A note on
"Separating principles below Ramsey's Theorem for Pairs"

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Abstract

In this note we present an adaptation of the forcing separating the Erdős Moser theorem (EM) from the stable Rasey theorem for pairs (SRT^2_2). We construct an ω-model of EM not model of a stable version of the thin set theorem for pairs ($\mathsf{STS}(2)$).

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We assume the reader is familiar with reverse mathematics (see [2] for a good survey) and the forcing separating EM from $\mathsf{STS}(2)$ by Lerman & al. [3]. This note does not even try to be self-contained and emphasis on the adaptations of the forcing from [3] needed for separating Erdős Moser theorem from a stable version of the thin set theorem for pairs over ω-models.

1. EM does not imply $\mathsf{STS}(2)$

Definition 1 (The Erdős-Moser theorem) A *tournament* $T = (D, T)$ consists of a set D and an irreflexive binary relation on D such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x, y)$ and $T(y, x)$ holds. A tournament T is *transitive* if the relation T is transitive in the usual sense. A *sub-tournament* of T is a tournament of the form $(E, E^2 \cap T)$ for an $E \subseteq D$. EM is the statement "for every infinite tournament there is an infinite transitive sub-tournament".

Definition 2 (The thin set theorem) Let $c : [\mathbb{N}]^n \to \mathbb{N}$ be a coloring function. A set H is *thin for c with witness a* if $a \notin c(H^n)$. H is *thin for c* if there is a witness a such that H is thin for c with witness a. $\mathsf{TS}(k)$ is statement: "Every coloring function $c : [\mathbb{N}]^k \to \mathbb{N}$ has an infinite set thin for c.". $\mathsf{STS}(k)$ is the restriction of $\mathsf{TS}(2)$ on stable colorings.

It has been proven in [1, Corollary 5.4] that for every k, $\mathsf{RCA}_0 \vdash \mathsf{RT}^k_2 \to \mathsf{TS}(k)$. We noticed that when considering stable functions, the proof still holds. Hence $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{STS}(2)$.

Definition 3 Fix sets A and B. A *partition map* $F^* : A \to B$ is a function from A to

$\mathcal{P}(B)$ such that

$$(\forall x \in B)(\forall y, z \in A)(x \in F(y) \cap F(z) \Rightarrow y = z).$$

There is a natural partial order between such maps:

$$F^* \leq G^* \quad \text{iff} \quad (\forall a \in \omega)(G^*(a) \subseteq F^*(a)).$$

We can also define an update operation defined as follows:

$$(F^* + (a \mapsto S_a))(x) = \begin{cases} F^*(x) \cup S_a & \text{if } x = a \\ F^*(x) & \text{otherwise} \end{cases}$$

Example 1 Let $c : [\mathbb{N}]^2 \to \mathbb{N}$ be a coloring function. There is a natural partition map F_c^* verifying

$$F_c^*(a) = \{x : (\forall^\infty y)(c(x, y) = a)\}.$$

Remark that for any infinite set H thin for c with witness a, $H \cap F_c^*(a) = \emptyset$. Hence F_c^* can be seen as the map of forbidden values if we want to create a set thin with a given witness.

1.1. Iteration forcing

Previous definitions and forcing conditions are similar to [3].

Definition 4 (4.8) A *requirement* is a set $\mathcal{K}^{X,F_c^*(x)}$ of finite transitive subtournaments of T_e^X which is closed under extensions and is defined by

$$\mathcal{K}^{X,F_c^*(x)} = \left\{F \in \mathbb{F}_e^X : \exists a \in F_c^*(x)(R_{\mathcal{K}}^X(F, a))\right\}$$

for an X-computable relation $R_{\mathcal{K}}^X(x, y)$.

Example 2 (4.9) For each m and x, we define the requirement

$$\mathcal{W}_m^{X,F_c^*(x)} = \left\{F \in \mathbb{F}_e^X : \exists a \in F_c^*(x)(\Phi_m^{(X \oplus F}(a) = 1)\right\}$$

Suppose a condition (F, I, S) used to construct our generic G satisfies $F \in \mathcal{W}_m^{X,F_c^*(x)}$. Because F is an initial segment of G, we have successfully diagonalized against $\Phi_m^{X \oplus G}$ computing an infinite set thin for c with witness x.

We can replace the set $F_c^*(x)$ by a set B. Usually B will be finite. We abuse notation and write $\mathcal{K}^{X,B}$ in this situation.

Definition 5 (4.10) We say $\mathcal{K}^X$ is *essential* below (F, I, S) if for every x there is a finite set $B > x$ and a level n such that whenever $E \in S(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F \cup F' \in \mathcal{K}^{X,B}$.

Definition 6 (4.11) We say $\mathcal{K}^{X,F_c^*(x)}$ is *uniformly dense* if whenever $\mathcal{K}^X$ is essential below (F, I, S), there is some level n such that whenever $E \in S(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F \cup F' \in \mathcal{K}^{X,F_c^*(x)}$.

Definition 7 (4.12) We say (F, I, S) *settles* $\mathcal{K}^{X,F_c^*(x)}$ if either $F \in \mathcal{K}^{X,F_c^*(x)}$ or there is an x such that whenever $E \in S(n)$ is on an infinite path through S and $F' \subseteq E$ is transitive, $F \cup F' \notin \mathcal{K}^{X,(x,\infty)}$.

We give one example to illustrate settling and prove one essential property of this notion.

Example 3 (4.13) Suppose (F, I, S) settles $\mathcal{W}_m^{X,F_c^*(x)}$. We claim that if (F, I, S) appears in a sequence defining a generic G, then $\Phi_m^{X \oplus G}$ is not a solution for c. If $(F, I, S) \in \mathcal{W}_m^{X,F_c^*(x)}$, then this claim was verified in Example 2. So, assume that (F, I, S) settles $\mathcal{W}_m^{X,F_c^*(x)}$ via the second condition in this definition and fix the witness x. We claim that for all $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$ and all $b > x$, $\Phi_m^{X \oplus \tilde{F}}(b) \neq 1$. It follows immediatly from this claim that $\Phi_m^{X \oplus G}$ is finite and hence is not a solution to c.

To prove this claim, fix $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$. Suppose of a contradiction that there is a $b > x$ such that $\Phi_m^{X \oplus \tilde{F}}(b) = 1$. Then $\exists b > x(\Phi_m^{X \oplus F}(b) = 1)$ and hence $\tilde{F} \in \mathcal{W}_m^{X,(x,\infty)}$.

Let $F' = \tilde{F} \setminus F$, so $F \cup F' \in \mathcal{W}_m^{X,(x,\infty)}$. Because $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$, we have $(\tilde{F} \setminus F) + S' = f' + S' \leq S$ and hence there is a level n and an $E \subseteq S(n)$ such that $F' \subseteq E$. Therefore, F' shows that our fixed x does not witness the second condition for (F, I, S) to settle $\mathcal{W}_m^{X,F_c^*(x)}$ giving the desired contradiction.

Lemma 1 (4.14) If (F, I, S) settles $\mathcal{K}^{X,F_c^*(x)}$ and $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$, then $(\tilde{F}, \tilde{I}, \tilde{S})$ settles $\mathcal{K}^{X,F_c^*}$.

The heart of this construction is the following theorem.

Theorem 1 (4.15) Let $\mathcal{K}^{X,F_c^*(x)}$ be a uniformly dense requirement and let (F, I, S) be a condition. There is an extension $(F', I', S') \leq (F, I, S)$ settling $\mathcal{K}^{X,F_c^*(x)}$.

We will show how Theorem 1 is used to construct our generic G and verify that $X \oplus G$ does not compute a solution to c and that for any index e' such that $\Phi_{e'}^{X \oplus G}$ defines a tournament, the associated requirements $\mathcal{K}^{X \oplus G, F_c^*}$ are uniformly dense.

To define G, let $\mathcal{K}_n^{X,F_c^*(x)}$, for $n \in \omega$ be a list of all the requirements. We define a sequence of conditions

$$(F_0, I_0, S_0) \geq (F_1, I_1, S_1) \geq \dots$$

by induction. Let $F_0 = \emptyset$, $I_0 = (-\infty, \infty)$ and $S_0(n) = \{[0, n]\}$. Assume (F_k, I_k, S_k) has ben defined. Let n be the least index such that $\mathcal{K}_n^{X,F_c^*(x)}$ is not settled by (F_k, I_k, S_k). Applying Theorem 1, we choose $(F_{k+1}, I_{k+1}, S_{k+1})$ so that it settles $\mathcal{K}_n^{X,F_c^*(x)}$. We define our generic by $G = \bigcup F_n$.

The next lemma shows that we eventually settle each condition that is not trivially satisfied.

Lemma 2 (4.16) Let $\mathcal{K}_n^{X,F_c^*(x)}$ be a requirement and let (F_j, I_j, S_j) be the sequence of conditions defining G. There is an index k such that (F_k, I_k, S_k) settles $\mathcal{K}_n^{X,F_c^*(x)}$.

We can now verify the properties of G starting with the fact that $X \oplus G$ does not compute a solution to c.

Lemma 3 (4.17) $X \oplus G$ does not compute a solution to c.

Proof. Fix an index m and we show that $\Phi_m^{X\oplus G}$ is not a solution to c using the requirement $\mathcal{W}_m^{X,F_c^*(x)}$. If $\Phi_m^{X\oplus G}(u)$ is never equal to 1 for any u, then $\Phi_m^{X\oplus G}$ does not compute an infinite set and we are done. Therefore assume that $\Phi_m^{X\oplus G}(u) = 1$ for some u. In this case $\mathcal{W}_m^{X,F_c^*(x)}$ is settled by some conditions (F_k, I_k, S_k) in the sequence defining G. In Example 3 we verified that if $\mathcal{W}_m^{X,F_c^*(x)}$ is settled by a condition in a sequence defining a generic G, then $\Phi_m^{X\oplus G}$ does not compute a solution c. $\square$

Next, we describe the requirements forcing uniform density at the next level. To specify a potential requirement at the next level, we need to fix three indices: an index e' for a potential infinite transitive tournament $T_{e'}^{X\oplus G}$ and an index for $\mathcal{R}_{\mathcal{K}}^{X\oplus G}$ (defining $\mathcal{K}^{X\oplus G,F_c^*}$). We regard the index for $\mathcal{R}_{\mathcal{K}}^{X\oplus G}$ as $\mathcal{K}$ and will represent this choice of index by indicating e' and $\mathcal{K}$. For each choice of these indices and each $q = (F_q, I_q, S_q)$, representing a potential condition in $\mathbb{Q}_{e'}^{X\oplus G}$, we will have a requirement $\mathcal{T}_{e',\mathcal{K},q}^{X}$.

Forcing definitions are the same as in [3]. The requirement $\mathcal{T}_{e',\mathcal{K},q}^{X,F_c^*(x)}$ consists of all finite transitive subtournaments F of T_e^X such that either

(C1) $F \Vdash q \notin \mathbb{Q}_{e'}^{X\oplus G}$; or

(C2) there is an $n \leq |F|$ such that $F \Vdash$ (q is a condition up to level n) and for all $E \in S_q^{X\oplus F}(n)$ and all partitions $E = E_0 \cup E_1$, there is an $i \in \{0,1\}$ and a transitive $F' \subseteq E_i$ such that $\exists a \in F_c^*(x)(\mathcal{R}_{\mathcal{K}}^{X\oplus F})(F_q \cup F', a))$.

Lemma 4 (4.18) Let $G = \bigcup F_k$ be a generic defined by a equence of conditions (F_k, I_k, S_k) and let e' be an index such that $T_{e'}^{X\oplus G}$ is an infinite tournament. Each requirement $\mathcal{K}^{X\oplus G,F_c^*(x)}$ is uniformly dense in $\mathbb{Q}_{e'}^{X\oplus G}$.

1.2. Ground forcing

We now carry out the ground level forcing to produce the coloring c. Our forcing conditions are pairs (c, F^*) where c is a coloring of two-element subsets of a finite domain $[0, |c|]$, and F^* is a partition map of support bounded by $\|c\|$. We say that $(c, F^*) \leq (c_0, F_0^*)$ if $c_0 \subseteq c$, $F^* \leq F_0^*$ and whenever $b \in F_0^*(a)$ and $x > |c_0|$, $c(b, x) = a$.

Clearly the set of (c, F^*) such that $i \in \bigcup Im(F^*)$ is dense, so we may ensure that the coloring given by a generic is stable. We need to ensure that our generic coloring does not compute a solution to itself.

Definition 8 We say $(c, F^*) \Vdash$ (Φ_e^G is finite) if $\exists k \forall (c_0, F_0^*) \leq (c, F^*) \forall x (\Phi_e^{c_0}(x) = 1 \to x \leq k)$. We say $(c, F^*) \Vdash$ (Φ_e^G is not thin with witness x) if $\exists a \in F^*(x)(\Phi_e^c(a) = 1)$.

Lemma 5 (4.25) For each index e and color x, the set of conditions which either force Φ_e^G is finite or force Φ_e^G is not thin with witness x is dense.

Proof. Fix an index e and a condition (c, F^*). If some extension of (c, F^*) forces Φ_e^G is finite, then we are done. Otherwise there is an $y > \|c\|$ and a condition (c_0, F^*) extending (c, F^*) such that $\Phi_e^{c_0}(y) = 1$. (Without loss of generality only the coloring changes.) The condition $(c_0, F^* + (x \mapsto \{y\}))$ extends (c, F^*) and forces is Φ_e^G not to be thin with witness x. $\square$

Finally, we need to force the requirements $\mathcal{K}^{G,F^*(G)}$ for any generic G to be uniformly dense in $\mathbb{Q}_e^G$. Fix an index e and a potential iterated forcing condition $p = (F_p, I_p, S_p)$ where F_p is a finite set, I_p is a pair of elements in F_p and S_p is the index for a potential family of subtournaments of T_e^G. Forcing notions remain the same as in original forcing.

Lemma 6 (4.26) Let $\mathcal{K}^{G,F_G^*(x)}$ be a potential requirement given by the indices i and i'. Then for any potential iterated forcing condition p, there is a dense set of conditions (c, F^*) such that:

- $(c, F^*) \Vdash p \notin \mathbb{Q}_e^G$; or
- $(c, F^*) \Vdash \mathcal{K}^G$ is not essential below p; or
- there is a level n such that $S_p^c(n)$ converges and whenever $E \in S_p^c(n)$ and $E = E_0 \cup E_1$ is a partition, there is a $j \in \{0, 1\}$ and a transitive $F' \subseteq E_j$ such that

$$\exists a \in F^*(x)(\Phi_i^c(F_p \cup F', a) = 1).$$

Proof. Fix a condition (c, F^*) and a potential iterated forcing condition $p = (F_p, I_p, S_p)$. If there is any $(c_0, F_0^*) \leq (c, F^*)$ forcing that $p \notin \mathbb{Q}_e^c$ there we are done, so assume not.

Suppose there is an extension $(c_0, F^*) \leq (c, F^*)$, a finite set $B > max(\|c\|, a_{\mathcal{K}}^{c_0}(F_p))$ and an n such that $S_p^{c_0}(n)$ converges and whenever $E \in S_p^{c_0}(n)$ and $E = E_0 \cup E_1$ is a partition, there is a $j \in \{0, 1\}$ and a transitive $F' \subseteq E_j$ such that

$$\exists b \in B(\Phi_i^{c_0}(F_p \cup F', b) = 1)$$

ie. $F_p \cup F' \in \mathcal{K}^{c_0,B}$. $(c_0, F^* + (a_{\mathcal{K}}^{c'} \mapsto B))$ is the desired condition.

Suppose there is no such (c_0, F^*). Then we claim that (c, F^*) already forces that $\mathcal{K}^G$ is not essential below p. Let $\tilde{c}$ be any completion of c to a stable coloring on ω, and suppose $\mathcal{K}^c$ were essential below p. Then there would be some $B > max(\|c\|, a_{\mathcal{K}}^c(F_p))$ and an n such that $S_p^c(n)$ converges and whenever $E \in S_p^c(n)$, every partition is as described above. In particular, there would be some finite initial segment of $\tilde{c}$ witnessing the necessary computations, contradicting our assumption. $\square$

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