

The role of randomness in reverse mathematics

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RRT₂² RCA₀ AMT
RT₂² COH ADS
OPT 1-GEN WKLO Π₁⁰G

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THEOREMS AS PROBLEMS

Look at “ordinary” theorems:

- ▶ (König's lemma)
Every **infinite, finitely branching tree** has an **infinite path**.
- ▶ (Ramsey's theorem)
Every **k -coloring** has an **infinite monochromatic subset**.
- ▶ (The atomic model theorem)
Every **complete atomic theory** has an **atomic model**.
- ▶ ...

THEOREMS AS PROBLEMS

Many theorems \mathbf{P} are of the form

$$(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of \mathbf{P} as a class of **problems**.

- ▶ An X such that $\Phi(X)$ holds is an **instance**.
- ▶ A Y such that $\Psi(X, Y)$ holds is a **solution** to X .

STRENGTH OF A THEOREM

Some theorems are more **effective** than others.

Theorem (Intermediate value theorem)

For every continuous function f over $[a, b]$ and every $y \in [f(a), f(b)]$, there is some $x \in [a, b]$ such that $f(x) = y$.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

STRENGTH OF A THEOREM

Provability strength

- ▶ Reverse mathematics
- ▶ Intuitionistic reverse mathematics

Computational strength

- ▶ Computable reducibility
- ▶ Uniform reducibility

Provability approach

REVERSE MATHEMATICS

Goal

Determine which axioms are required to prove **ordinary** theorems in reverse mathematics.

- ▶ Simpler proofs
- ▶ More insights

Subsystems of second-order arithmetic.

BASE THEORY RCA_0

- ▶ Basic Peano axioms
- ▶ Σ_1^0 induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ_1^0 formula of L_2

- ▶ Δ_1^0 comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X.\forall n.(x \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ_1^0 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π_1^0 formula of L_2 .

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures **computable** mathematics

RCA_0 has model $\mathcal{M} = \{\omega, S, <, +, \cdot\}$ where

- ▶ ω is the set of the standard integers
- ▶ $S = \{X \in 2^\omega : X \text{ is computable}\}$ is the second-order part

NON-PROVABILITY OVER RCA_0

Let P be a statement.

How to prove that $\text{RCA}_0 \not\vdash P$?

A method: Exhibit a computable instance I of P which admits no computable solution.

NON-PROVABILITY OVER RCA_0

Let \mathcal{M} be the model of RCA_0 whose second-order part are the computable sets.

- ▶ $\mathcal{M} \models \text{RCA}_0$;
- ▶ Because I is computable, $I \in \mathcal{M}$;
- ▶ Because I does not have a computable solution, $\mathcal{M} \not\models P$.

Therefore $\text{RCA}_0 \not\vdash P$.

PROBABILISTIC SOLUTIONS

Are there **probabilistic** algorithms to solve instances with no **computable** solution?

PROBABILISTIC SOLUTIONS

Definition (n -RAN)

“For every set X , there is a **Martin-Löf random** real relative to $X^{(n-1)}$ ”.

Given a statement P , does $\text{RCA}_0 \vdash n\text{-RAN} \rightarrow P$ for some n ?

PROBABILISTIC SOLUTIONS

Usually not

Definition (No randomized algorithm)

A statement P has the **NRA property** if it has a **computable** instance I such that

$$\mu\{X : X \text{ computes a solution to } I\} = 0$$

PROBABILISTIC SOLUTIONS

If P has the NRA property
then $\text{RCA}_0 \not\vdash n\text{-RAN} \rightarrow P$ for every n .

If P has the NRA property and $\text{RCA}_0 \vdash Q \rightarrow P$
then Q has the NRA property.

PROBABILISTIC SOLUTIONS

Many **weak** statements not provable over RCA_0 have the **NRA property**.

INTUITION

- ▶ Many proofs of a computable P -instance with no computable solutions are **diagonalizations**.
- ▶ Many diagonalizations can be done by **block**, defeating positive measure of oracles.

DIAGONAL NON-COMPUTABILITY

Definition (Diagonal non-computability)

A function f is **DNC relative to X** if $(\forall e)[f(e) \neq \Phi_e^X(e)]$

- ▶ **Simplest example** of non-computable function.
- ▶ Cantor's **diagonal argument**.
- ▶ **Unifying framework** for comparing theorems.

DIAGONAL NON-COMPUTABILITY

Theorem

The following are computably equivalent:

- ▶ *DNC functions relative to X*
- ▶ *Infinite subset of X -Martin-Löf randoms*
- ▶ *Escaping X -c.e. sets of computably bounded size*

n -DNC

For every set X , there is a function DNC relative to $X^{(n-1)}$.

DIAGONAL NON-COMPUTABILITY

Theorem

The following are computably equivalent:

- ▶ *$\{0, 1\}$ -valued DNC functions relative to X*
- ▶ *Computing an infinite path through an X -computable infinite binary tree*
- ▶ *Choosing between two $\Pi_2^{0,X}$ statements*

n -DNC₂

For every set X , there is a $\{0, 1\}$ -valued function DNC relative to $X^{(n-1)}$.

DIAGONAL NON-COMPUTABILITY

Theorem

$\text{RCA}_0 \vdash n\text{-RAN} \rightarrow n\text{-DNC}$

Hint: To define $f(n)$, pick a number **at random** in $[0, 2^{n+2}]$.

Theorem (Jockusch & Soare)

$n\text{-DNC}_2$ has the NRA property.

Hint: A finite range enables us to apply the **pigeonhole principle** and defeat a block of oracles.

Ramsey's theorem

RAMSEY'S THEORY

Given some **size s** , every **sufficiently large** collection of objects has a sub-collection of size s , whose objects satisfy some **structural properties**.

RAMSEY'S THEOREM

Definition

Given a coloring $f : [\mathbb{N}]^n \rightarrow k$, a set H is *f-homogeneous* if there exists a color $i < k$ such that $f([H]^n) = i$.

RT_k^n (Ramsey's theorem)

Every coloring $f : [\mathbb{N}]^n \rightarrow k$ has an infinite *f-homogeneous* set.

COHESIVENESS

Definition

Given a sequence of sets R_0, R_1, \dots , an infinite set C is \vec{R} -cohesive if for every i , $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

COH (Cohesiveness)

Every sequence of sets R_0, R_1, \dots has an \vec{R} -cohesive set.

COHESIVENESS

Theorem (Jockusch & Stephan)

The following are computably equivalent

- ▶ COH
- ▶ *For every set X , there is a set whose jump computes a $\{0, 1\}$ -valued function DNC relative to X' .*

Corollary (Jockusch & Soare)

COH has the NRA property.

THE ATOMIC MODEL THEOREM

AMT (Atomic model theorem)

Every complete atomic theory has an atomic model.

Theorem (Hirschfeldt, Shore, Slaman & Conidis)

The following are computably equivalent:

- ▶ **AMT**
- ▶ *For every Δ_2^0 function f , there exists a function g such that $f(x) \leq g(x)$ for infinitely many x .*

THE ATOMIC MODEL THEOREM

Theorem (Kurtz)

AMT has the NRA property.

Hint: \emptyset' is uniformly almost everywhere dominating.

THE RAINBOW RAMSEY THEOREM

Definition (k -bounded function)

A coloring function $\mathbb{N}^n \rightarrow \mathbb{N}$ is k -bounded if $|\{x \in \mathbb{N}^n : f(x) = c\}| \leq k$ for every color c .

RRT_k^n (Rainbow Ramsey theorem)

For every k -bounded coloring function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ there is an infinite set H such that $f \upharpoonright H^n$ is injective.

THE RAINBOW RAMSEY THEOREM

Theorem (Csimá & Mileti)

$$\text{RCA}_0 \vdash 2\text{-RAN} \rightarrow \text{RRT}_2^2$$

Theorem (Miller)

$$\text{RCA}_0 \vdash \text{RRT}_2^2 \leftrightarrow 2\text{-DNC}$$

Hint: The set of “bad” one-point extensions is a computably bounded \emptyset' -c.e. set.

THE RAINBOW RAMSEY THEOREM

Theorem (Bienvenu, Patey & Shafer)

RRT_2^3 has the NRA property.

Hint: RRT_2^3 implies the [atomic model theorem](#) over RCA_0 .

THE FINITE INTERSECTION PROPERTY

Definition

A sequence of set A_0, A_1, \dots has the **FIP** if the intersection of finitely many sets is non-empty.

FIP (Finite intersection property)

Every sequence of sets has a **maximal** subsequence having the FIP.

- ▶ Equivalent to the axiom of choice in set theory.

THE FINITE INTERSECTION PROPERTY

Definition

Fix a set of strings S . A real G **meets** S if it has some initial segment in S . A real G **avoids** S if it has an initial segment with no extension in S . A real X is **n -generic** if it meets or avoids every Σ_n^0 set of strings.

n -GEN (n -genericity)

For every set X , there is a real n -generic relative to X .

Theorem (Cholak, Downey, Diamondstone, Greenberg, Igusa & Turetsky)

$\text{RCA}_0 \vdash \text{FIP} \leftrightarrow 1\text{-GEN}$

THE FINITE INTERSECTION PROPERTY

Theorem (Kurtz, Kautz)

$RCA_0 \vdash 2\text{-RAN} \rightarrow \text{FIP}$

Hint: Use a fireworks argument.

Is 2-RAN needed? What about 2-DNC?

CONCLUSION

- ▶ Few theorems studied in reverse mathematics and not provable over RCA_0 admit probabilistic algorithms.
- ▶ All known examples have natural computability-theoretic characterization and admit a universal instance.
- ▶ Is 1-genericity a reverse mathematical consequence of the rainbow Ramsey theorem for pairs?

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QUESTIONS

Thank you for listening!