Controlling iterated jumps of Ramsey-type theorems

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RAMSEY-TYPE HIERARCHIES

Ramsey's theorem

Definition
Given a coloring $f : [\mathbb{N}]^n \to k$, a set H is f-homogeneous if there exists a color $i < k$ such that $f([H]^n) = i$.

RT^n_k : Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set.

Rainbow Ramsey theorem

Definition
A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is k-bounded if each color is used at most k times. A set H is an f-rainbow if f is injective on $[H]^n$.

RRT^n_k : Every k-bounded coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has an infinite f-rainow.

RAMSEY-TYPE HIERARCHIES

Thin set theorem

Definition
Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, a set H is f-thin if $f([H]^n)$ avoids i.

TS^n : Every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has an infinite f-thin set.

Free set theorem

Definition
Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, a set H is f-free if for every $\sigma \in [H]^n$, $f(\sigma) \in H \to f(\sigma) \in \sigma$.

FS^n : Every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has an infinite f-free set.

THE HIERARCHIES OVER COMPUTABLE REDUCIBILITY

Definition (Jockusch's bounds)

For every $n \geq 2$,

(i) Every computable P^n-instance has a Π^0_n solution.

(ii) There is a computable P^n-instance with no Σ^0_n solution.

If a hierarchy satisfies Jockusch's bounds, then it is strict over computable reducibility.

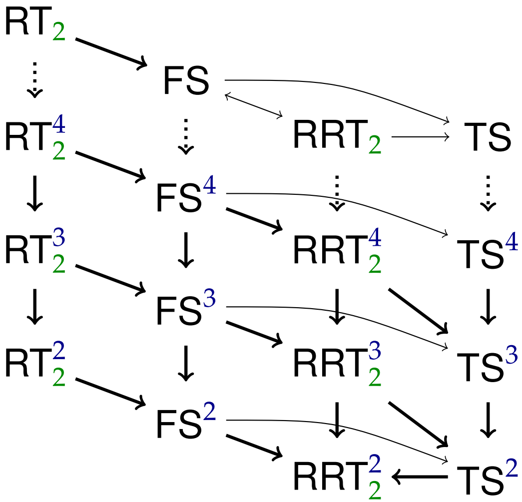
THE HIERARCHIES OVER COMPUTABLE REDUCIBILITY

Theorem

The following satisfy Jockusch's bounds.

- *Ramsey's theorem* (Jockusch)
- *The rainbow Ramsey theorem* (Csima, Mileti)
- *The free set theorem* (Cholak, Giusto, Hirst, Jockusch)
- *The thin set theorem* (Cholak, Giusto, Hirst, Jockusch)

THE HIERARCHIES OVER COMPUTABLE REDUCIBILITY



RAMSEY-TYPE HIERARCHIES

What about reverse mathematics?

RAMSEY'S THEOREM IN REVERSE MATHS

Theorem (Simpson)

For every $n \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \leftrightarrow \mathsf{ACA}_0$.

Theorem (Seetapun)

$\mathsf{RCA}_0 \wedge \mathsf{RT}^2_2 \nvdash \mathsf{ACA}_0$

RT^k_2, $k \geq 3$

$\downarrow$

RT^2_2

RAMSEY-TYPE HIERARCHIES IN REVERSE MATHS

Theorem (Wang)

None of FS, RRT_2 *and* TS *imply* ACA_0 *over* RCA_0.

Theorem (Cholak, Jockusch, Slaman)

Every computable RT^2_2*-instance admits a low*$_2$ *solution. The same holds for* FS^2, RRT^2_2 *and* TS^2.

Theorem (Wang)

Every computable RRT^3_2*-instance admits a low*$_3$ *solution.*

RAMSEY-TYPE HIERARCHIES IN REVERSE MATHS

Definition (Strong Jockusch's bounds)

For every $n \geq 2$,

(i) Every computable P^n-instance has a low$_n$ solution.

(ii) There is a computable P^n-instance with no Σ^0_n solution.

If a hierarchy satisfies strong Jockusch's bounds, then it is strict over reverse mathematics.

RAMSEY-TYPE HIERARCHIES IN REVERSE MATHS

Do FS, RRT_2 or TS satisfy strong Jockusch's bounds?

RAMSEY-TYPE HIERARCHIES IN REVERSE MATHS

Do FS, RRT_2 or TS satisfy strong Jockusch's bounds?

I don't know :'(

MATHIAS FORCING

Solutions to Ramsey-type theorems are built using variants of Mathias forcing.

$$(F, X)$$

Initial segment Reservoir

F is finite, X is infinite and $max(F) < min(X)$.

LOW$_n$-NESS

Make a $\emptyset^{(n)}$-effective construction which decides $\Sigma^{0,G}_n$ formulas.

$$c \Vdash \varphi(G)$$

Condition Arithmetic formula

If $\mathcal{F}$ sufficiently generic filter, then
$\varphi(G)$ holds iff $c \Vdash \varphi(G)$ for some $c \in \mathcal{F}$.

FORCING RELATION

Fix a condition $c = (F, X)$ and an arithmetic formula $\varphi(G)$.

Base case:
If φ is Σ^0_0, then $c \Vdash \varphi$ iff $\varphi(F)$ holds.

Induction rules:

(i) $c \Vdash (\exists n)\varphi(n)$ iff $c \Vdash \varphi(n)$ for some $n \in \omega$.

(ii) $c \Vdash \neg\varphi$ iff $d \nVdash \varphi$ for all $d \leq c$.

FORCING RELATION

computable Mathias forcing = computable reservoir.

Theorem (Cholak, Dzhafarov, Hirst, Slaman)

Fix a condition $c = (F, X)$ and an arithmetic formula $\varphi(G)$.

(1) *If φ is Σ^0_0, then the relation $c \Vdash \varphi$ is computable.*

(2) *If φ is Σ^0_1, Π^0_1 or Σ^0_2, then so is the relation $c \Vdash \varphi$.*

(3) *For $n \geq 2$, if φ is Π^0_n, then the relation $c \Vdash \varphi$ is Π^0_{n+1}.*

(4) *For $n \geq 3$, if φ is Σ^0_n, then the relation $c \Vdash \varphi$ is Σ^0_{n+1}.*

APPROACH

Define more precise conditions thanks to a refined analysis of the argument.

COHESIVENESS

Definition
An infinite set C is $\vec{R}$-cohesive for a sequence of sets $R_0, R_1, \ldots$ if for each $i \in \omega$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

COH: "Every sequence of sets $\vec{R}$ has an $\vec{R}$-cohesive set."

The $\vec{R}$-cohesive set C is

- p-cohesive if $\vec{R}$ are all the primitive recursive sets
- r-cohesive if $\vec{R}$ are all the computable sets

COHESIVENESS

Fix $R_0, R_1, \ldots$ Given some condition (F, X),

(S1) the extension step:

- take an element x from X and add it to F
- new condition: $(F \cup \{x\}, X \setminus [0, x])$

(S2) the cohesiveness step:

- choose an infinite $X \cap R_i$ or $X \cap \overline{R}_i$
- new condition: $(F, X \cap R_i)$ or $(X \cap \overline{R}_i)$

COHESIVENESS

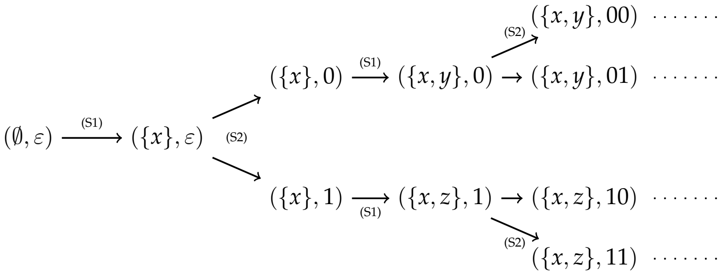
- After 0 coh step, we obtain $(F, \omega \setminus [0, max(F)])$
- After 1 coh step, we obtain one of the following
  - $(F, R_0 \setminus [0, max(F)])$
  - $(F, \overline{R}_0 \setminus [0, max(F)])$
- After 2 coh steps, we obtain one of the following
  - $(F, R_0 \cap R_1 \setminus [0, max(F)])$
  - $(F, \overline{R}_0 \cap R_1 \setminus [0, max(F)])$
  - $(F, R_0 \cap \overline{R}_1 \setminus [0, max(F)])$
  - $(F, \overline{R}_0 \cap \overline{R}_1 \setminus [0, max(F)])$
- And so on...

COHESIVENESS

- Define $R_\varepsilon = \omega$.
- Given a string σ of length i,
  - $R_{\sigma 0} = R_\sigma \cap \overline{R}_i$
  - $R_{\sigma 1} = R_\sigma \cap R_i$

$$(F, \sigma) \text{ denotes } (F, R_\sigma \setminus [0, max(F)])$$

COHESIVENESS



COHESIVENESS

Definition
A tuple $c = (F, \sigma)$ is valid iff R_σ is infinite.

Definition

$$\mathcal{T}(\vec{R}) = \{\sigma \in 2^{<\omega} : |R_\sigma| \geq |\sigma|\}$$

- $\mathcal{T}(\vec{R})$ is an infinite, Δ^0_2 binary tree.
- (F, σ) is valid iff σ is extendible in $\mathcal{T}(\vec{R})$

COHESIVENESS

A condition is a tuple (F, σ, T) such that

(a) F is a finite set

(b) T is an infinite, $\emptyset'$-p.r. subtree of $\mathcal{T}(\vec{R})$

(c) $\sigma \in 2^{<\omega}$ is a stem of T

A condition (E, τ, S) extends (F, σ, T) iff

(i) $F \subseteq E$, $E \setminus F \subseteq R_\sigma \setminus [0, max(E)]$

(ii) $\sigma \preceq \tau$

(iii) $S \subseteq T$

FORCING Σ^0_1 AND Π^0_1 FORMULAS

Fix a precondition $c = (F, \sigma, T)$ (drop "T is infinite").

Fix a Σ^0_0 formula $\varphi(G, x)$.

(i) $c \Vdash (\exists x)\varphi(G, x)$ iff $\varphi(F, w)$ holds for some $w \in \omega$

(ii) $c \Vdash (\forall x)\varphi(G, x)$ iff $\varphi(E, w)$ holds for every $w \in \omega$ and every set E satisfying (F, σ).

FORCING ARITHMETIC FORMULAS

Fix a condition $c = (F, \sigma, T)$.

Fix an arithmetic formula $\varphi(G)$.

(iii) If $\varphi = (\exists x)\psi(x)$ where $\psi \in \Pi^0_{n+1}$ then $c \Vdash \varphi$ iff there is a $w < |\sigma|$ such that $c \Vdash \psi(w)$

(iv) If $\varphi = (\forall x)\psi(x)$ where $\psi \in \Sigma^0_1$ then $c \Vdash \varphi$ iff for every $\tau \in T$, every E satisfying (F, τ) and every $w < |\tau|$, $(E, \tau, T^{[\tau]}) \nVdash \neg\psi(w)$

(v) If $\varphi = \neg\psi(x)$ where $\psi \in \Sigma^0_{n+3}$ then $c \Vdash \varphi$ iff $d \nVdash \psi$ for every $d \leq c$.

THE FORCING RELATION

Lemma

Suppose that $\mathcal{F}$ is a sufficiently generic filter and let G be the corresponding generic real. Then for each arithmetic formula $\varphi(G)$, $\varphi(G)$ holds iff $c \Vdash \varphi(G)$ for some $c \in \mathcal{F}$.

Lemma

Fix a condition c and an arithmetic formula $\varphi(G)$. If $\varphi(G)$ is a Σ^0_n (Π^0_n) formula then so is the relation $c \Vdash \varphi(G)$.

WHO'S NEXT?

We can do the same for

- The Erdős-Moser theorem $(\vec{F}, T, \mathcal{C})$
- Stable Ramsey's theorem for pairs $(\vec{F}, T, \mathcal{C})$
- ...

CONCLUSION

- The free set, thin set and rainbow Ramsey hierarchies are combinatorially weak.
- Eventhough conceptually simple, controlling iterated jumps is syntactically heavy.
- In order to control iterated jumps, we must avoid wasting properties.

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QUESTIONS

Thank you for listening!