How randomly rainbows appear !

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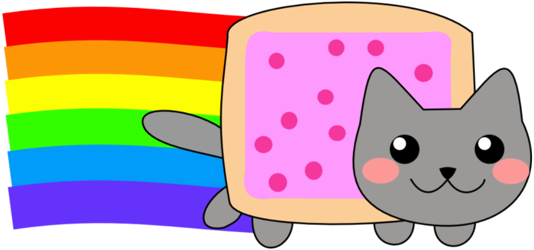

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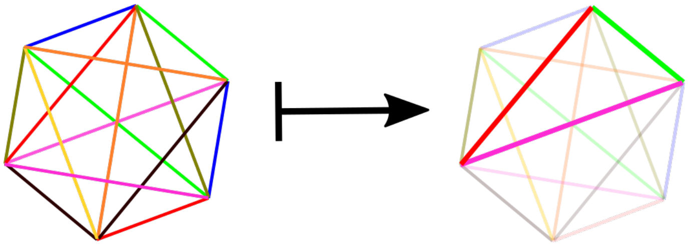
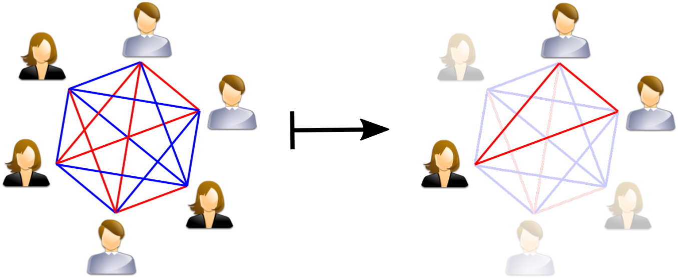
RAINBOW RAMSEY THEOREM

RRT^n_k Every k-bounded coloring of $[\mathbb{N}]^n$ admits an infinite rainbow.



RAMSEY'S THEOREM

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.



$$\mathsf{RRT}^n_k \leq_c \mathsf{RT}^n_k$$

(Galvin)

Let $\prec$ be a well-ordering of $[\mathbb{N}]^n$ and $f \in \mathsf{RRT}^n_k$

$$g(\vec{x}) = |\{\vec{y} \in [\mathbb{N}]^n : \vec{y} \prec \vec{x} \text{ and } f(\vec{x}) = f(\vec{y})\}|$$

RRT^2_k admits probabilistic solutions

(Cisma and Mileti)

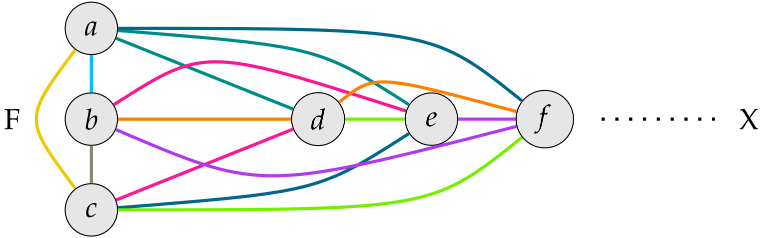
$$(F, X)$$

Initial segment ↗ ↖ Reservoir

$$(\forall x \in X)[F \cup \{x\} \text{ is a rainbow}]$$

x is bad if

$$(\forall^\infty s \in X)(\exists y \in F)[f(x,s) = f(y,s)]$$



If f is 2-bounded then X contains at most $|F|$ bad elements !

$$(\emptyset, \omega) = (F_0, X_0) \geq (F_1, X_1) \geq (F_2, X_2) \geq \dots$$

At stage s

- Pick x at random among the 10^x first elements of X_s
- Set $F_{s+1} = F \cup \{x\}$, and
  $X_{s+1} = \{y \in X_s : F \cup \{x, y\} \text{ is a rainbow}\}$

$H = \bigcup_s F_s$ is likely to be a rainbow

For every RRT^2_k-instance f,
$\mu\{X : X \oplus f \text{ computes an } f\text{-rainbow } \} > 0$

(Csima and Mileti)

There is a computable RRT^3_2-instance f such that,
$\mu\{X : X \text{ computes an } f\text{-rainbow } \} = 0$

(P.)

Let $(X_e)_{e\in\mathbb{N}}$ be a uniform family of sets.
A function $f : \mathbb{N}^2 \to \mathbb{N}$ is $(X_e)_{e\in\mathbb{N}}$-escaping if

$$(\forall e)(\forall n)[|X_e| \leq n \to f(e,n) \notin X_e]$$

Let $h : \mathbb{N} \to \mathbb{N}$ be a function.
A function $f : \mathbb{N} \to \mathbb{N}$ is h-diagonalizing if

$$(\forall x)[f(x) \neq h(x)]$$

The following are computably equivalent :

- RRT^2_2
- Any family $(X_e)_{e\in\mathbb{N}}$ of Σ^0_2 sets has an escaping function
- Any partial Δ^0_2 function has a diagonalizing function

(J. Miller)

The measure of a tree $T \subseteq 2^{<\mathbb{N}}$ is

$$\mu(T) = \lim_s \frac{|\{\sigma \in T : |\sigma| = s\}|}{2^{-s}}$$

A set H is homogeneous for a tree $T \subseteq 2^{<\mathbb{N}}$ if

$$(\forall n)[\{\sigma \in T : H \upharpoonright n \subseteq \sigma \vee H \upharpoonright n \subseteq \overline{\sigma}\} \text{ is infinite}]$$

The following are computably equivalent :

- RRT^2_2
- Any family $(X_e)_{e\in\mathbb{N}}$ of Σ^0_2 sets has an escaping function
- Any partial Δ^0_2 function has a diagonalizing function
- Any Δ^0_2 tree of positive measure has an infinite homogeneous set

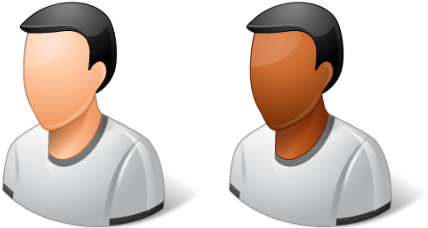
(Bienvenu, Miller, P., Shafer)

Towards a stable rainbow Ramsey theorem

What is a stable k-bounded coloring ?

Given $x < s$, see $f(\{x, s\})$ as the state of person x at time s

Two people x and y are together at time s if $f(\{x, s\}) = f(\{y, s\})$

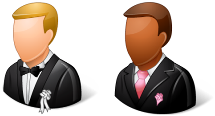


x is wise if

$$(\forall y \neq x)[\{s : f(x,s) = f(y,s) \wedge f(x,s+1) \neq f(y,s+1)\} \text{ is finite}]$$

x and y get married if

$$(\forall^\infty s)[f(\{x,s\}) = f(\{y,s\})]$$



x becomes a monk if

$$(\forall^\infty s)[f(\{x,s\}) \text{ is unique}]$$

A 2-bounded coloring $f : [\mathbb{N}]^2 \to \mathbb{N}$ is

weakly rainbow-stable

if everybody is wise

rainbow-stable

if everybody becomes of monk or get married

The following are computably equivalent :

- Any rainbow-stable coloring has an infinite rainbow
- Any family $(X_e)_{e \in \mathbb{N}}$ of Σ^0_2 finite sets whose sizes are uniformly Δ^0_2 has an escaping function
- Any Δ^0_2 function has a diagonalizing function
- Any Δ^0_2 tree of Δ^0_2 positive measure has an infinite homogeneous set

(P.)

Factorizing proofs

A degree **d** bounds P if every computable P-instance has a solution bounded by d.

The only Δ^0_2 degree bounding stable Ramsey's theorem for pairs is $\mathbf{0}'$

(Mileti)

The only Δ^0_2 degree bounding the stable rainbow Ramsey theorem for pairs is $\mathbf{0}'$

(P.)

Every DNC function over $\mathbf{0}'$
is of hyperimmune degree

(Csima and Mileti)

There is a Δ^0_2 function whose escaping
functions are of hyperimmune degree

(P.)

A sequence X is Π^0_1-generic if for all Σ^0_2 sets G, either $X \in G$, or X is in some Π^0_1 set disjoint from G.

A sequence X is Π^0_1-generic iff
it is of hyperimmune-free degree

(Monin)

There is a Δ^0_2 function whose escaping functions are of hyperimmune degree

(P.)

- Let P be a low PA degree
- Let f be Δ^0_2 with no P-computable escaping function
- No Π^0_1-generic computes an f-escaping function

$$U = \{X \in 2^{\mathbb{N}} : (\exists e)\Psi^X(e) \uparrow \vee \Psi^X(e) = f(e)\}$$

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