The strength of Ramsey's theorem under reducibilities

Ludovic PATEY
PPS, Paris 7

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STRENGTH OF A THEOREM

Some theorems are more effective than others.

Theorem (Intermediate value theorem)

For every continuous function f *over* $[a, b]$ *and every* $y \in [f(a), f(b)]$, *there is some* $x \in [a, b]$ *such that* $f(x) = y$.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

STRENGTH OF A THEOREM

Provability strength

- Reverse mathematics
- Intuitionistic reverse mathematics

Computational strength

- Computable reducibility
- Uniform reducibility

Provability approach

REVERSE MATHEMATICS

Goal

Determine which axioms are required to prove ordinary theorems in reverse mathematics.

- Simpler proofs
- More insights

Subsystems of second-order arithmetic.

BASE THEORY RCA_0

- Basic Peano axioms
- Σ^0_1 induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \to \varphi(n+1))) \to \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

- Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X.\forall n.(x \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

STANDARD MODELS OF RCA_0

An ω-structure is a structure $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$ where

(i) ω is the set of standard natural numbers

(ii) $<$ is the natural order

(iii) $+$ and $\cdot$ are the standard operations over natural numbers

(iv) $\mathcal{S} \subseteq \mathcal{P}(\omega)$

An ω-structure is fully specified by its second-order part $\mathcal{S}$.

STANDARD MODELS OF RCA_0

Definition (Turing ideal)

A Turing ideal $\mathcal{I}$ is a collection of subsets of ω which is closed under

(i) the Turing reduction: $(\forall X \in \mathcal{I})(\forall Y \leq_T X)[Y \in \mathcal{I}]$

(ii) the effective join: $(\forall X, Y \in \mathcal{I})[X \oplus Y \in \mathcal{I}]$.

STANDARD MODELS OF RCA_0

Fix an ω-structure $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$.

$$\mathcal{M} \models \mathsf{RCA}_0 \qquad \equiv \qquad \mathcal{S} \text{ is a Turing ideal.}$$

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures computable mathematics

RCA_0 a minimal ω-model $\mathcal{M} = \{\omega, \mathcal{I}, <, +, \cdot\}$
where $\mathcal{I}$ is the set of all computable subsets of ω.

Computational approach

THEOREMS AS PROBLEMS

Many theorems P are of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of P as a class of problems.

- An X such that $\Phi(X)$ holds is an instance.
- A Y such that $\Psi(X, Y)$ holds is a solution to X.

THEOREMS AS PROBLEMS

Examples:

- (König's lemma)
  Every infinite, finitely branching tree has an infinite path.
- (Ramsey's theorem)
  Every k-coloring has an infinite monochromatic subset.
- (The atomic model theorem)
  Every complete atomic theory has an atomic model.
- ...

COMPUTABLE REDUCIBILITY

Definition (Computable reducibility)

A theorem P *is computably reducible to a theorem* Q *if every* P*-instance* I *computes a* Q*-instance* J *such that for every solution* X *to* J*,* $X \oplus I$ *computes a solution to* I*.*

Intuition:

If $\mathsf{P} \leq_c \mathsf{Q}$ then solving Q is harder than solving P.

COMPUTABLE REDUCIBILITY

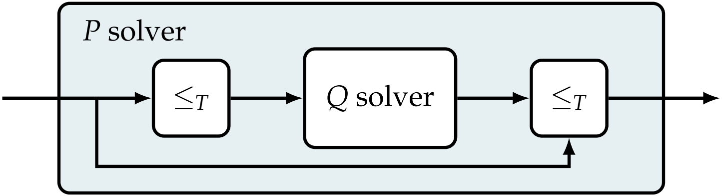


Figure: Computable reducibility

PROVABILITY VS COMPUTATIONAL APPROACH

If we forget induction,

$$\mathsf{P} \leq_c \mathsf{Q}$$

can be seen as

$$\mathsf{RCA}_0 \vdash \mathsf{Q} \rightarrow \mathsf{P}$$

where only one application of Q is allowed.

Ramsey's theorem

RAMSEY'S THEORY

Given some size s, every sufficiently large collection of objects has a sub-collection of size s, whose objects satisfy some structural properties.

RAMSEY'S THEOREM

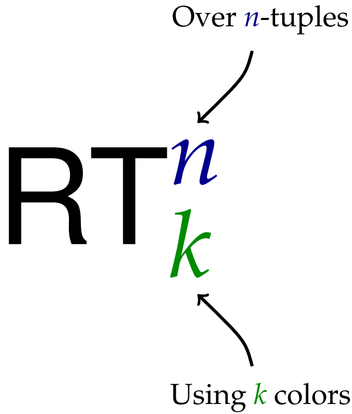
Definition

Given a coloring $f : [\mathbb{N}]^n \to k$, a set H is f-homogeneous if there exists a color $i < k$ such that $f([H]^n) = i$.

Definition (Ramsey's theorem)

Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set.

RAMSEY'S THEOREM



Over n-tuples

$$\mathsf{RT}^{n}_{k}$$

Using k colors

RAMSEY'S THEOREM

Fix the number of colors k.

RAMSEY'S THEOREM FOR *n*-TUPLES

Theorem (Jockusch, 1972)

Every computable coloring $f : [\mathbb{N}]^n \to k$ *has a* Π^0_n *infinite* f*-homogeneous set.*

Theorem (Jockusch, 1972)

For every $n \geq 3$*, there is a computable coloring* $f : [\mathbb{N}]^n \to k$ *such that every infinite* f*-homogeneous set computes* $\emptyset^{(n-2)}$*.*

RAMSEY'S THEOREM FOR n-TUPLES

Theorem (Simpson, 2009)

For each $n, m \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^m_k$.

What about RT^2_k ?

RAMSEY'S THEOREM FOR PAIRS

Theorem (Seetapun, 1995)

For every computable coloring $f : [\mathbb{N}]^2 \to k$ *and every non-computable set* C*, there is an infinite* f*-homogeneous set* $H \not\geq_T C$*.*

Corollary

RT^2_k *does not imply* RT^3_k *over* RCA_0.

HOW MANY APPLICATIONS?

When $3 \leq m < n$, the proof of

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^m_k \to \mathsf{RT}^n_k$$

involves multiple applications of RT^m_k.

How many applications of RT^m_k are necessary?

HOW MANY APPLICATIONS?

Theorem (Jockusch, 1972)

For every $n \geq 2$*, there is a computable coloring* $f : [\mathbb{N}]^n \to k$ *with no* Σ^0_n *infinite* f*-homogeneous set.*

Corollary

For every $n \geq 2$*,* $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^{n+1}_k$*.*

At least 2 applications of RT^n_k are necessary to prove RT^{n+1}_k.

HOW MANY APPLICATIONS?

Theorem (Cholak, Jockusch, Slaman, 2001)

For every $n \geq 2$*, every set* $P \gg \emptyset^{(n-1)}$*, and every computable coloring* $f : [\mathbb{N}]^n \to k$*, there is an infinite* f*-homogeneous set* H *such that* $H' \leq_T P$*.*

- At most 3 applications of RT^3_k are necessary to prove RT^4_k
- Exactly 2 applications of RT^n_k are necessary to prove RT^{n+1}_k whenever $n \geq 4$.

SUMMARY FOR A FIXED k

$$\begin{tikzcd}
 & {} \arrow[d, dotted] \\
 & \mathsf{RT}^4_k \arrow[d] \\
\mathsf{RT}^n_k, n \geq 3 \arrow[d] & \mathsf{RT}^3_k \arrow[d] \\
\mathsf{RT}^2_k & \mathsf{RT}^2_k
\end{tikzcd}$$

Over RCA_0 Over $\leq_c$

RAMSEY'S THEOREM

Fix the size of tuples n.

RAMSEY'S THEOREM

Theorem (Folklore)

For every $k, \ell \geq 2$*,* $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^n_\ell$

Proof for $k = \ell^2$.

- Take a coloring $f : [\mathbb{N}]^n \to \ell^2$
- Define $g : [\mathbb{N}]^n \to \ell$ by merging colors by blocks of size ℓ
- Apply RT^n_ℓ to g to obtain H such that $|f([H]^2)| \leq \ell$.
- Apply again RT^n_ℓ to f restricted to H.

☐

HOW MANY APPLICATIONS?

Theorem (P.)

For every $k > \ell \geq 2$, $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_\ell$.

Theorem (P.)

For every $k > \ell \geq 2$, *there is a* Δ^0_n *partition* $A_0 \cup \cdots \cup A_{k-1} = \mathbb{N}$ *such that every computable* RT^n_ℓ*-instance has a homogeneous set which computes no infinite subset of one of the* A*'s.*

A HARD Δ^0_2 PARTITION

Definition

A function f is Y-hyperimmune if f is not dominated by any Y-computable function. A set X is Y-hyperimmune if its principal function p_X is.

If $\overline{X}$ is Y-hyperimmune, then every infinite Y-computable set intersects X.

A HARD Δ^0_2 PARTITION

Lemma (Folklore)

This is a Δ^0_2 partition $A_0 \cup \cdots \cup A_{k-1} = \mathbb{N}$ such that the $\overline{A}$'s are hyperimmune.

If suffices to show that every computable RT^2_ℓ-instance has a homogeneous set H such that $\overline{A}_i$ is H-hyperimmune for at least two i's.

COHESIVENESS

Definition

Given a sequence of sets $R_0, R_1, \ldots,$ an infinite set C is $\vec{R}$-cohesive if $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$ for each $i \in \mathbb{N}$.

Definition (Cohesiveness)

Every countable sequence of sets $\vec{R}$ admits an $\vec{R}$-cohesive set.

COHESIVENESS AND RT^2_ℓ

- Fix computable instance $f : [\mathbb{N}]^2 \to \ell$ of RT^2_ℓ.
- Define $R_{x,i} = \{y : f(x,y) = i\}$.
- Take an $\vec{R}$-cohesive set C.
- Let $B_i = \{x \in C : \lim_{y \in C} f(x,y) = i\}$

Any infinite subset of one of the B's computes an infinite f-homogeneous set.

RT^2_ℓ AND HYPERIMMUNITY

We need to prove hyperimmunity preservation results for

- Cohesiveness
- Non-effective RT^1_ℓ

PRESERVATION OF HYPERIMMUNITY

Definition

A Π^1_2 statement P admits preservation of hyperimmunity if for each set Z, each sequence of Z-hyperimmune sets $A_0, A_1, \ldots,$ and each P-instance $X \leq_T Z$, there is a solution Y to X such that the A's are $Y \oplus Z$-hyperimmune.

Preservation of hyperimmunity $\neq$ hyperimmune-free solutions

Theorem (Jockusch & Stephan)

If $R_0, R_1, \dots$ are the primitive recursive sets then every $\vec{R}$-cohesive is hyperimmune.

Theorem (P.)

COH *admits preservation of hyperimmunity.*

MATHIAS FORCING

$$(F, X)$$

Initial segment

Reservoir

F is finite, X is infinite and $max(F) < min(X)$.

MATHIAS FORCING

A condition (E, Y) extends (F, X) if

(a) $F \subseteq E$
(b) $Y \subseteq X$
(c) $E \setminus F \subseteq X$

A set G satisfies (F, X) if $F \subseteq G$ and $G \setminus F \subseteq X$.

COH ADMITS PRESERVATION OF HYPERIMMUNITY

- Fix a Z and a sequence of Z-hyperimmune sets $A_0, A_1, \dots$
- Fix a Z-computable sequence $R_0, R_1, \dots$

We build an $\vec{R}$-cohesive set with Mathias conditions (F, X) where the A's are $X \oplus Z$-hyperimmune.

COH ADMITS PRESERVATION OF HYPERIMMUNITY

Lemma

For every condition c and every pair of indices e, i, there is an extension d of c which forces $\Phi_e^{G \oplus Z}$ not to dominate p_{A_i}.

Proof (Part I).

- Fix $c = (F, X)$.
- Define $f(x) = \begin{cases} \Phi_e^{(F \cup E) \oplus Z}(x) & \text{for some } E \subseteq X \\ \uparrow & \text{otherwise} \end{cases}$
- f is partial $X \oplus Z$-computable.

□

COH ADMITS PRESERVATION OF HYPERIMMUNITY

Lemma

For every condition c and every pair of indices e, i, there is an extension d of c which forces $\Phi_e^{G \oplus Z}$ *not to dominate* p_{A_i}*.*

Proof (Part II).

- If f is partial, then c forces $\Phi_e^{G \oplus Z}$ to be partial.
- If f is total, then $f(x) \leq p_{A_i}(x)$ for some x.
    - Let E be such that $f(x) = \Phi_e^{(F \cup E) \oplus Z}(x)$
    - $(F \cup E, X \setminus [0, max(E)])$ forces $f(x) \leq p_{A_i}(x)$

□

NON-EFFECTIVE RT^1_ℓ

Lemma

Δ^0_2-RT^1_ℓ does not admit preservation of hyperimmunity.

Proof.

Take $C_0 \cup \cdots \cup C_{\ell-1} = \mathbb{N}$ be hyperimmune sets.
If $H \subseteq C_i$, then p_H dominates p_{C_i}, so C_i is not H-hyperimmune. □

Definition

Given two integers $u, \ell \geq 1$, we let $\pi(u, \ell)$ denote the unique $a \geq 1$ such that $u = a \cdot \ell - b$ for some $b \in [0, \ell)$.

If you have u pigeons in ℓ pigeonholes, one of the holes has at least $\pi(u, \ell)$ pigeons.

PRESERVATION OF HYPERIMMUNITY

Theorem (P.)

Fix some $k \geq 1$ *and* $\ell \geq 2$ *and* k *hyperimmune sets* $A_0, \dots, A_{k-1}$. *For every* ℓ*-partition* $B_0 \cup \dots \cup B_{\ell-1} = \omega$, *there exists an infinite subset* H *of some* B_i *such that* $\pi(k, \ell)$ *sets among the* A*'s are* H*-hyperimmune.*

Build a set G by Mathias forcing, and let $H = G \cap B_i$ for some $i < \ell$.

NON-EFFECTIVE RT^1_ℓ

Lemma

For every condition c and every pair of indices e, i, there is an extension d of c which forces $\Phi_e^{(G \cap B_j) \oplus Z}$ *not to dominate* p_{A_i} *for some* $j < \ell$.

HOW MANY APPLICATIONS?

Theorem

For every $k > \ell \geq 2$, $\mathsf{RT}^2_k \not\leq_c \mathsf{RT}^2_\ell$.

Proof (Part I).

- Define a Δ^0_2 partition $A_0 \cup \cdots \cup A_{k-1} = \mathbb{N}$ such that the $\overline{A}$'s are hyperimmune.
- Consider its Δ^0_2 approximation function as a computable instance of RT^2_k.

□

HOW MANY APPLICATIONS?

Theorem

For every $k > \ell \geq 2$, $\mathsf{RT}^2_k \not\leq_c \mathsf{RT}^2_\ell$.

Proof (Part II).

- Fix computable instance $f : [\mathbb{N}]^2 \to \ell$ of RT^2_ℓ.
- Construct an $\vec{R}$-cohesive set C such that the $\overline{A}$'s are hyperimmune relative to C.
- Let $B_i = \{x \in C : \lim_{y \in C} f(x, y) = i\}$
- Take an infinite subset H of some B_i such that $\pi(k, \ell)$ among the A's are $H \oplus C$-hyperimmune.

$\square$

HOW MANY APPLICATIONS?

Theorem (P.)

For every $k > \ell \geq 2$, $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_\ell$.

Proof.

By induction over $k \geq 2$ using prehomogeneous sets. ☐

SUMMARY FOR A FIXED n

$$\begin{tikzcd}
& {} \arrow[d, dotted] \\
& \mathsf{RT}^n_4 \arrow[d] \\
& \mathsf{RT}^n_3 \arrow[d] \\
\mathsf{RT}^n_k, k \geq 2 & \mathsf{RT}^n_2 \\
\text{Over } \mathsf{RCA}_0 & \text{Over } \leq_c
\end{tikzcd}$$

COUNTING APPLICATIONS

Question

How many applications needed to prove that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_5$*?*

Take a Δ^0_2 5-partition $A_0 \cup \cdots \cup A_4 = \mathbb{N}$ whose complements are hyperimmune.

# of apps of RT^2_2	# of i's such that $\overline{A}_i$ is hyperimmune
0	5
1	$\pi(5,2) = 3$
2	$\pi(3,2) = 2$
3	$\pi(2,2) = 1$

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k}$$

Using k colors

RAMSEY’S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

THIN SET THEOREM

$$\mathsf{TS}^{n}_{k} \qquad \mathsf{RT}^{n}_{k,k-1}$$

ALLOWING MORE COLORS

Theorem (Wang, 2014)

Fix some n and some sufficiently large k's. For every instance f of TS^n_k and every non-computable set C, there is an infinite solution to f which does not compute C.

Corollary

For every n and sufficiently large k, TS^n_k does not imply RT^3_2 over RCA_0.

ALLOWING MORE COLORS

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)
$\mathsf{RCA}_0 \vdash \mathsf{TS}^{ns+1}_{k^s} \to \mathsf{TS}^{n+1}_{k}$

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)
$\mathsf{RCA}_0 \vdash \mathsf{TS}^{n+2}_{2^n} \to \mathsf{TS}^{3}_{2}$

ALLOWING MORE COLORS

Tuples	Strong avoidance	Computes $\emptyset'$
TS^1_k	$k \geq 2$	never
TS^2_k	$k \geq 3$	$k = 2$
TS^3_k	$k \geq 7$	$k \leq 4$

Does any of TS^3_5 or TS^3_6 admit strong cone avoidance?

ALLOWING MORE COLORS

Theorem (P.)

For every $k \geq 2$,

- TS^2_{k+1} *admits preservation of k hyperimmunities.*
- TS^2_{k} *does not admit preservation of k hyperimmunities.*

Corollary (P.)

For every $k \geq 2$, TS^2_{k+1} *does not imply* TS^2_{k} *over* RCA_0.

ALLOWING MORE COLORS

Fix some $\ell \geq 2$.

Theorem (P.)

For every n and sufficiently large k's,
TS^n_k *admits preservation of* ℓ *hyperimmunities.*

Corollary (P.)

For every n and sufficiently large k's,
TS^n_k *does not imply* TS^2_ℓ *over* RCA_0.

SUMMARY FOR $n = 2$

$$\begin{tikzcd}
\mathsf{RT}^2_2 \arrow[d] \\
\mathsf{TS}^2_3 \arrow[d] \\
\mathsf{TS}^2_4 \arrow[d, dotted] \\
{}
\end{tikzcd}$$

Over RCA_0

CONCLUSION

- Computable reducibility gives a more fine-grained analysis than reverse mathematics.
- Ramsey's theorem is not robust for computable reducibility.
- Changing the number of allowed colors has a great impact on the strength of Ramsey's theorem.

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QUESTIONS

Thank you for listening!