Mathématiques à rebours et un Lemme de König Faible de Type Ramsey

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Le contexte général

Le programme dit des « mathématiques à rebours » (reverse mathematics) est un programme de fondation des mathématiques ayant pour objet l'analyse de la force logique des théorèmes mathématiques les plus standards. Le cadre logique dans lequel se placent les mathématiques à rebours est celui des sous-systèmes de l'arithmétique du second ordre, et notamment les systèmes de base RCA_0, WKL_0 et ACA_0. Ces systèmes correspondent en calculabilité à divers niveaux de difficulté. RCA_0 est le système correspondant aux mathématiques calculables. WKL_0 correspond à l'utilisation de la compacité, c'est-à-dire à la possibilité d'extraire de toute classe Π^0_1 non-vide de réels un élément particulier (ce qui n'est pas constructif en général). Enfin, ACA_0 correspond à l'existence pour tout réel du problème de l'arrêt associé (dont la non-calculabilité est bien connue et est à la base de toute la théorie moderne de la calculabilité). La référence en la matière est le livre de Stephen Simpson *Subsystems of Second Order Arithmetic*, qui sera bientôt rejoint par le livre en cours d'écriture de Damir Dzhafarov et Carl Mummert.

Le problème étudié

On s'intéresse dans ce rapport à des sous-systèmes entre RCA_0 et WKL_0 fondés sur un principe hybride combinant le *lemme de König* (existence d'un chemin pour tout arbre infini à branchement fini) et le *théorème de Ramsey* (tout coloriage fini de l'ensemble des n-parties de ω admet un sous-ensemble infini monochromatique) en un principe d'existence dans tout arbre infini à branchement fini d'un sous-ensemble infini d'un chemin de l'arbre.

La classe RKL ainsi définie a été récemment introduite par Stephen Flood dans son article *Reverse Mathematics and a Ramsey-type König's lemma*. L'introduction de nouveaux principes définissant de nouvelles classes permet souvent d'améliorer la compréhension d'un système existant en prouvant l'équivalence d'un nouveau principe avec le systeme. La multiplicité des principes pour définir un même système permet de simplifier les preuves en

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choisissant le principe convenant le mieux à une preuve naturelle pour un problème donné. Les mathématiques à rebours ont montré que les principaux théorèmes étaient équivalents à des systèmes faibles, c'est à dire pour la plupart en dessous de ACA_0. L'attention des mathématiciens s'est donc naturellement portée sur l'enrichissement des systèmes faibles pour affiner la compréhension de la difficulté des théorèmes existants.

En définissant un nouveau principe, Flood a ouvert une série de questions sur le positionnement du système induit ainsi que de ses variantes dans la hiérarchie des systèmes existants. S'il y a apporté une réponse partielle en démontrant des implications relatives à ce système, la question de la caractérisation du système reste ouverte.

La contribution proposée

Afin de mieux cerner l'expressivité d'un principe, la démarche canonique consiste à étudier des variantes du principe en faisant varier des paramètres.

Ce rapport introduit WRKL, une variante de RKL basée sur le même principe mais restreint aux arbres de mesure positive, restriction déjà étudiée dans le cadre du *lemme de König* en le système $WWKL_0$. La variante WRKL apparaît être équivalente à une classe déjà existante nommée DNC.

Nous introduisons également la variante RKL+ imposant des contraintes plus fortes sur la nature du sous-ensemble infini d'un chemin de l'arbre, et prouvons que cette variante est équivalente au système WKL_0. Sachant que RKL a été prouvé différent de WKL_0 par Flood, nous mettons en exergue par notre variante l'importance de la contrainte imposée dans l'expressivité du système.

Les arguments en faveur de sa validité

Afin de s'assurer de la robustesse de notre approche ainsi que mieux comprendre les raisons profondes de l'équivalence entre les deux principes, nous fournissons deux preuves : une probabiliste et une purement combinatoire. La simplicité des preuves fournies permet au lecteur attentif de se convaincre de leur validité.

En outre, les principes considérés comportent de fortes connexions avec la théorie de l'aléatoire de Martin-Löf et le résultat d'équivalence entre WRKL et DNC correspond à un théorème préexistant en aléatoire.

Le bilan et les perspectives

Le système RKL ayant été introduit très récemment, il n'était pas possible d'évaluer a priori la difficulté de la question posée. Bien que la question de la nature exacte du système RKL reste toujours ouverte, nous avons amélioré la compréhension du principe en question en prouvant respectivement la faiblesse d'une de ses restrictions et la force d'une de ses variantes. Le système est supposé disjoint de toute autre classe existante. Sa séparation de WKL_0 a été prouvée par Flood. Il reste à prouver sa séparation du système DNC ainsi que de $WWKL_0$.

Reverse Mathematics and a Weak Ramsey-Type König's Lemma

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Abstract

In this report, we present a weakened version of **RKL** (see [4]) and characterize it in terms of *diagonally non computable functions* using two proofs: a probabilistic and a combinatorial one. We also try to handle the expressivity of **RKL** by studying its variant $\mathbf{RKL}^+$, proving its equivalence to the existence of a DNC_2 function. However the separation between **DNC** and **RKL** remains still open.

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1 Introduction

Reverse mathematics is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics. Set theory being widely used as foundational formalism for defining mathematics, the axioms choosen for giving a characterization of the strength of everyday theorems are naturally set existence axioms. We focus on the language of second order arithmetic, because that language is the weakest one that is rich enough to express and develop the bulk of core mathematics.

As a remarkable empirical observation, most theorems in “ordinary” mathematics appear to involve only a few principles – weak König's lemma, arithmetical comprehension, arithmetical transfinite recursion... – and they live mainly in low classes. Furthermore, most common theorems are proven to be equivalent to those axioms.

In section 1.1 we will introduce a few notations, then we will define the notion of subsystem of second order logic in section 1.2. Then we will introduce the weakest and the most foundamental subsystem $\mathbf{RCA_0}$ in section 1.3 as it will be used as a basis to prove implications between other systems. In sections 1.4, 1.5, 1.6 and 1.7 we will present definitions of subsystems useful for the understanding of our results. We introduce them by class of principles together with their main results. Then in section 2 we prove the equivalence of **WRKL** and **DNC** in two ways: a probabilistic one (section 2.1) and a combinatorial one (section 2.2). In section 3 we put our focus on **RKL** by discussing about its separation proof from $\mathbf{WKL_0}$ (section 3.1) and proving the equivalence between its variants $\mathbf{RKL}^+$, $\mathbf{RKL_h}$ and $\mathbf{WKL_0}$ (sections 3.2 and 3.3).

1.1 Prerequisites and conventions

The reader is assumed to know basics about computability theory – S^m_n theorem, Rice's theorem, Kleene's recursion theorem –. A good introductory book is *Computability Theory* from S.B. Cooper [3]. Notions about reverse mathematics are welcome. The reference book in this domain is *Subsystems of second order arithmetic* from S.G. Simpson [17].

We now fix some notations and introduce a few definitions.

Turing machines We write *Turing functionnals* in upper case greek letters Φ, Ψ... Φ^S denotes the Turing machine with oracle S. It is a well known result that Turing machines are enumerable. Let $(\Phi_e)_{e\in\mathbb{N}}$ be an enumeration of all partial computable functions. We denote by W_e the domain of Φ_e.

Strings, sequences A *string* is an element of $2^{<\omega}$. The empty string is written ϵ. We write $|\sigma|$ the length of string σ. We denote by $\preceq$ the prefix relation over strings, ie.

$$\preceq \stackrel{def}{=} \left\{(\sigma_1,\sigma_2) \in 2^{<\omega} \times 2^{<\omega} \ : \ |\sigma_1| \leq |\sigma_2| \wedge \forall i \leq |\sigma_1|\,,\, \sigma_1(i) = \sigma_2(i)\right\}$$

We will denote by Γ^i_n the set of strings having a i at position n:

$$\Gamma^i_n \stackrel{def}{=} \left\{s \in 2^{<\omega} \ : \ |s| > n \wedge s(n) = i\right\}$$

Trees, paths A *binary tree* T is a subset of $\omega^{<\omega}$ closed under prefixes. In the remaining of this report and unless mention, we will consider only infinite binary trees with $T \subseteq 2^{<\omega}$. A *path* in T is a sequence $P \in 2^\omega$ such that all finite prefixes of P is in T. We will sometimes identify P to a function $P : \omega \to \{0,1\}$ such that $P(n) = 1$ iff the nth bit of P is 1. We write $[[T]]$ for the set of paths of T. The *measure* of a binary tree T is defined as follows.

$$\mu(T) \stackrel{def}{=} \lim_{n\to\infty} \frac{\mathtt{card}\left\{\sigma \in T \ : \ |\sigma| = n\right\}}{2^n}$$

Note that this limit always exists. A tree T has *positive measure* if $\mu(T) > 0$.

1.2 Subsystems of $\mathbf{Z_2}$

Since in ordinary mathematics the objects studied are almost always countable or separable, it would seem appropriate to consider a language in which countable objects occupy center stage. For this reason, the language of study is second order arithmetic. We will introduce L_2, the langage of second order arithmetic, then we will define $\mathbf{Z_2}$, it's formal system and will explain notions of subsystems of $\mathbf{Z_2}$

Definition 1 (Language of Second Order Arithmetic L_2) The *language of second order arithmetic* is a two-sorted language – ie. it manipulates two kinds of objects: numbers and sets of numbers –. Variables of the numbers sort are denoted by $x, y, z \dots$ whereas variables of sets sort are denoted by capital letters: $X, Y, \dots$. The *numerical terms* of L_2 are defined as follows:

$$t ::= 0 \mid 1 \mid x \mid t_1 + t_2 \mid t_1 \cdot t_2$$

Here $+$ and $\cdot$ are intended to denote addition and multiplication over natural numbers. *Formulas* of L_2 are defined as follows:

$$f ::= t_1 = t_2 \mid t_1 < t_2 \mid t_1 \in X \mid \forall x.f \mid \exists x.f \mid \forall X.f \mid \exists X.f \mid \neg f \mid f_1 \vee f_2$$

$\forall x.f$ and $\exists x.f$ are intended to denote universal and existential quantification over numbers, whereas $\forall X.f$ and $\exists X.f$ are quantifications over sets. Formulas used as the base case of this inductive definition are called *atomic formulas*. Other connectives are defined in the usual way – $f_1 \wedge f_2$ is a notation for $\neg(\neg f_1 \vee \neg f_2)$, $f_1 \Rightarrow f_2$ for $\neg f_1 \vee f_2$, $f_1 \Leftrightarrow f_2$ for $(f_1 \Rightarrow f_2) \wedge (f_2 \Rightarrow f_1) \dots$–. A *sentence* is a formula without free variables.

Definition 2 (Second Order Arithmetic $\mathbf{Z_2}$) The axioms of $\mathbf{Z_2}$ are the following:

(i) Basic axioms:

$$\begin{array}{ll} n+1 \neq 0 & m+1 = n+1 \Rightarrow m = n \\ m+0 = m & m + (n+1) = (m+n)+1 \\ m \cdot 0 = 0 & m \cdot (n+1) = (m \cdot n) + m \\ \neg m < 0 & m < n+1 \Leftrightarrow (m < n \vee m = n) \end{array}$$

(ii) Induction axiom:

$$(0 \in X \wedge \forall n.(n \in X \Rightarrow n+1 \in X)) \Rightarrow \forall n.(n \in X)$$

(iii) Comprehension scheme:

$$\exists X.\forall n.(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Definition 3 (L_2-Structure, model) A L_2*-structure* is an ordered 7-tuple

$$M = \langle |M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M <_M \rangle$$

where $|M|$ is the set in which range number variables. $\mathcal{S}_M$ is a set of subsets of $|M|$ serving as the range of the set variables. $+_M$ and $\cdot_M$ are binary operations over $|M|$, 0_M and 1_M are distinguished elements of $|M|$. $<_M$ is a binary relation over $|M|$. Formulas of L_2 are interpreted in a L_2-structure in the obvious way.

A *model* of Z_2 is a L_2-structure satisfying axioms of second order arithmetic. An ω-*model* of Z_2 is a L_2-structure of the form

$$\langle \omega, \mathcal{S}, +, \cdot, 0, 1, < \rangle$$

where ω is the set of natural integers, $\mathcal{S} \subseteq \mathcal{P}(\omega)$, $+$ and $\cdot$ are usual operations over integers and $<$ is the natural ordering, such that $\mathcal{S}$ verifies comprehension scheme. Note that basic axioms and induction axiom are valid in any ω-model. As only $\mathcal{S}_M$ varies, we sometimes simply denote an ω-model by its set $\mathcal{S}$.

Definition 4 (Subsystem of $\mathbf{Z_2}$) A *subsystem* T *of* $\mathbf{Z_2}$ is a formal system based on language L_2 whose axioms are theorems of $\mathbf{Z_2}$. A *model* of T is any L_2-structure verifying axioms of T.

We will be especially interested in weakenings of induction axiom and comprehension scheme. The following sections will detail some particularily important subsystems which play an role in our results.

1.3 The system $\mathbf{RCA_0}$

$\mathbf{RCA_0}$ is the weakest of our studied subsystems and is especially important because almost all our theorems will be proved in $\mathbf{RCA_0}$. The acronym RCA stands for Recursive Comprehension Axiom, because $\mathbf{RCA_0}$ asserts the existence of any set computable in a few oracles $B_1, B_2, \dots$.

Definition 5 (Σ^0_1, Π^0_1 and Δ^0_1 formulas) A L_2 formula is Σ^0_1 if it is of the form $\exists n.\phi$ where ϕ is a formula with only bounded quantifiers. We can define dually Π^0_1 formulas with universal quantifier.

A formula is Δ^0_1 if it can be expressed equivalently by a Σ^0_1 and a Π^0_1 formula. Notice that the notion of Δ^0_1 is semantic whereas Σ^0_1 and Π^0_1 are defined syntactically.

These notions are closely related to Computability Theory as follows

Theorem 1 [Post's theorem] A set A is *computably enumerable* (resp. *computable*) in $B_1, B_2, \dots$ iff it is definable by a Σ^0_1 formula (resp. Δ^0_1 formula) with parameters $B_1, B_2, \dots$.

Definition 6 (System $\mathbf{RCA_0}$) The axioms of $\mathbf{RCA_0}$ are the following:

(i) Basic axioms of $\mathbf{Z_2}$

(ii) Σ^0_1 Induction axiom:

$$(\varphi(0) \wedge \forall n.(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

(iii) Δ^0_1 Comprehension axiom:

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X.\forall n.(x \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

Theorem 1 together with axioms of $\mathbf{RCA_0}$ justify its name Recursive Comprehension Axiom. $\mathbf{RCA_0}$ corresponds intuitively to "computable mathematics". In fact we have a nice characterization of ω-models of $\mathbf{RCA_0}$ in terms of Computability Theory:

Theorem 2 (ω-models of $\mathbf{RCA_0}$) [17] A set $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is an ω-model of $\mathbf{RCA_0}$ iff

(i) $\mathcal{S} \neq \emptyset$

(ii) A and $B \in \mathcal{S}$ implies $A \oplus B \in \mathcal{S}$

(iii) $A \in \mathcal{S}$ and $B \leq_T A$ implies $B \in \mathcal{S}$

where $\leq_T$ is the *Turing reduction* and $A \oplus B$ is the *computable join*, ie

$$A \oplus B \stackrel{def}{=} \{2n \ : \ n \in A\} \cup \{2n+1 \ : \ n \in B\}$$

In particular, $\mathbf{RCA_0}$ has a minimal ω-model which is the set of computable sets of natural numbers. All other subsystems presented in this report will be extensions of $\mathbf{RCA_0}$ with some principles.

1.4 Weak König's Lemmas

An important class of principles is about the existence of a path in an infinite tree. *König's lemma* in its full generality asserts the existence of a path in any infinite tree finitely branching. It can be proven withing $\mathbf{Z_2}$ and even weaker subsystems.

Given a tree T, the sets of paths of T is written $[[T]]$ and is called a Π^0_1 *class*. This notion has been extensively studied since Simpson suggested it could be a natural generalization of Turing degrees. We can define two notion of reduction between two Π^0_1 classes.

Definition 7 (Mučnik and Medvedev reductions) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two Π^0_1 classes.

$\mathcal{C}_1$ is *Mučnik-reducible* (or weakly-reducible) to $\mathcal{C}_2$ if for every set $X_2 \in \mathcal{C}_2$ there is a set $X_1 \in \mathcal{C}_1$ such that $X_1 \leq_T X_2$. We note $\mathcal{C}_1 \leq_w \mathcal{C}_2$.

$\mathcal{C}_1$ is *Medvedev-reducible* (or strongly-reducible) to $\mathcal{C}_2$ if there is a functionnal Ψ such that for every set $X \in \mathcal{C}_2$, $\Psi^X \in \mathcal{C}_1$. We note $\mathcal{C}_1 \leq_s \mathcal{C}_2$.

Those notions of reductions express intuitively that being given any set of one class we can produce (uniformly or not) a set in a lower class. Each reduction induce a different notion of degree in which we can embed Turing degrees. See [6] for a good survey on Mučnik and Medvedev degrees.

The strongest system related to König's lemma is $\mathbf{ACA_0}$, standing for *arithmetical comprehension axiom*. $\mathbf{ACA_0}$ appears to be equivalent in $\mathbf{RCA_0}$ to the unrestricted König's lemma. It can also be seen as a convervative extension of first order arithmetic as a L_1-sentence is a theorem of $\mathbf{ACA_0}$ iff it is a theorem of $\mathbf{Z_1}$.

Definition 8 (System $\mathbf{ACA_0}$) The axioms of $\mathbf{RCA_0}$ are the following:

(i) Basic axioms of $\mathbf{Z_2}$

(ii) Σ^0_1 Induction axiom.

(iii) Arithmetical Comprehension axiom:

$$\exists X. \forall n.(x \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any arithmetical formula of L_2 – ie. without set quantifier – in which X does not occur freely.

$\mathbf{ACA_0}$ is a strong class and a lot of theorems live in weaker classes. In our study, we will be interested in weaker notions of König's lemma, restricted to infinite subtrees of $2^{<\omega}$ trees. Notice that the restriction to subtrees of $2^{<\omega}$ is different from the restriction to binary trees as the latter is still equivalent to $\mathbf{ACA_0}$.

Definition 9 (System $\mathbf{WKL_0}$) The axioms of $\mathbf{WKL_0}$ are those of $\mathbf{RCA_0}$ augmented with the principle "every infinite subtree of $2^{<\omega}$ has a path".

$\mathbf{WKL_0}$ corresponds to the notion of compactness and is for example equivalent to the Heine/Borel theorem: *Every covering of the closed unit interval* $0 \leq x \leq 1$ *by a sequence of open intervals has a finite subcovering.* We can obtain a model of $\mathbf{WKL_0}$ by taking the set of computable sets augmented with low sets – ie. sets X such that $X' \equiv_T \emptyset'$ – because of the low basis theorem [10] stating that any non-empty Π^0_1 class has a low member.

An even weaker notion of König's lemma is the restriction to infinite trees of positive measure.

Definition 10 (System $\mathbf{WWKL_0}$) The axioms of $\mathbf{WKL_0}$ are those of $\mathbf{RCA_0}$ augmented with the principle "every infinite subtree of $2^{<\omega}$ of positive measure has a path".

For the reader who is familiar with the theory of algorithmic randomness, $\mathbf{WWKL_0}$ is closely related to Martin-Löf randoms, as any Π^0_1 class of positive measure contains all MLR up to prefixes [12]. $\mathbf{WWKL_0}$ is then equivalent to the existence of a MLR and the latter are used to construct a model of $\mathbf{WWKL_0}$. This connection to the theory of algorithmic randomness will be continued when we will study Ramsey-like König's lemmas. See Nies book [14] for a good introduction to randomness.

All previously presented systems are proven to form a strict hierarchy:

Theorem 3 (Simpson et al. [18], [17])

$$\mathbf{RCA_0} \subsetneq \mathbf{WWKL_0} \subsetneq \mathbf{WKL_0} \subsetneq \mathbf{ACA_0}$$

1.5 Ramsey's Theorems

Another class of principles concerns Ramsey's theorems which is a generalization of the pigeonhole principle. Given $n \in \omega$, let $[N]^n$ denote the collection of subsets of ω of size n.

Definition 11 (System $\mathbf{RT}$, $\mathbf{RT^n_k}$) The axioms of $\mathbf{RT}$ are those of $\mathbf{RCA_0}$ augmented with the principle "given n and $k \in \omega$, for every function (called a coloring) $f \in \{0, \dots, k-1\}^{[N]^n}$, there is an infinite set $H \subseteq \omega$ which is given one color by f".

$\mathbf{RT^n_k}$ is the restriction of $\mathbf{RT}$ to a fixed n and k.

A simple argument of color blindness shows that for any fixed $n > 0$, $k_1 > 1$ and $k_2 > 1$, $\mathbf{RT^n_{k_1}} \Leftrightarrow \mathbf{RT^n_{k_2}}$. For almost all values of n and k, the strength of $\mathbf{RT^n_k}$ is known as states the following theorem.

Theorem 4 (Simpson [17])

(i) For each $n \geq 3$ and $k \geq 2$ (both n and k fixed), $\mathbf{RT^n_k}$ is equivalent to $\mathbf{ACA_0}$ over $\mathbf{RCA_0}$.

(ii) $\mathbf{RT}$ is not provable in $\mathbf{ACA_0}$.

Hirst proved in his PhD thesis [7] that $\mathbf{RT^1_k}$ is provable in $\mathbf{RCA_0}$ for each k. However, the case of $\mathbf{RT^2_2}$ remained a long-standing problem. In 1995, Seetapun proved in [16] that $\mathbf{RT^2_2}$ does not imply $\mathbf{ACA_0}$. Then Cholak, Jockusch, and Slaman proved in 2001 that $\mathbf{RT^2_2}$ is not provable in $\mathbf{WKL_0}$ [2]. It is only in 2011 that Liu proved $\mathbf{RT^2_2}$ does not imply $\mathbf{WKL_0}$ [13].

1.6 Ramsey-Type Weak König's Lemmas

In his paper *Reverse mathematics and a Ramsey-type König's Lemma*, Stephen Flood emitted the idea of combining König's lemma and Ramsey principles by creating the notion of *homogeneous set for a path in a tree.*

Definition 12 (Homogeneous set) A set H is *homogeneous for* $\sigma \in 2^{<\omega}$ *with color* $c \in \{0,1\}$ if $\sigma(x) = c$ for each $x \in H$ s.t. $x < |\sigma|$. H is *homogeneous for a path trough* T if $\exists c \in \{0,1\}$ s.t. H is homogeneous for σ with color c for arbitrarily long $c \in T$.

Definition 13 (System **RKL**) The axioms of **RKL** are those of $\mathbf{RCA_0}$ augmented with the principle "each binary tree T has an infinite set which is homogeneous for a path through T."

He proved the following relations between **RKL** and existings systems:

Theorem 5 (Flood [4]) The following statements are true:

(i) $\mathbf{RKL} < \mathbf{RT^2_2}$

(ii) $\mathbf{RKL} < \mathbf{WKL_0}$

(iii) $\mathbf{DNC} \leq \mathbf{RKL}$

The question wether **DNC** implies **RKL** in $\mathbf{RCA_0}$ is still open. In order to understand better the expressiveness of the existence of homogeneous set for a path in a tree, we have been introducing two restrictions of **RKL** and will give a characterization of them in further sections.

As the notion of path existence has been studied in terms of tree of positive measure through the system $\mathbf{WWKL_0}$, it seems natural to wonder how powerful the existence of Ramsey-type König's principle is with the same restriction.

Definition 14 (System **WRKL**) **WRKL** is obtained from **RKL** by considering only trees of positive measure.

As **WRKL** is a weakening of **RKL**, we can define a stronger statement still restricted to trees of positive measure and wonder wether is will add enough strength to be equivalent to **RKL**.

Definition 15 (System $\mathbf{WRKL^+}$) $\mathbf{WRKL^+}$ is obtained from **WRKL** by considering only homogeneous sets of color 0.

1.7 Diagonally Non-Computable functions

Any principle can be used to define a new axiomatic system. A wide range of studied systems are defined by the extension of $\mathbf{RCA_0}$ verifying a given principle. The whole difficulty of reverse mathematics consists of relating different principles by proving whether they are equivalent, one strictly implies the other or wether they are unrelated. DNC-like functions provide a uniform principle definition framework using function-based principles. Therefore they help understanding the relations between systems as they are described under the same aspect.

Rice's theorem [15] states that no set of partial function verifying non trivial properties is computable. In this case, these principles state the existence of such sets for some given properties. Here are a few of them. *DNC* stands for *Diagonally Non Computable* and *FPF* for *Fixpoint-Free Function.*

- **DNC**: For every set X, there exists a total function $f \in \omega^\omega$ such that for each Turing index e, $f(e) \neq \Phi_e^X(e)$
- $\mathbf{DNC_k}$: For every set X, there exists a total function $f \in k^\omega$ such that for each Turing index e, $f(e) \neq \Phi_e^X(e)$
- $\mathbf{DNC_h}$ (where h is a computable function): For every set X, there exists a h-bounded total function $f \in \omega^\omega$ such that for each Turing index e, $f(e) \neq \Phi_e^X(e)$
- **FPF**: For every set X, there exists a total function $f \in \omega^\omega$ such that for each Turing index e, $\Phi_{f(e)}^X \neq \Phi_e^X$

Manipulation of such principles is quite well understood and there is a bunch of separation and equality results between function-based principles.

Theorem 6 (Jockusch, Lerman, Soare & Solovay [9]) $\mathbf{RCA_0} \vdash \mathbf{DNC} = \mathbf{FPF}$

Theorem 7 (Jockusch [8]) For all $k \geq 2$ and $f \in \mathbf{DNC_{k+1}}$, there exists a functionnal Γ such that $\Gamma^f \in \mathbf{DNC_k}$. However the reduction is not uniform.

In other words $\mathbf{DNC_k} \leq_w \mathbf{DNC_{k+1}}$ but $\mathbf{DNC_k} \not\leq_s \mathbf{DNC_{k+1}}$.

Theorem 8 (Ambos-Spies, Kjos-Hanssen, Lempp & Slaman [1]) For all computable function h, there exists a computable function g such that $\mathbf{DNC_g} \subsetneq \mathbf{DNC_h}$

The new goal becomes then to express a given system as an extension of $\mathbf{RCA_0}$ verifying a function-based principle to reuse our separation tools on it. For example we have a nice characterization of $\mathbf{WKL_0}$:

Theorem 9 (Jockusch [8]) $\mathbf{RCA_0} \vdash \mathbf{WKL_0} = \mathbf{DNC_2}$

2 The system WRKL

We will first prove that $\mathbf{RCA_0} \vdash \mathbf{DNC} \Rightarrow \mathbf{WRKL}^+$ by two approaches: a probabilistic one and a combinatorial one. The proof requires the following general purpose lemma saying that if we are given a DNC function, for each size-bounded c.e. set we can uniformly compute an number outside of it.

Lemma 1 ($\mathbf{RCA_0}$) Let f be a DNC function, g a computable function, $(W_e)_{e\in\mathbb{N}}$ an enumeration of c.e. sets such that $\mathtt{card}(W_e) \leq g(e)$ for all e. Then there is a f-computable function h such that $W_{h(e)}$ is infinite and $W_{h(e)} \cap W_e = \emptyset$.

Proof. Let e be an enumeration index. We construct the Turing machines $\Phi_{e_1}, \dots, \Phi_{e_{g(e)}}$ as follows

$$\forall n,\, \Phi_{e_i}(n) \overset{def}{=} \begin{cases} \Phi_x(i) & \text{if } x \text{ is the } i\text{th element of } W_e \\ \uparrow & \text{if } \mathtt{card}(W_e) < i \end{cases}$$

If $f(e_i) = n$ then for any x such that $\Phi_x(e) = n$, x is different from the ith element enumerated by W_e. So it suffices to choose for $h(e)$ such that for all $x \in W_{h(e)}$, $\Phi_x(i) = f(e_i)$. $\square$

We now give the main proofs about $\mathbf{DNC} = \mathbf{WRKL}$ assuming the following lemma which will be proven by two ways in further sections.

Lemma 2 ($\mathbf{RCA_0}$) There are computable functions g and $h \in \omega^\omega$ such that for each binary tree T of measure $\mu(T) > 2^{-m}$,

$$\mathtt{card}\left\{n \in \omega \ : \ \mu(T \cap \Gamma^0_n) \leq 2^{-g(m)}\right\} < h(m)$$

Proof. See lemma 5 or 9. $\square$

Theorem 10 ($\mathbf{RCA_0}$) $\mathbf{DNC} \Rightarrow \mathbf{WRKL}^+$

Proof. Let f be a DNC function and T a binary tree of measure $\mu(T) > 2^{-m}$. Let g and h be the functions of lemma 2. Using f, we will construct a f-computable strictly increasing sequence of integers (u_n) and a decreasing sequence of trees $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ of positive measure verifying the two properties:

$$\mathcal{P}_i : \mu(T_i) > 2^{-g^i(m)} \qquad \mathcal{Q}_i : T_i = T \bigcap_{j\leq i} \Gamma^0_{u_j}$$

At stage i, assume we have a tree T_i verifying above properties. Let's consider the following p.c. function:

$$\Phi_e(n) \downarrow \Leftrightarrow \mu(T_i \cap \Gamma^0_n) \leq 2^{-g^{i+1}(m)}$$

By lemma 2, $\mathtt{card}(W_e) < h(g^i(m))$ and using lemma 1 we can compute a value $u_{i+1} \notin W_e \cup \{u_j \ : \ j \leq i\}$. We set then $T_{i+1} = T_i \cap \Gamma^0_n$ and begin stage $i+1$.

For the sake of contradiction, if $H \overset{def}{=} \{u_i \ : \ i \in \omega\}$ is not an infinite homogeneous set with color 0 for a path in T, then there exists an i such that $H_i \overset{def}{=} \{u_j \ : \ j \leq i\}$ does not form a homogeneous set with color 0 for a path in T. And we derive the following contradiction:

$$0 = \mu(T \bigcap_{j\leq i} \Gamma^0_{u_j}) = \mu(T_i) > 2^{-g^i(m)}$$

$\square$

Theorem 11 (**RCA₀**) The following classes are equivalent

(i) **WRKL**

(ii) $\mathbf{WRKL}^+$

(iii) **DNC**

Proof. (ii) $\Rightarrow$ (i) is obvious. (ii) $\Rightarrow$ (iii) is the proof of **RKL** $\Rightarrow$ **DNC** from Flood [4] by noticing that the only use of the **RKL** existence axiom is on a tree of positive measure. (iii) $\Rightarrow$ (ii) is theorem 10. $\square$

For the reader who knows the theory of algorithmic randomness, this result is confirmed by the following theorem:

Theorem 12 (Kjos-Hanssen [11], Greenberg & Miller [5]) The following are equivalent:

(i) A computes a DNC function.

(ii) A computes an infinite subset of a Martin Löf random.

$\mathbf{WWKL_0}$ is known to be equivalent to the existence of a MLR as Kučera proved in [12] that any Π^0_1 class of positive measure contains all MRL up to prefix modification and as there exists a Π^0_1 class containing only MLRs. Hence intuitively **WRKL** corresponds to the existence of an infinite subset of a MRL and hence is equivalent to the existence of a DNC function.

Note that we could have slightly modified our proof to ensure that the constructed set is homogeneous both for a path with color 0 and for a path with color 1 in the tree.

The core of the proof of **WRKL** = **DNC** relies on lemma 2 saying that the set of bad candidates for an homogeneous set is finite and moreover uniformly bounded. This lemma is proved using a probabilistic approach (lemma 5) and a combinatorial one (lemma 8). We will now develop each approach.

2.1 Probabilistic proof of WRKL = DNC

Here is the intuition of the probabilistic approach: If we fix a set of positions and choose a binary sequence uniformly, we are unlikely to have much more than half of 1's in the chosen positions. However we are given a Π^0_1 class $\mathcal{C}$ such that if a sequence is in $\mathcal{C}$, the probability of having much more 1's than 0's at the chosen positions is very high. We can hence construct an upper bound of the measure of $\mathcal{C}$ in function of the number of such positions. As the size of the tree is bounded, the set is finite and the proof gives a computable bound.

Lemma 3 ($\mathbf{RCA_0}$) Let X be a random variable following a uniform distribution over binary sequences. Let $\{x_1, \dots, x_k\}$ be a set of k integers and $X_i \overset{def}{=} X(x_i)$, ie. the value of the x_ith digit of X. Then the following holds:

$$\mathbb{P}\left[\sum_{i=1}^{k} X_i \geq \frac{3k}{4}\right] \leq 2\mathbf{e}^{-\frac{k}{16}}$$

Proof. We consider the mutually independant random variables $Y_i = 2X_i - 1$. Let $Y = \sum_{i=1}^{k} Y_i = 2\sum_{i=1}^{k} X_i - k$. Using $Var(Y) = E(Y^2) - E(Y)^2$ and by a simple induction we obtain $Var(Y) = k$. By Chernoff bound

$$\mathbb{P}\left[\sum_{i=1}^{k} X_i \geq \frac{3k}{4}\right] \leq \mathbb{P}\left[\left|2\sum_{i=1}^{k} X_i - k\right| \geq \frac{k}{2}\right] = \mathbb{P}\left[|Y| \geq \frac{\sqrt{k}}{2}\sqrt{k}\right] \leq 2\mathbf{e}^{-\frac{k}{16}}$$

□

Lemma 4 ($\mathbf{RCA_0}$) Let X be a random variable following a uniform distribution over binary sequences. Let T be an event of positive probability, Γ_n^0 the event "there is a 0 at n-th position of X" and $S = \{x_1, \ldots, x_k\}$ a set of k integers such that $\mathbb{P}\left[\Gamma_{x_i}^0 | T\right] \leq \frac{1}{2}$. We write $X_i = X(x_i)$. Then the following holds:

$$\mathbb{P}\left[\sum_{i=1}^{k} X_i \geq \frac{3k}{4} \; |T\right] > \frac{1}{2}$$

Proof. Let $Y_i = 1 - X_i$. So $E[Y_i|T] < \frac{1}{2}$. Let $Y = \sum_{i=1}^k Y_i$. We have $E[Y|T] = \sum_{i=1}^k E[X_i|T] < \frac{k}{2}$. Using Markov inequality

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{k} X_i < \frac{3k}{4} \; |T\right] &= \mathbb{P}\left[k - \sum_{i=1}^{k} X_i \geq \frac{k}{4} \; |T\right] = \mathbb{P}\left[\sum_{i=1}^{k}(1 - X_i) \geq \frac{k}{4} \; |T\right] \\
&= \mathbb{P}\left[Y \geq \frac{3k}{4} \; |T\right] \leq \frac{4E[Y \; |T]}{k} < \frac{1}{2}
\end{aligned}$$

And we obtain the desired lower bound.

$$\mathbb{P}\left[\sum_{i=1}^{k} X_i \geq \frac{3k}{4} \; |T\right] = 1 - \mathbb{P}\left[\sum_{i=1}^{k} X_i < \frac{3k}{4} \; |T\right] > \frac{1}{2}$$

□

Lemma 5 ($\mathbf{RCA_0}$) Let T be a binary tree of measure $\mu(T) > 2^{-n}$ and $k \in \omega$ such that $2\mathbf{e}^{-\frac{k}{16}} \leq \frac{1}{2}$ and

$$S \overset{def}{=} \left\{n \in \omega \;:\; \mu(T \cap \Gamma_n^0) \leq 2^{-n-1}\right\}$$

Then $\texttt{card}(S) < k$.

Proof. Let X be a random variable following a uniform distribution over binary sequences. By abuse of notation, T also denotes the event "$X \in [[T]]$". If there exists a subset $\{x_1, \ldots, x_k\}$ of S of size k, then we can define the random variables $X_i \overset{def}{=} X(x_i)$. Let E be the event "$\sum_{i=1}^k X_i \geq \frac{3k}{4}$". Using lemmas 3 and 4 we obtain the following contradiction.

$$\frac{1}{2} \geq 2\mathbf{e}^{-\frac{k}{16}} \geq \mathbb{P}\left[E\right] \geq \mathbb{P}\left[T \cap E\right] = \mathbb{P}\left[T\right]\mathbb{P}\left[E|T\right] > \frac{1}{2}$$

□

2.2 Combinatorial proof of WRKL = DNC

In this part, will computably bound the measure of a tree T in function of the size of the set of levels where we can't choose to build an homogeneous set for a path of T. We reduce this problem to the question of a maximal set of strings having less 0's in their columns than a given value. We bound the size of the maximal set and deduce from it a bound of the measure of a tree. Then by fixing the measure of the tree, we can extract a bound of the size of wrong levels for an homogeneous set.

Lemma 6 ($\mathbf{RCA_0}$) Let T be a binary tree, $m \in \omega$ and $S = \{x_1, \ldots, x_k\}$ a set of k integers such that $\mu(T \cap \Gamma_{x_i}^0) \leq 2^{-m}$ for each i. Then there exists a binary tree T' of the same measure such that

$$\forall n < \texttt{card}(S),\; \mu(T' \cap \Gamma_n^0) \leq 2^{-m}$$

Proof. We can obtain a tree T' from T by giving a Δ^0_1 formula which will flip the bits of S and the first $\texttt{card}(S)$ bits of T. To ensure that we can flip the coins, we only consider strings of length greater than the maximum value of S, ie. x_k an complete the tree with all strings of length smaller than x_k. The measure remains the same.

$$T' \stackrel{def}{=} 2^{<x_k} \cup \left\{ s \in 2^{<\omega} \ : \ \exists t \in T, \ |t| = |s| \geq x_k \wedge \forall i < |s|, \, s(i) = \begin{cases} t(x_i) & \text{if } i < k \\ t(j) & \text{if } i = x_j \\ t(i) & \text{otherwise} \end{cases} \right\}$$

□

Lemma 7 ($\mathbf{RCA_0}$) Let $k, m, N, M \in \omega$ and S' be a set of strings of length N such that $k \leq N$ and

$$\sum_{j=0}^{M-1} \binom{k-1}{j} \geq 2^{k-m} \qquad \forall i < k, \, \texttt{card}\{\sigma \in S' \ : \ \sigma[i] = 0\} \leq 2^{N-m}$$

Then

$$\texttt{card}(S') \leq \sum_{j=0}^{M} \binom{k}{j} \times 2^{N-k}$$

Proof. Being given a string σ of length N, we denote by $0(\sigma)$ the number of zeros in the k first positions. For a set S of strings of length N, we denote by $0(S)$ the total number of 0's in the k first positions. Let $S_{\leq M}$ be the set of strings of length N having at most M 0's in the first k positions. The following property holds:

$$\forall i < k, \, \texttt{card}\{\sigma \in S_{\leq M} \ : \ \sigma[i] = 0\} = \sum_{j=0}^{M-1} \binom{k-1}{j} \times 2^{N-k} \geq 2^{N-m}$$

So for any set S' verifying required properties, we have $0(S') \leq 0(S_{\leq M})$. If $\texttt{card}(S' - S_{\leq M}) > \texttt{card}(S_{\leq M} - S')$, as for any $\sigma_1 \in S' - S_{\leq M}$ and $\sigma_2 \in S_{\leq M}$ we have $0(\sigma_1) \geq 0(\sigma_2)$ then $0(S' - S_{\leq M}) > 0(S_{\leq M} - S')$ and we have the following contradiction:

$$0(S') = 0(S' - S_{\leq M}) + 0(S' \cap S_{\leq M}) > 0(S_{\leq M} - S') + O(S' \cap S_{\leq M}) = 0(S_{\leq M})$$

So $\texttt{card}(S' - S_{\leq M}) \leq \texttt{card}(S_{\leq M} - S')$ and hence

$$\begin{aligned} \texttt{card}(S') &= \texttt{card}(S' \cap S_{\leq M}) + \texttt{card}(S' - S_{\leq M}) \\ &\leq \texttt{card}(S' \cap S_{\leq M}) + \texttt{card}(S_{\leq M} - S') \\ &\leq \texttt{card}(S_{\leq M}) = \textstyle\sum_{j=0}^{M} \binom{k}{j} \times 2^{N-k} \end{aligned}$$

□

The intuition behind lemma 7 is very simple: for any $M \in \omega$, the set $S_{\leq M}$ is optimal, in the sense that each of the k first columns contains the same number of 0's – which is computable in function of M using combinatorics – and each string in $S_{\leq M}$ contains the less posible number of 0's. So any other set would contains either more 0's in a column and hence wouldn't be a solution, or it would contain strings with more 0's and hence it would be a smaller set.

Example 1 Let's illustrate lemma 7 with $k = 4$, $N = 5$, $m = 2$. We can take $M = 2$ as

$$(\sum_{j=0}^{M-1} \binom{k-1}{j}) \times 2^{N-k} = (\binom{3}{0} + \binom{3}{1}) \times 2 = 8 \geq 2^3 = 2^{N-m}$$

The choice of M is so that the set $S_{\leq M}$ has at least $2^{N-m} = 8$ 0's in each k first columns:

$\overbrace{}^{N}$	$\overbrace{}^{k}$				
11110	1011 0	11100	**0**1010	10010	
11111	10111	11101	**0**1011	10011	11000
01110	11010	**0**0110	**0**1100	10100	11001
01111	11011	**0**0111	**0**1101	10101	

Notice that this set has cardinality

$$\texttt{card}(S_{\leq M}) = \sum_{j=0}^{M} \binom{k}{j} \times 2^{N-k} = (\binom{4}{0} + \binom{4}{1} + \binom{4}{2}) \times 2 = 22$$

The lemma claims that there can't be a set bigger than $S_{\leq M}$ verifying required properties, because $S_{\leq M}$ has already at least the maximum number of 0's in each k first columns, using strings with the least number of 0's in each. So the total number of 0's of a set verifying required properties must be less than $8 \times 4 = 32$, and if there are strings in such a set outside $S_{\leq M}$, then it would have more 0's and so there must be less such strings to stay under the total bound 32.

Notice that in this particular case, $S_{\leq M}$ is a solution and hence its cardinality is the optimal bound, but sometimes we have strictly more 0's in the k first columns of $S_{\leq M}$ than the allowed amount. The reasoning remains valid, but the bound isn't any more tight.

Lemma 8 ($\mathbf{RCA_0}$) Let T be a binary tree, $m, M \in \omega$ and S a finite set such that

$$\sum_{j=0}^{M-1} \binom{\texttt{card}(S) - 1}{j} \geq 2^{\texttt{card}(S)-m} \qquad \forall n \in S,\ \mu(T \cap \Gamma_n^0) \leq 2^{-m}$$

Then

$$\mu(T) \leq \frac{\sum_{j=0}^{M} \binom{\texttt{card}(S)}{j}}{2^{\texttt{card}(S)}}$$

Proof. Set $\texttt{card}(S) = k$. Using lemma 6 we can assume that $S = \{0, \ldots, k-1\}$. By hypothesis the following holds: $\sum_{j=0}^{M-1} \binom{k-1}{j} \geq 2^{k-m}$. For each $N \in \omega$, $S_N \overset{def}{=} \{\sigma \in T \ : \ |\sigma| = N\}$. If there exists an $i < k$ such that for all N, $\texttt{card}\,\{\sigma \in S_N \ : \ \sigma[i] = 1\} > 2^{N-m}$. Then

$$\mu(T \cap \Gamma_i^0) \overset{def}{=} \lim_{N \to \infty} \frac{\texttt{card}\,\{\sigma \in T \ : \ |\sigma| = N \wedge \sigma[i] = 0\}}{2^N} > \lim_{N \to \infty} \frac{2^{N-m}}{2^N} = 2^{-m}$$

which would contradict the hypothesis. Therefore there exists an $N \in \omega$ such that

$$\forall i < k,\ \texttt{card}\,\{\sigma \in S_N \ : \ \sigma[i] = 1\} \leq 2^{N-m}$$

Then by lemma 7

$$\texttt{card}(S_N) \leq \sum_{j=0}^{M} \binom{k}{j} \times 2^{N-k}$$

And hence

$$\mu(T) \overset{def}{=} \lim_{N \to \infty} \frac{S_N}{2^N} \leq \frac{\sum_{j=0}^{M} \binom{k}{j} \times 2^{N-k}}{2^N} = \frac{\sum_{j=0}^{M} \binom{\texttt{card}(S)}{j}}{2^{\texttt{card}(S)}}$$

□

Lemma 9 ($\mathbf{RCA_0}$) Let T be a binary tree of measure $\mu(T) > 2^{-n}$. We can uniformly compute $k \in \omega$ such that

$$\texttt{card}\left\{n \in \omega \ : \ \mu(T \cap \Gamma^0_n) \leq 2^{-k}\right\} < k$$

Proof. Let $k, M \in \omega$ such that $M \geq 1$ and $\sum_{j=0}^{M} \binom{k}{j} \leq 2^{k-n}$ Let's assume for the sake of contradiction that there exists a set S of size k such that $\forall n \in S$, $\mu(T \cap \Gamma^0_n) \leq 2^{-k}$. By setting $m = k$, we have $\sum_{j=0}^{M-1} \binom{k-1}{j} \geq 1 = 2^{\texttt{card}(S)-m}$ and using lemma 8 we can derive the following contradiction.

$$2^{-n} < \mu(T) \leq \frac{\sum_{j=0}^{M} \binom{\texttt{card}(S)}{j}}{2^{\texttt{card}(S)}} \leq 2^{-n}$$

□

3 The system RKL

In this section we will introduce some variations of **RKL** in order to understand better which aspects of its principle is responsible of its computational power.

3.1 RKL vs $\mathbf{WKL_0}$

The proof of **RKL** $<$ $\mathbf{WKL_0}$ given by Flood in [4] exploits a result from Liu [13] saying that $\mathbf{RT^2_2}$ does not imply $\mathbf{WKL_0}$: as **RKL** $\leq$ $\mathbf{RT^2_2}$ we can't have **RKL** $=$ $\mathbf{WKL_0}$. However, the result from Liu has been a long standing open question and the proof involved very powerful technics. There might exist a direct proof of **RKL** $<$ $\mathbf{WKL_0}$ with would give more insight about what happens really.

A technic could be to define a stronger variant of **RKL** and characterize it in terms of a weaker system than $\mathbf{WKL_0}$. In the two following sections, we will study two strengthened versions of **RKL** which happend to be equivalent to $\mathbf{WKL_0}$. Such a result remains of interest as it gives rise to the importance of some details of the definition to have exactly the power of **RKL**. Hence we can rule out any attempt of characterization by a principle which could construct a set as stated in one of the variants above.

3.2 RKL with fixed color

Theorem 11 shows that in the case of trees of positive measure, assuming the existence of an infinite homogeneous set of any color for a path is similar to assume its existence with a fixed color. As the question of separation between **WRKL** and **RKL** remains still open, it is natural to ask wether it is still the case when we remove the assumption of positive measure.

Definition 16 (System $\mathbf{RKL^+}$) $\mathbf{RKL^+}$ asserts "each infinite binary tree T with no computable path has an infinite set which is homogeneous with color 0 for a path through T.

We need to ensure that the tree has no computable path as otherwise, it could have only path with finite number of left-branching. If such a case happens, the tree has a computable path as we can hard-code the prefix of the path until there remains only 1's.

Theorem 13 ($\mathbf{RCA_0}$) $\mathbf{RKL^+ \Leftrightarrow WKL_0}$

Proof. $\mathbf{WKL_0} \Rightarrow \mathbf{RKL^+}$ is obvious because a non-computable path contains an infinity of 0 and we can then extract an infinite homogeneous set with color 0. Let's consider the other direction.

We define a tree whose paths have zeros only at positions corresponding to the initial segment of a DNC_2 function. As we can compute a function from an infinite subset of its initial segments, we can compute a DNC_2 function from an infinite homogeneous set for a path through this tree with color 0.

$$T \stackrel{def}{=} \left\{ \begin{array}{ll} s \in 2^{<\omega} \ : \ \exists s' \in 2^{|s|}, \forall i \leq |s|, & \forall j \leq i, \ \Phi_j(j)[i] \neq s'(j) \\ & \wedge \ s(i) = \left\{ \begin{array}{ll} 0 & \text{if } i \preceq \sigma' \\ 1 & \text{otherwise} \end{array} \right. \end{array} \right\}$$

The formula says that given a guess s' of an initial segment of a DNC_2 function, we add strings whose only bits at 0 are those which corresponds to a prefix of s'. So any path in T will be a set of initial segments of a DNC_2 function.

Le H be an infinite homogeneous set for a path in T with color 0. Let e be a Turing index and let k be such that $\Phi_e(e)[k] = \Phi_e(e)$. We search for an $i \in H$ such that $|i| \geq e$.

As H is homogeneous for a path in T, $i \preceq s_1 \prec s_2 \prec s_3 \dots$ and hence $|s_k| \geq k$. By construction of T the following holds

$$\forall j \leq |s_k| \, \Phi_j(j)[k] \neq s'(j)$$

Then $\Phi_e(e) = \Phi_e(e)[k] \neq s'(e)$. □

However, this result isn't enough to conclude that **WRKL** and **RKL** are distinct.

3.3 RKL with bounded sparsity

The homogeneous sets of **RKL** can be arbitrarily sparse. One might wonder wether restricting this sparsity by imposing a computable bound to the nth element of the set would make the system strictly stronger. In fact, such a variation raises the new system to the power of $\mathbf{WKL_0}$.

Definition 17 (System $\mathbf{RKL_h}$) Let $h \in \omega^\omega$ be a computable function. $\mathbf{RKL_h}$ asserts "each infinite binary tree T has two infinite sets H_1 and H_2 such that

(i) H_1 is homogeneous with color 0 for a path through T.

(ii) H_2 is homogeneous with color 1 for a path through T.

(iii) $\forall n \in \omega$, $\texttt{card}((H_1 \cup H_2) \restriction h(n)) \geq n$.

Theorem 14 ($\mathbf{RCA_0}$) For any strictly increasing computable function h, $\mathbf{RKL_h} \Leftrightarrow \mathbf{WKL_0}$

Proof. Let T be a binary tree. We will construct a "elongated" tree so that the homogeneous subset will give each of the bits of a path in T. We will create redundancy in bits of T by define a sequence $(u_n)_{n\in\omega}$ such that either H_1 or H_2 will have an element in $[u_n, u_{n+1})$. One may be tempted to choose $[h(n-1), h(n))$ but there is a tricky point: nth is assumed to be lower then $h(n)$, but it can be also lower then $h(n-1)$ and hence $(H_1 \cup H_2) \cap [h(n-1), h(n))$ might be empty. So we must introduce more redundancy so that we can use the pigeonhole principle. Let's consider the following sequence:

$$u_n \stackrel{def}{=} \begin{cases} u_0 = 0 \\ u_{n+1} = h(u_n + 1) \end{cases}$$

This sequence is chosen such that $(H_1 \cup H_2) \cap [u_n, u_{n+1})$ is never empty for any n. This is an easy consequence of the pigeonhole principle which is provable in $\mathbf{RCA_0}$.

$$T' \stackrel{def}{=} \left\{ s' \in 2^{<\omega} \ : \ \exists s \in T, \ |s'| = u_{|s|} \wedge \forall i < |s|, \forall j < u_{i+1}, \, j \geq u_i \Rightarrow s'(j) = s(i) \right\}$$

For example, if $h : n \mapsto 2n$, then a string 10101 in T will correspond in T' to

$$\overbrace{11}^{u_1-u_0}\ \overbrace{0000}^{u_2-u_1}\,\overbrace{11111111}^{u_3-u_2}\overbrace{0000000000000000}^{u_4-u_3}\overbrace{11111111111111111111111111111111}^{u_4-u_3}$$

□

We conjecture that imposing a bound to **WRKL** would give a system equivalent to $\mathbf{DNC_h}$ for a computable function h uniformly related to the bound.

4 Conclusion

Although the exact strength of **RKL** – ie. its relation to **DNC** and $\mathbf{WWKL_0}$ – remains an open problem, we have deepened our comprehension of Ramsey-type König's lemma principle by characterizing some of its variants in terms of existing systems. It's restriction to classes of positive measure (**WRKL**) is equivalent to the existence of a diagonally non computable function (**DNC**). It's strengthenings $\mathbf{RKL^+}$ by fixing the color or $\mathbf{RKL_h}$ by bounding the sparsity of the homogeneous sets makes it equivalent to weak König's lemma ($\mathbf{WKL_0}$). Using connections between reverse mathematics and randomness theory, we have given another way of proving the computability of an infinite subset of a Martin Löf random using a DNC function.

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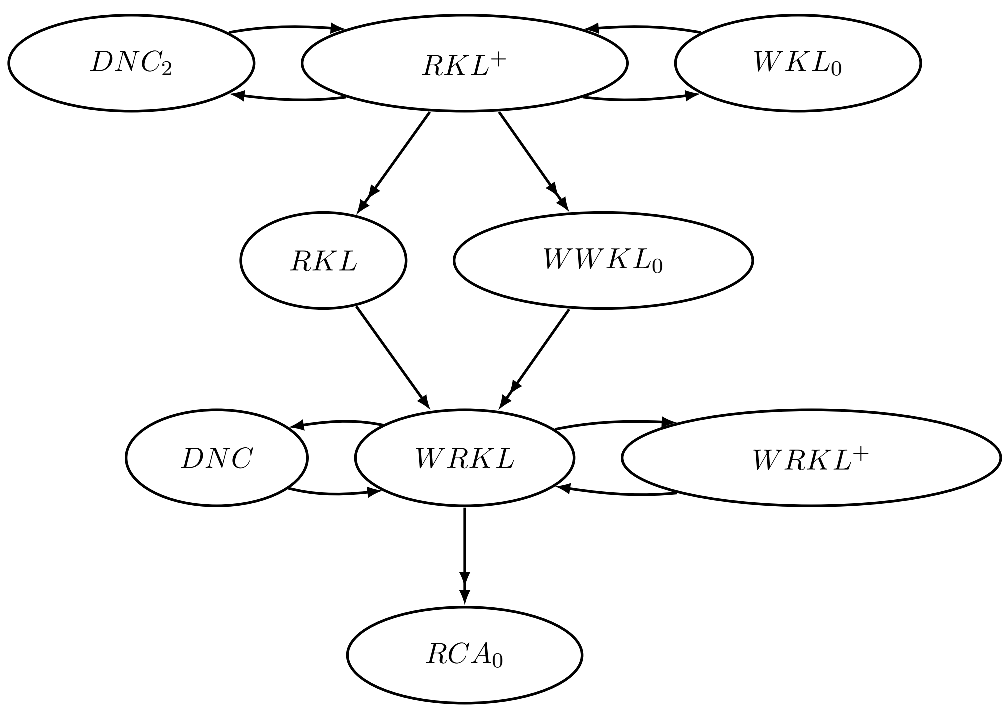

Figure 1: Summary of classes considered

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