ER^2_2, and let $g\colon [\mathbb{Q}]^2 \to 2$ be defined by $g(x,y) = 1$ if and only if $f(x) = f(y)$. Provably in RCA_0 every one-to-one function from an infinite set is unbounded. Then by ER^2_2 there exists a dense 1-homogeneous set for g, which is homogeneous for f.

Now assume $\mathsf{I}\Sigma^0_2$ and define $A_i = f^{-1}(i)$ for $i < n$. As $\mathbb{Q} = \bigcup_{i<n} A_i$ is positive, by lemma 2.3, there exists $i < n$ such that A_i is positive. From A_i we can compute a dense i-homogeneous set.

2) is trivial. □

As in [3], the proof of our separation result consists of a few lemmas. We start by first adapting [3, Lemma 3.4] (see Lemma 3.3 below). The combinatorial core of the proof is based on the following.

Lemma 3.2 ($\mathsf{I}\Sigma_1$) For each $e < n$, let Γ_e consist of $4n$ pairwise disjoint intervals of $\mathbb{Q}$. Then there exist $2n$ pairwise disjoint intervals $\langle I_{e,i} \colon e < n, i < 2\rangle$ such that $I_{e,i} \in \Gamma_e$ for all $e < n$ and $i < 2$.

Proof. Let Γ_e, $e < n$, be given. Consider the following recursive procedure. At each stage we define $\Gamma_{e,s}$ for $e < n$ and Δ_s as follows. At stage 0, $\Gamma_{e,0} = \Gamma_e$ and $\Delta_0 = \langle\rangle$. At stage $s+1$, if $|\Delta_s| = 2n$ or $\Gamma^s = \bigcup_{e<n} \Gamma_{e,s}$ is empty, we are done. Otherwise search for $I \in \Gamma^s$ minimal with respect to inclusion (such an interval exists by $\mathsf{I}\Sigma_1$). Add I to Δ_s, that is, $\Delta_{s+1} = \Delta_s{}^\frown I$. Let e be such that $I \in \Gamma_{e,s}$. If Δ_s already contains an interval in Γ_e, let $\Gamma_{e,s+1} = \emptyset$, otherwise let $\Gamma_{e,s+1} = \Gamma_{e,s} \setminus \{I\}$. For all $j \neq e$, let $\Gamma_{j,s+1} = \{J \in \Gamma_{j,s} \colon I \cap J = \emptyset\}$. Notice that by the choice of I as minimal, at most two intervals from each $\Gamma_{j,s}$ with $j \neq e$ have nonempty intersection with J.

By $\mathsf{I}\Sigma_1$ (indeed $\mathsf{I}\Sigma_0$) it is easy to show that, for all $s < 2n+1$, Δ_s consists of s disjoint intervals from $\bigcup_{e<n} \Gamma_e$ with at most two intervals from the same Γ_e, that every interval in Δ_s is disjoint from any interval in Γ^s, and that if Δ_s does not contain 2 intervals from Γ_e, then $\Gamma_{e,s}$ contains at least $4n - 2s$ intervals. In particular, Δ_{2n} is as desired. □

Lemma 3.3 ($\mathsf{RCA_0}$) For every real X there exists an X-recursive function $d \colon \mathbb{N} \times \mathbb{Q} \to 2$ such that for all n and $e < n$, if W^X_e is a dense set of $\mathbb{Q}$, then there exist two disjoint intervals I_0, I_1 such that $W^X_e \cap I_i$ is infinite for all $i < 2$ and $d(n,x) = i$ for all $i < 2$ and for almost every $x \in I_i$.

Proof sketch. Our strategy to defeat n-many dense sets $\{A_e \colon e < n\}$ is to choose $2n$ pairwise disjoint intervals $I_{e,0}, I_{e,1}$ for $e < n$ so that each $I_{e,i}$ has end-points in A_e, and assign color i to the interval $I_{e,i}$ for all $e < n$ and $i < 2$. As we want to diagonalize against n-many potential dense sets of the form W^X_e for $e < n$ and we cannot decide uniformly in n which ones are dense, we act only when some W^X_e outputs $4n+1$ points. We then specify a set Γ_e of $4n$ disjoint intervals with end-points in W^X_e and from each Γ_e currently defined we choose intervals $I_{e,0}$ and $I_{e,1}$ as in Lemma 3.2. Every time we act, our choice of $I_{e,0}$ and $I_{e,1}$ might change, but this happens at most n-many times. As the actual construction is essentially the one in the proof of [3, Lemma 3.4], we leave the details to the reader. □

The next lemma is the key part of the whole argument (see [3, Proposition 3.5]).

Lemma 3.4 Let M be a model of $\mathsf{RCA_0} + \neg\mathsf{I}\Sigma^0_2$. Then for some real $X \in M$ there is an X-recursive (in the sense of M) coloring f of $\mathbb{Q}$ into M-finitely many colors such that no X-recursive dense set is homogeneous for f.

Proof. Let $X \in M$ witness the failure of $\mathsf{I}\Sigma^0_2$. Then there exists an X-recursive function $h \colon \mathbb{N}^2 \to \mathbb{N}$ such that for some number a, the range of the partial function $h(y) = \lim_{s\to\infty} h(y,s)$ is unbounded on $\{y \colon y < a\}$ (see also [3, Lemma 3.6]). Define $f \colon \mathbb{Q} \to 2^a$ by

$$f(x) = \langle d(h(y,x),x) \colon y < a\rangle,$$

where $d(n,x)$ is the function of Lemma 3.3. Let W^X_e be a dense set of $\mathbb{Q}$. We aim to show that W^X_e is not homogeneous for f. Let $y < a$ such that $h(y) > e$. Observe that for almost every $x \in \mathbb{Q}$ the yth bit of $f(x)$ is $d(h(y),x)$. As $e < h(y)$, let I_0 and I_1 be two intervals as in Lemma 3.3. Now for sufficiently large $x_0 \in W^X_e \cap I_0$ and $x_1 \in W^X_e \cap I_1$ we have $d(h(y,x_i),x_i) = d(h(y),x_i) = i$, and hence $f(x_0) \neq f(x_1)$. □

We can finally prove the analogue of [3, Corollary 3.8], which is the main result.

Theorem 3.5 Let P be a Π^1_1 sentence. Then $\mathsf{RCA_0} + P \vdash \mathsf{ER}^1$ if and only if $\mathsf{RCA_0} + P \vdash \mathsf{I}\Sigma^0_2$. In particular, $\mathsf{RCA_0} + \mathsf{B}\Sigma^0_2 \nvdash \mathsf{ER}^1$.

Proof. The argument is the same as in the proof of [3, Theorem 3.7]. As $\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2 \vdash \mathsf{ER}^1$, we just need to prove one implication. Suppose that $\mathsf{RCA_0} + P \nvdash \mathsf{I}\Sigma^0_2$, and let M be a model of $\mathsf{RCA_0} + P$ where $\mathsf{I}\Sigma^0_2$ fails. By Lemma 3.4, for some real $X \in M$, there exists an X-recursive instance of ER^1 with no X-recursive solutions. Let M' be the submodel of M with the same first-order part as M and second-order part consisting of the reals recursive in X (in the sense of M). Therefore ER^1 fails in M'. Since M' has same first-order part as M, M' satisfies the Π^1_1

sentence P. As the reals of M' are the ones recursive in a given real of M, M satisfies RCA_0. Thus $\mathsf{RCA}_0 + P \not\vdash \mathsf{ER}^1$. □

4. ER^2_2 does not computably reduce to RT^2_2

Many proofs of $\mathsf{Q} \to \mathsf{P}$ over RCA_0 make use only of one Q-instance to solve a P-instance. This is the notion of computable reducibility.

Definition 4.1 (Computable reducibility) Fix two Π^1_2 statements P and Q. P is *computably reducible* to Q (written $\mathsf{P} \leq_c \mathsf{Q}$) if every P-instance X_0 computes a Q-instance X_1 such that for every solution Y to X_1, $Y \oplus X_0$ computes a solution to X_0.

Proving that $\mathsf{P} \leq_c \mathsf{Q}$ is not sufficient to deduce that $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$. One needs to prove that this reducibility can be formalized within RCA_0, and in particular that Σ^0_1-induction is sufficient to prove its validity. The fine-grained nature of computable reducibility enables one to exhibit distinctions between statements which would not have been revealed in reverse mathematics. For example, RT^2_k and RT^2_{k+1} are equivalent over RCA_0 whereas $\mathsf{RT}^2_{k+1} \not\leq_c \mathsf{RT}^2_k$ [16].

This notion of reducibility can be also seen as an intermediary step to tackle difficult separations [4]. Proving that $\mathsf{P} \not\leq_c \mathsf{Q}$ is simpler than separating Q from P over ω-models. Lerman, Solomon and Towsner [12] introduced a framework to separate Ramsey-type statements over ω-models, in which they transform a one-step diagonalization, that is, computable non-reducibility, into a separation in the sense of reverse mathematics. In this section, we prove that the Erdős-Rado theorem for pairs does not reduce to Ramsey's theorem for pairs in one step.

Theorem 4.2 $\mathsf{ER}^2_2 \not\leq_c \mathsf{RT}^2$.

Interestingly, this diagonalization does not seem to be easily generalizable to a separation over ω-models. A reason is that the fairness property ensured by the ER^2_2-instance does not seem to be preserved by weak König's lemma. This is hitherto the first example of a computable non-reducibility of a principle P to RT^2 which is not generalizable to a proof that RT^2_2 does not imply P over RCA_0.

The remainder of this section is devoted to a proof of Theorem 4.2. The notion of fairness presented below may have some ad-hoc flavor. It has been obtained by applying the main ideas of the framework of Lerman, Solomon and Towsner [12, 14]. Thanks to an analysis of the combinatorics of Ramsey's theorem for pairs and the Erdős-Rado theorem for pairs, we prove our computable non-reducibility result by constructing an instance of ER^2_2 ensuring the density of the diagonalizing conditions in the forcing notion of RT^2_2. Then we abstract the diagonalization to any Σ^0_1 formula, to get rid of the specificities of the forcing notion of RT^2_2 in the notion of fairness preservation. See [15] for a detailed example of the various steps of this framework, leading to a separation of RT^2_2 from the tree theorem for pairs over RCA_0.

Definition 4.3 (Simple partition) A *simple partition* $\mathrm{int}_{\mathbb{Q}}(S)$ is a finite sequence of open intervals $(-\infty, x_0), (x_0, x_1), \dots, (x_{n-1}, +\infty)$ for some set of rationals $S = \{x_0 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} x_{n-1}\}$. We set $\mathrm{int}_{\mathbb{Q}}(\emptyset) = \{\mathbb{Q}\}$. A simple partition $I_0, \dots, I_{n-1}$ *refines* another simple partition $J_0, \dots, J_{m-1}$ if for every $i < n$, there is some $j < m$ such that $I_i \subseteq J_j$. Given two simple partitions $I_0, \dots, I_{n-1}$ and $J_0, \dots, J_{m-1}$, the product $\vec{I} \otimes \vec{J}$ is the simple partition

$$\{I \cap J : I \in \vec{I} \wedge J \in \vec{J}\}$$

One can easily see that $\mathrm{int}_{\mathbb{Q}}(S)$ refines $\mathrm{int}_{\mathbb{Q}}(T)$ if $T \subseteq S$ and that $\mathrm{int}_{\mathbb{Q}}(S \cup T) = \mathrm{int}_{\mathbb{Q}}(S) \otimes \mathrm{int}_{\mathbb{Q}}(T)$. Note that every simple partition has a finite description, since the set S and each rational has a finite description. Also note that a simple partition is not a true partition of $\mathbb{Q}$ since the endpoints do not belong to any interval. However, we have $S \cup \bigcup \mathrm{int}_{\mathbb{Q}}(S) = \mathbb{Q}$.

Definition 4.4 (Matrix) An *m-by-n matrix* M is a rectangular array of rationals $x_{i,j} \in \mathbb{Q}$ such that $x_{i,j} <_{\mathbb{Q}} x_{i,k}$ for each $i < m$ and $j < k < n$. The ith *row* $M(i)$ of the matrix M is the n-tuple of rationals $x_{i,0} < \dots < x_{i,n-1}$. The simple partition $\mathrm{int}_{\mathbb{Q}}(M)$ is defined by $\bigotimes_{i<m} \mathrm{int}_{\mathbb{Q}}(M(i))$. In particular, $\bigotimes_{i<m} \mathrm{int}_{\mathbb{Q}}(M(i))$ refines the simple partition $\mathrm{int}_{\mathbb{Q}}(M(i))$ for each $i < m$.

It is important to notice that an m-by-n matrix is formally a 3-tuple $\langle m, n, M\rangle$ and not only the matrix itself M. This distinction becomes important when dealing with the degenerate cases. An m-by-0 matrix M and a 0-by-n matrix N are both empty. However, they have different sizes. In particular, we shall define the notion of M-type for a matrix, and this definition will depend on the number of columns of the matrix M, which is 0 for M, and n for N. Notice also that, for a degenerate matrix M, the simple partition $\mathrm{int}_{\mathbb{Q}}(M)$ is the singleton $\{\mathbb{Q}\}$.

Given a simple partition $\vec{I}$, we want to classify the k-tuples of rationals according to which interval of $\vec{I}$ they belong to. This leads to the notion of $(\vec{I}, k)$-type.

Definition 4.5 (Type) Given a simple partition $I_0, \dots, I_{n-1}$ and some $k \in \omega$, an *$(\vec{I}, k)$-type* is a tuple $T_0, \dots, T_{k-1}$ such that $T_i \in \vec{I}$ for each $i < k$. Given an m-by-n matrix M, an *M-type* is an $(\mathrm{int}_{\mathbb{Q}}(M), n)$-type.

We now state two simple combinatorial lemmas which will be useful later. The first trivial lemma simply states that each m-tuple of rationals (different from the endpoints of a simple partition) belongs to a type.

Lemma 4.6 For every simple partition $I_0, \dots, I_{n-1}$ and every k-tuple of rationals $x_0, \dots, x_{k-1} \in \bigcup_{i<n} I_i$, there is an $(\vec{I}, k)$-type $T_0, \dots, T_{k-1}$ such that $x_j \in T_j$ for each $j < k$.

Proof. Fix k rationals $x_0, \dots, x_{k-1}$. For each $i < k$, there is some interval $T_i \in \vec{I}$ such that $x_i \in T_i$ since $x_i \in \bigcup_{j<n} I_j$. The sequence $T_0, \dots, T_{k-1}$ is the desired $(\vec{I}, k)$-type. □

The next lemma is a consequence of the pigeonhole principle.

Lemma 4.7 For every m-by-n matrix M and every M-type $T_0, \dots, T_{n-1}$, there is an m-tuple of intervals $J_0, \dots, J_{m-1}$ with $J_i \in \mathrm{int}_{\mathbb{Q}}(M(i))$ such that

$$(\bigcup_{j<n} T_j) \cap (\bigcup_{i<m} J_i) = \emptyset$$

Proof. Let $T_0, \dots, T_{n-1}$ be an M-type. For every $i < m$ and $j < n$, there is some $J \in \mathrm{int}_{\mathbb{Q}}(M(i))$ such that $T_j \subseteq J$. Since $|\mathrm{int}_{\mathbb{Q}}(M(i))| = n + 1$, there is an interval $J_i \in \mathrm{int}_{\mathbb{Q}}(M(i))$ such that $(\bigcup_{j<n} T_j) \cap J_i = \emptyset$. □

Definition 4.8 (Formula, valuation) Given an m-by-n matrix M, an *M-formula* is a formula $\varphi(\vec{U}, \vec{V})$ with distinguished (finite coded) set variables U_j for each $j < n$ and $V_{i,I}$ for each $i < m$ and $I \in \mathrm{int}_{\mathbb{Q}}(M(i))$. An *$M$-valuation* $(\vec{R}, \vec{S})$ is a tuple of finite sets $R_j \subseteq \mathbb{Q}$ for each $j < n$ and $S_{i,I} \subseteq I$ for each $i < m$ and $I \in \mathrm{int}_{\mathbb{Q}}(M(i))$. The M-valuation $(\vec{R}, \vec{S})$ is of type $\vec{T}$ for some M-type $T_0, \dots, T_{n-1}$ if moreover $R_j \subseteq T_j$ for each $j < n$. The M-valuation $(\vec{R}, \vec{S})$ *satisfies* φ if $\varphi(\vec{R}, \vec{S})$ holds.

Given some valuation $(\vec{R}, \vec{S})$ and some integer s, we write $(\vec{R}, \vec{S}) > s$ to say that for every $x \in (\bigcup \vec{R}) \cup (\bigcup \vec{S})$, $x > s$. Following the terminology of [12], we define the notion of essentiality for a formula (an abstract requirement), which corresponds to the idea that there is room for diagonalization since the formula is satisfied by valuations which are arbitrarily far.

Definition 4.9 (Essential formula) Given an m-by-n matrix M, an M-formula φ is *essential* if for every $s \in \omega$, there are an M-type $\vec{T}$ and an M-valuation $(\vec{R}, \vec{S}) > s$ of type $\vec{T}$ such that $\varphi(\vec{R}, \vec{S})$ holds.

The notion of fairness is defined accordingly. If some formula is essential, that is, leaves enough room for diagonalization, then there is an actual valuation which will diagonalize against the ER^2_2-instance.

Definition 4.10 (Fairness) Fix two sets $A_0, A_1 \subseteq \mathbb{Q}$. Given an m-by-n matrix M, an M-valuation $(\vec{R}, \vec{S})$ *diagonalizes* against A_0, A_1 if $\bigcup \vec{R} \subseteq A_1$ and for every $i < m$, there is some $I \in \mathrm{int}_{\mathbb{Q}}(M(i))$ such that $S_{i,I} \subseteq A_0$. A set X is *fair* for A_0, A_1 if for every $m, n \in \omega$, every m-by-n matrix M and every $\Sigma^{0,X}_1$ essential M-formula, there is an M-valuation $(\vec{R}, \vec{S})$ diagonalizing against A_0, A_1 such that $\varphi(\vec{R}, \vec{S})$ holds.

Of course, if $Y \leq_T X$, then every $\Sigma^{0,Y}_1$ formula is $\Sigma^{0,X}_1$. As an immediate consequence, if X is fair for some A_0, A_1 and $Y \leq_T X$, then Y is fair for A_0, A_1.

Now that we have introduced the necessary terminology, we create a non-effective instance of $\mathsf{a}\text{-}\mathsf{ER}^1_2$ which will serve as a bootstrap for fairness preservation. Remember that *erps* asserts that for every partition $A_0 \cup A_1 = \mathbb{Q}$ of the rationals there exists either an infinite subset of A_0 or a dense subset of A_1.

Lemma 4.11 For every set C, there exists a $\Delta^{0,C}_2$ partition $A_0 \cup A_1 = \mathbb{Q}$ such that C is fair for A_0, A_1.

Proof. The proof is a no-injury priority construction. Let $M_0, M_1, \dots$ be an enumeration of all m-by-n matrices and $\varphi_0, \varphi_1, \dots$ be an effective enumeration of all $\Sigma^{0,C}_1$ M_k-formulas for every $m, n \in \omega$. We want to satisfy the following requirements for each pair of integers e, k.

> $\mathcal{R}_{e,k}$: If the M_k-formula φ_e is essential, then $\varphi_e(\vec{R}, \vec{S})$ holds for some M_k-valuation $(\vec{R}, \vec{S})$ diagonalizing against A_0, A_1.

The requirements are ordered via the standard pairing function $\langle \cdot, \cdot \rangle$. The sets A_0 and A_1 are constructed by a C'-computable list of finite approximations $A_{i,0} \subseteq A_{i,1} \subseteq \dots$ such that all elements added to $A_{i,s+1}$ from $A_{i,s}$ are strictly greater than the maximum of $A_{i,s}$ (in the $\mathbb{N}$ order) for each $i < 2$. We then let $A_i = \bigcup_s A_{i,s}$ which will be a $\Delta^{0,C}_2$ set. At stage 0, set $A_{0,0} = A_{1,0} = \emptyset$. Suppose that at stage s, we have defined two disjoint finite sets $A_{0,s}$ and $A_{1,s}$ such that

(i) $A_{0,s} \cup A_{1,s} = [0, b]_{\mathbb{N}}$ for some integer $b \geq s$
(ii) $\mathcal{R}_{e',k'}$ is satisfied for every $\langle e', k' \rangle < s$

Let $\mathcal{R}_{e,k}$ be the requirement such that $\langle e, k \rangle = s$. Decide C'-computably whether there are some M_k-type $\vec{T}$ and some M_k-valuation $V = (\vec{R}, \vec{S}) > b$ of type $\vec{T}$ such that $\varphi_e(V)$ holds. If so, C-effectively fetch $\vec{T} = T_0, \dots, T_{n-1}$ and such a $(\vec{R}, \vec{S}) > b$. Let d be an upper bound (in the $\mathbb{N}$ order) on the rationals in $(\vec{R}, \vec{S})$. By Lemma 4.7, for each $i < m$, there is some $J_i \in \mathrm{int}_{\mathbb{Q}}(M(i))$ such that

$$(\bigcup_{j<n} T_j) \cap (\bigcup_{i<m} J_i) = \emptyset$$

Set $A_{0,s+1} = A_{0,s} \cup \bigcup_{i<m} J_i \cap (b, d]_{\mathbb{N}}$ and $A_{1,s+1} = [0, d]_{\mathbb{N}} \setminus A_{0,s+1}$. This way, $A_{0,s+1} \cup A_{1,s+1} = [0, d]_{\mathbb{N}}$. By the previous equation, $\bigcup_{j<n} T_j \cap (b, d]_{\mathbb{N}} \subseteq [0, d]_{\mathbb{N}} \setminus A_{0,s+1}$ and the requirement $\mathcal{R}_{e,k}$ is satisfied. If no such M_k-valuation is found, the requirement $\mathcal{R}_{e,k}$ is vacuously satisfied. Set $A_{0,s+1} = A_{0,s} \cup \{b+1\}$ and $A_{1,s+1} = A_{1,s}$. This way, $A_{0,s+1} \cup A_{1,s+1} = [0, b+1]_{\mathbb{N}}$. In any case, go to the next stage. This finishes the construction. □

Lemma 4.12 If X is fair for some sets $A_0, A_1 \subseteq \mathbb{Q}$, then X computes neither an infinite subset of A_0, nor a dense subset of A_1.

Proof. Since fairness is downward-closed under Turing reducibility, it suffices to prove that if X is infinite and fair for A_0, A_1, then it intersects both A_0 and A_1.

We first prove that X intersects A_1. Let M be the 0-by-1 matrix and $\varphi(U)$ be the $\Sigma^{0,X}_1$ M-formula which holds if $U \cap X \neq \emptyset$. The only M-type is $\mathbb{Q}$ and since X is infinite, φ is essential.

By fairness of X, there is an M-valuation R diagonalizing against A_0, A_1 such that $\varphi(R)$ holds. By definition of diagonalization, $R \subseteq A_1$. Since $R \cap X \neq \emptyset$, this shows that $X \cap A_1 \neq \emptyset$.

We now prove that X intersects A_0. Let M be the 1-by-0 matrix and $\varphi(V)$ be the $\Sigma^{0,X}_1$ M-formula which holds if $V \cap X \neq \emptyset$. The M-formula φ is essential since X is infinite. By fairness of X, there is an M-valuation S diagonalizing against A_0, A_1 such that $\varphi(S)$ holds. By definition of diagonalization, $S \subseteq A_0$. Since $S \cap X \neq \emptyset$, this shows that $X \cap A_0 \neq \emptyset$. □

Note that we did not use the fact that X is dense to make sure it intersects A_0. Density will be useful in the proof of Theorem 4.14.

Definition 4.13 A *Scott set* is a set $\mathcal{S} \subseteq 2^\omega$ such that

(i) $(\forall X \in \mathcal{S})(\forall Y \leq_T X)[Y \in \mathcal{S}]$
(ii) $(\forall X, Y \in \mathcal{S})[X \oplus Y \in \mathcal{S}]$
(iii) Every infinite, binary tree in $\mathcal{S}$ has an infinite path in $\mathcal{S}$.

Theorem 4.14 Let $A_0, A_1 \subseteq \mathbb{Q}$ and $\mathcal{S}$ be a Scott set whose members are all fair for A_0, A_1. For every set $C \in \mathcal{S}$, every C-computable coloring $f : [\omega]^2 \to k$, there is an infinite f-homogeneous set H such that $H \oplus C$ computes neither an infinite subset of A_0, nor a dense subset of A_1.

Proof. The proof is by induction over the number of colors k. The case $k = 1$ is ensured by Lemma 4.12. Fix a set $C \in \mathcal{S}$ and let $f : [\omega]^2 \to k$ be a C-computable coloring. If f has an infinite f-thin set $H \in \mathcal{S}$, that is, an infinite set over which f avoids at least one color, then $H \oplus C$ computes a coloring $g : [\omega]^2 \to k-1$ such that every infinite g-homogeneous set computes relative to $H \oplus C$ an infinite f-homogeneous set. Since $H \oplus C \in \mathcal{S}$, by induction hypothesis, there is an infinite g-homogeneous set H_1 such that $H_1 \oplus H \oplus C$ computes neither an infinite subset of A_0, nor a dense subset of A_1. So suppose that f has no infinite f-thin set in $\mathcal{S}$.

We construct k infinite sets $G_0, \dots, G_{k-1}$. We need therefore to satisfy the following requirements for each $p \in \omega$.

$$\mathcal{N}_p : \qquad (\exists q_0 > p)[q_0 \in G_0] \qquad \wedge \dots \wedge \qquad (\exists q_{k-1} > p)[q_{k-1} \in G_{k-1}]$$

Furthermore, we want to ensure that one of the G's computes neither an infinite subset of A_0, nor a dense subset of A_1. To do this, we will satisfy the following requirements for every k-tuple of integers $e_0, \dots, e_{k-1}$.

$$\mathcal{Q}_{\vec{e}} : \qquad \mathcal{R}^{G_0}_{e_0} \qquad \vee \dots \vee \qquad \mathcal{R}^{G_{k-1}}_{e_{k-1}}$$

where $\mathcal{R}^H_e$ holds if $W^{H \oplus C}_e$ is neither an infinite subset of A_0, nor a dense subset of A_1.

We construct our sets $G_0, \dots, G_{k-1}$ by forcing. Our conditions are variants of Mathias conditions $(F_0, \dots, F_{k-1}, X)$ such that each X is an infinite set in $\mathcal{S}$, each F_i is a finite set with $\max(F_i) < \min(X)$, and the following property holds:

(P) $(\forall i < k)(\forall x \in X)[F_i \cup \{x\}$ is f-homogeneous with color $i]$

A condition $d = (E_0, \dots, E_{k-1}, Y)$ *extends* $c = (F_0, \dots, F_{k-1}, X)$ if (E_i, Y) Mathias extends (F_i, X) for every $i < k$. We now prove the progress lemma, stating that we can force the G's to be infinite. This is where we use the fact that there is no infinite f-thin set in $\mathcal{S}$.

Lemma 4.15 For every condition $c = (F_0, \dots, F_{k-1}, X)$, every $i < k$ and every $p \in \omega$ there is some extension $d = (E_0, \dots, E_{k-1}, Y)$ such that $E_i \cap (p, +\infty)_\mathbb{N} \neq \emptyset$.

Proof. Fix c, i and p. If for every $x \in X \cap (p, +\infty)_\mathbb{N}$ and almost every $y \in X$, $f(x, y) \neq i$, then X computes an infinite f-thin set, contradicting our hypothesis. Therefore, there is some $x \in X \cap (p, +\infty)_\mathbb{N}$ such that $f(x, y) = i$ for infinitely many $y \in X$. Let Y be the collection of such y's. The condition $(F_0, \dots, F_{i-1}, F \cup \{x\}, F_{i+1}, \dots, F_k, Y)$ is the desired extension. □

We now prove the core lemma stating that we can satisfy each $\mathcal{Q}$-requirement. A condition c *forces* a requirement $\mathcal{Q}$ if $\mathcal{Q}$ holds for every set G satisfying c.

Lemma 4.16 For every condition $c = (F_0, \dots, F_{k-1}, X)$ and every k-tuple of indices $\vec{e}$, there is an extension $d = (E_0, \dots, E_{k-1}, Y)$ forcing $\mathcal{Q}_{\vec{e}}$.

Proof. We can assume that $W_{e_i}^{F_i \oplus C}$ has already outputted at least k elements and is either included in A_0 or in A_1 for each $i < k$. Indeed, if c has no extension satisfying this condition, then c forces $W_{e_i}^{G_i \oplus C}$ to be finite or not to be a valid solution for some $i < k$ and therefore forces $\mathcal{Q}_{\vec{e}}$. For each $i < k$, we associate the label $\ell_i < 2$ and the number p_i such that $W_{e_i}^{F_i \oplus C}$ is the $(p_i + 1)$th set of this form included in A_{ℓ_i}.

Let n be the number of sets $W_{e_i}^{F_i \oplus C}$ which are included in A_0, and let M be the $(k-n)$-by-n matrix such that the jth row is composed of the n first elements already outputted by the set $W_{e_i}^{F_i \oplus C}$ where $p_i = j$ and $\ell_i = 1$. In other words, $M(j)$ are the n first elements outputted by the jth set $W_{e_i}^{F_i \oplus C}$ included in A_1.

Let $\varphi(\vec{U}, \vec{V})$ be the $\Sigma_1^{0, X \oplus C}$ formula which holds if there is a finite set $Z \subseteq X$ such that for every k-partition $Z_0 \cup \dots \cup Z_{k-1} = Z$, there are some $i < k$ and some set $E \subseteq Z_i$ which is f-homogeneous with color i and such that either $\ell_i = 0$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap U_{p_i} \neq \emptyset$, or $\ell_i = 1$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap V_{p_i, I} \neq \emptyset$ for each $I \in \mathrm{int}_{\mathbb{Q}}(M(p_i))$. We have two cases.

In the first case, $\varphi(\vec{U}, \vec{V})$ is essential. Since $X \oplus C$ is fair for A_0, A_1, there is an M-valuation $(\vec{R}, \vec{S})$ diagonalizing against A_0, A_1 such that $\varphi(\vec{R}, \vec{S})$ holds. By compactness and definition of diagonalization against A_0, A_1, there is a finite subset $D \subset X$ such that for every k-partition $D_0 \cup \dots \cup D_{k-1} = D$, there are some $i < k$ and some finite set $E \subseteq D_i$ which is f-homogeneous with color i and such that either $\ell_i = 0$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap A_1 \neq \emptyset$, or $\ell_i = 1$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap A_0 \neq \emptyset$.

Each $y \in X \setminus D$ induces a k-partition $D_0 \cup \dots \cup D_{k-1}$ of D by setting $D_i = \{x \in D : f(x, y) = i\}$. Since there are finitely many possible k-partitions of D, there are a k-partition $D_0 \cup \dots \cup D_{k-1} = D$ and an infinite X-computable set $Y \subseteq X$ such that

$$(\forall i < k)(\forall x \in D_i)(\forall y \in Y)[f(x, y) = i]$$

We furthermore assume that $min(Y)$ is larger than the use of the computations. Let $i < k$ and $E \subseteq D_i$ be the f-homogeneous set with color i such that either $\ell_i = 0$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap A_1 \neq \emptyset$, or $\ell_i = 1$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap A_0 \neq \emptyset$. The condition $(F_0, \dots, F_{i-1}, F_i \cup E, F_{i+1}, \dots, F_{k-1}, Y)$ is an extension of c forcing $\mathcal{Q}_{\vec{e}}$ by the ith side.

In the second case, there is some threshold $s \in \omega$ such that for every M-type $\vec{T}$, there is no M-valuation $(\vec{R}, \vec{S}) > s$ of type $\vec{T}$ such that $\varphi(\vec{R}, \vec{S})$ holds. By compactness, it follows that for every M-type $\vec{T}$, the $\Pi_1^{0, X \oplus C}$ class $\mathcal{C}_{\vec{T}}$ of all k-partitions $Z_0 \cup \dots \cup Z_{k-1} = X$ such that for every $i < k$ and every finite set $E \subseteq Z_i$ which is f-homogeneous with color i, either $\ell_i = 0$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap T_{p_i} \cap (s, +\infty)_{\mathbb{N}} = \emptyset$, or $\ell_i = 1$ and $W_{e_i}^{(F_i \cup E) \oplus C} \cap I \cap (s, +\infty)_{\mathbb{N}} = \emptyset$ for some $I \in \mathrm{int}_{\mathbb{Q}}(M(p_i))$ is non-empty. Since $\mathcal{S}$ is a Scott set, for each M-type $\vec{T}$, there is a k-partition $\vec{Z}^{\vec{T}} \in \mathcal{C}_{\vec{T}}$ such that $\bigoplus_{\vec{T}} \vec{Z}^{\vec{T}} \oplus X \oplus C \in \mathcal{S}$.

If there are some M-type $\vec{T}$ and some $i < k$ such that $\ell_i = 1$ and $Z_i^{\vec{T}}$ is infinite, then the condition $(F_0, \dots, F_{k-1}, Z_i^{\vec{T}})$ extends X and forces $W_{e_i}^{G_i \oplus C}$ not to be dense. So suppose that it is not the case. Let $Y \in \mathcal{S}$ be an infinite subset of X such that for each M-type $\vec{T}$, there is some $i < k$ such that $Y \subseteq Z_i^{\vec{T}}$. Note that by the previous assumption, $\ell_i = 0$ for every such i. We claim that the condition $(F_0, \dots, F_{k-1}, Y)$ forces $W_{e_i}^{G_i \oplus C}$ to be finite for some $i < k$ such that $\ell_i = 0$. Suppose for the sake of contradiction that there are some rationals $x_0, \dots, x_{n-1} > s$ such that $x_{p_i} \in W_{e_i}^{G_i \oplus C}$ for each $i < k$ where $\ell_i = 0$. Since $x_0, \dots, x_{n-1} > s$, $x_0, \dots, x_{n-1} \in \bigcup \mathrm{int}_{\mathbb{Q}}(M)$. Therefore, by Lemma 4.6, let $\vec{T}$ be the unique M-type such that $x_j \in T_j$ for each $j < n$. By assumption, there is some $i < k$ such that $Y \subseteq Z_i^{\vec{T}}$ and $\ell_i = 0$. By definition of $Z_i^{\vec{T}}$, $W_{e_i}^{G_i \oplus C} \cap T_{p_i} \cap (s, +\infty)_{\mathbb{N}} = \emptyset$, contradicting $x_{p_i} \in W_{e_i}^{G_i \oplus C}$. □

Using Lemma 4.15 and Lemma 4.16, define an infinite descending sequence of conditions $c_0 = (\emptyset, \dots, \emptyset, \omega) \geq c_1 \geq \dots$ such that for each $s \in \omega$

(i) $|F_{i,s}| \geq s$ for each $i < k$
(ii) c_{s+1} forces $\mathcal{Q}_{\vec{e}}$ if $s = \langle e_0, \dots, e_{k-1} \rangle$

where $c_s = (F_{0,s}, \dots, F_{k-1,s}, X_s)$. Let $G_i = \bigcup_s F_{i,s}$ for each $i < k$. The G's are all infinite by (i) and G_i does not compute an $\mathsf{a\text{-}ER}^1_2$-solution to the A's for some $i < k$ by (ii). This finishes the proof of Theorem 4.14. □

We are now ready to prove the main theorem.

Proof of Theorem 4.2. By the low basis theorem [11], there is a low set P of PA degree. By Scott [18], every PA degree bounds a Scott set. Let $\mathcal{S}$ be a Scott set such that $X \leq_T P$ for every $X \in \mathcal{S}$. By Lemma 4.11, there is a $\Delta^{0,P}_2$ (hence Δ^0_2) partition $A_0 \cup A_1 = \mathbb{Q}$ such that P is fair for A_0, A_1. In particular, every set $X \in \mathcal{S}$ is fair for A_0, A_1 since fairness is downward-closed under the Turing reducibility.

By Schoenfield's limit lemma [20], there is a computable function $h : [\mathbb{Q}]^2 \to 2$ such that for each $x \in \mathbb{Q}$, $\lim_s h(x,s)$ exists and $x \in A_{\lim_s h(x,s)}$. Note that for every infinite set D 0-homogeneous for h, $D \subseteq A_0$, and for every dense set D 1-homogeneous for h, $D \subseteq A_1$.

Fix a computable RT^2-instance $f : [\omega]^2 \to k$. In particular, $f \in \mathcal{S}$. By Theorem 4.14, there is an infinite f-homogeneous set H such that H computes neither an infinite subset of A_0, nor a dense subset of A_1. Therefore, H computes no ER^2_2-solution to h. □

5. Discussion and questions

This Erdős-Rado theorem shares an essential feature with another strengthening of Ramsey's theorem for pairs already studied in reverse mathematics: the tree theorem for pairs [2, 3, 5, 15].

Definition 5.1 (Tree theorem) We denote by $[2^{<\mathbb{N}}]^n$ the collection of *linearly ordered* subsets of $2^{<\mathbb{N}}$ of size n. A set $S \subseteq 2^{<\mathbb{N}}$ is *order isomorphic* to $2^{<\mathbb{N}}$ (written $S \cong 2^{<\mathbb{N}}$) if there is a bijection $g : 2^{<\mathbb{N}} \to S$ such that for all $\sigma, \tau \in 2^{<\mathbb{N}}$, $\sigma \preceq \tau$ if and only if $g(\sigma) \preceq g(\tau)$. Given a coloring $f : [2^{<\mathbb{N}}]^n \to k$, a tree S is f-homogeneous if $S \cong 2^{<\mathbb{N}}$ and $f{\restriction}[S]^n$ is monochromatic. TT^n_k is the statement "Every coloring $f : [2^{<\mathbb{N}}]^n \to k$ has an f-homogeneous tree."

Both TT^2_2 and ER^2_2 lie between the arithmetic comprehension axiom and RT^2_2, but more than that, they share a *disjoint extension commitment.* Let us try to explain this informal notion with a case analysis.

Suppose we want to construct a computable RT^1_2-instance $f : \mathbb{N} \to 2$ which diagonalizes against two opponents W^f_0 and W^f_1. After some finite amount of time, each opponent W^f_i will have outputted a finite approximation of a solution to f, that is, a finite f-homogeneous set F_i. The two opponents share a common strategy. W^f_0 tries to build an infinite f-homogeneous set H_0 for color 0, and W^f_1 tries to build an infinite f-homogeneous set H_1 for color 1. It is therefore difficult to defeat both opponents at the same time, since if from now on we set $f(x) = 1$, W^f_1 will succeed in extending F_1 to an infinite f-homogeneous set, and if we always set $f(x) = 0$, W^f_0 will succeed with its dual strategy.

Consider now the same situation, where we want to construct a computable TT^1_2-instance $f : 2^{<\mathbb{N}} \to 2$. After some time, the opponent W^f_0 will have outputted a finite tree $S_0 \cong 2^{<b}$ which is f-homogeneous for color 0, and the opponent W^f_1 will have done the same with a finite tree $S_1 \cong 2^{<b}$ f-homogeneous for color 1. The main difference with the RT^1_2 case is that each opponent will *commit to extend* each leaf of his finite tree S_i into an infinite tree isomorphic to $2^{<\mathbb{N}}$. In particular, for each tree S_i, the sets X_σ of nodes extending the leaf $\sigma \in S_i$ are *pairwise disjoint.* Therefore, each opponent commits to extend its partial solution to disjoint sets. Moreover, by asking b to be large enough, each opponent will commit to extend enough pairwise disjoint sets so that we can choose two of them for each opponent and operate the diagonalization without any conflict.

This combinatorial property works in the same way for ER^1-instances. Indeed, in this case, each opponent will commit to extend its partial solution to pairwise disjoint intervals due to the density requirement of an ER^1-solution. Since the combinatorial arguments of the Erdős-Rado theorem and the tree theorem for pairs are very similar, one may wonder whether they are equivalent in reverse mathematics.

Question 5.2 How do ER^2_2 and TT^2_2 compare over RCA_0?

The failure of Seetapun's argument for ER^2_2 comes from this disjoint extension commitment feature. In particular, it is hard to find a forcing notion for ER^2_2 whose conditions are extendible.

Question 5.3 Does ER^2_2 imply ACA_0 over RCA_0?

ER^1 and TT^1 have the same state of the art due to their common combinatorial flavor. However, when looking at their statements for pairs, ER^2_2 and TT^2_2 have a fundamental difference: ER^2_2 has only a half disjoint extension commitment feature. This weaker property prevents one from separating RT^2_2 from ER^2_2 over RCA_0 by adapting the argument of TT^2_2 in [15].

Question 5.4 Does RT^2_2 imply ER^2_2 over RCA_0?

We have seen in section 3 that the separation of $\mathsf{B}\Sigma^0_2$ from ER^1 is directly adaptable from the separation of $\mathsf{B}\Sigma^0_2$ from TT^1 from Corduan, Groszek, and Mileti [3], since the combinatorial core of this separation comes from this shared disjoint extension commitment. It is natural to conjecture that the status of ER^1 with respect to $\mathsf{I}\Sigma^0_2$ will be the same as TT^1.

Question 5.5 Does ER^1 imply $\mathsf{I}\Sigma^0_2$ over RCA_0?

It is worth mentioning that $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ proves a strengthening of both TT^1 and ER^1, namely the statement "For every n and every $f\colon 2^{<\mathbb{N}} \to n$ there exists a strong copy S of the full binary tree such that f is constant on S", where by *strong copy* we mean an isomorphic copy of $2^{<\mathbb{N}}$ with respect to order and minima. It is easy to see that a strong copy computes a dense set of $2^{<\mathbb{N}}$, when $2^{<\mathbb{N}}$ is equipped with the standard dense linear ordering on binary strings, i.e., the only linear order such that $\{\tau\colon \tau \succeq \sigma^\frown 0\} <_{\mathbb{Q}} \sigma <_{\mathbb{Q}} \{\tau\colon \tau \succeq \sigma^\frown 1\}$ for all $\sigma \in 2^{<\mathbb{N}}$. It is likely that if we can separate TT^1 or ER^1 from $\mathsf{I}\Sigma^0_2$, then we can already separate this stronger statement by essentially the same proof.

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Mathematical Institute, Tohoku University, Japan
E-mail address: `frittaion@math.tohoku.ac.jp`
URL: `http://www.math.tohoku.ac.jp/ frittaion/`

Laboratoire PPS, Paris Diderot University, France
E-mail address: `ludovic.patey@computability.fr`
URL: `http://ludovicpatey.com/`