The weakness of Ramsey's theorem under omniscient reductions

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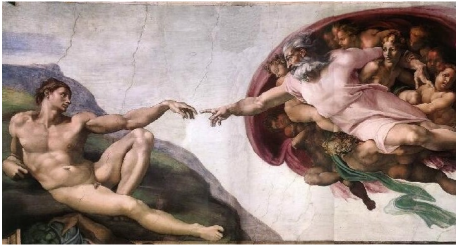
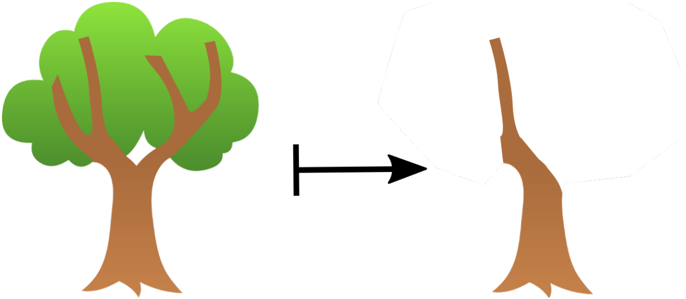


May 22, 2016

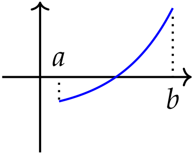
Many theorems can be seen as problems.

König's lemma
Every infinite, finitely branching tree admits an infinite path.

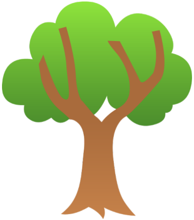


Some theorems are more effective than others.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.

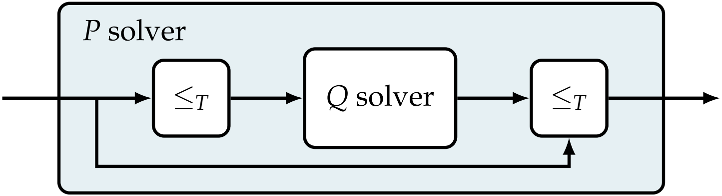


König's lemma
Every infinite, finitely branching tree admits an infinite path.



COMPUTABLE REDUCTION

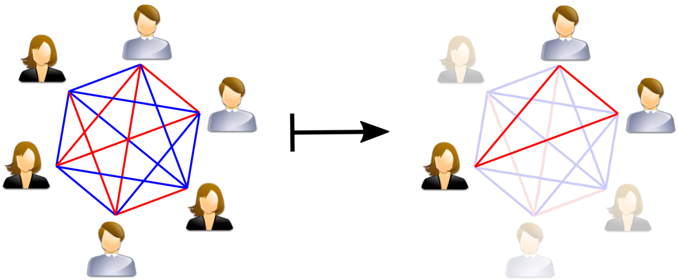
"Q is at least as hard as P"



$$P \leq_c Q$$

RAMSEY'S THEOREM

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.



$$\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_\ell$$

whenever $k > \ell \geq 2$ and $n \geq 2$.

(P.)

Definition

A problem P preserves m among n hyperimmunities if for every n-tuple of hyperimmune sets $A_0, \dots, A_{n-1}$ and every computable P-instance X, there is a solution Y to X such that at least m among the A's are Y-hyperimmune.

RT^2_ℓ preserves 2 among k hyperimmunities, but RT^2_k does not.

$$\mathsf{RT}^1_k =_c \mathsf{RT}^1_\ell$$

whenever $k, \ell \geq 1$.

Refining $\leq_c$

Weihrauch reduction
Consider the uniformity
of reductions

Strong computable reduction
Removes access
to the instance

$$\mathsf{RT}^1_k =_c \mathsf{RT}^1_\ell$$

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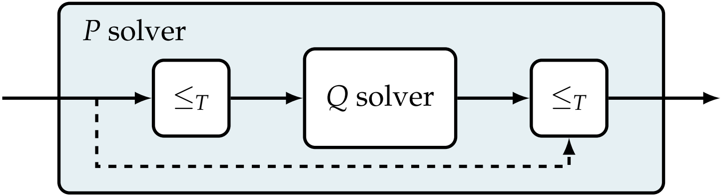


$$P \leq_{sc} Q$$

$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{RT}^1_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov)

Definition

A problem P strongly preserves m among n hyperimmunities if for every n-tuple of hyperimmune sets $A_0, \ldots, A_{n-1}$ and every P-instance X, there is a solution Y to X such that at least m among the A's are Y-hyperimmune.

RT^1_ℓ strongly preserves 2 among k hyperimmunities, but RT^1_k does not.

$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{RT}^1_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov)

The RT^1_k-instance witnessing it
defeats all RT^1_ℓ-instances.

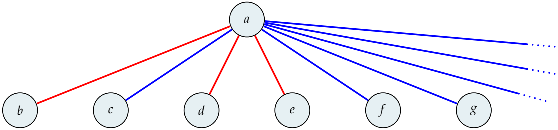
(Hirschfeldt, Jockusch, P.)

$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov, P., Solomon, Westrick)

SRT^2_k : Restriction of RT^2_k to stable colorings.



$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov, P., Solomon, Westrick)

The RT^1_k-instance witnessing it defeats all SRT^2_ℓ-instances.

WKL : Restriction of König's lemma to binary trees.

$$\mathsf{WKL} \leq_c \mathsf{RT}^n_k$$

whenever $k \geq 2$ and $n \geq 3$.

(Jockusch)

$$\mathsf{WKL} \not\leq_c \mathsf{RT}^2_k$$

whenever $k \geq 1$.

(Liu)

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$$\mathsf{WKL} \not\leq_c \mathsf{RT}^2_k$$

whenever $k \geq 1$.

(Liu)

Definition

- A function f is a modulus of a set A if every function dominating f computes A.
- A set A is computably encodable if for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ computing A.

A is computably encodable $\Leftrightarrow$ A admits a modulus $\Leftrightarrow$ A is hyperarithmetic

(Solovay, Groszek and Slaman)

$$\mathsf{WKL} \not\leq_{sc} \mathsf{RT}^n_k$$

whenever $n, k \geq 1$.

(Hirschfeldt, Jockusch)

The WKL-instance witnessing it defeats all RT^n_k-instances.

WWKL : Restriction of WKL to trees of positive measure.

$$\mathsf{WWKL} \leq_c \mathsf{RT}^n_k$$

whenever $k \geq 2$ and $n \geq 3$.

(Jockusch)

$$\mathsf{WWKL} \not\leq_c \mathsf{RT}^2_k$$

whenever $k \geq 1$.

(Liu)

Definition

- A function f is a Π^0_1 modulus of a set $\mathcal{C} \subseteq \omega^\omega$ if $\mathcal{C}$ has a non-empty g-computably bounded $\Pi^{0,g}_1$ subset for every $g \geq f$.

- A set $\mathcal{C} \subseteq \omega^\omega$ is Π^0_1 encodable if for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ such that $\mathcal{C}$ admits a non-empty X-computably bounded $\Pi^{0,X}_1$ subset.

$\mathcal{C}$ is Π^0_1 encodable $\Leftrightarrow$ $\mathcal{C}$ admits a Π^0_1 modulus
$\Leftrightarrow$ $\mathcal{C}$ has a non-empty Σ^1_1 subset

(Monin, P.)

$$\mathsf{WWKL} \not\leq_{sc} \mathsf{RT}^n_k$$

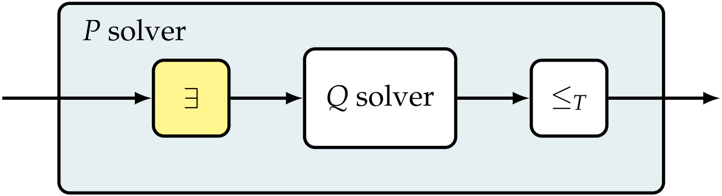
whenever $n, k \geq 1$.

(Monin, P.)

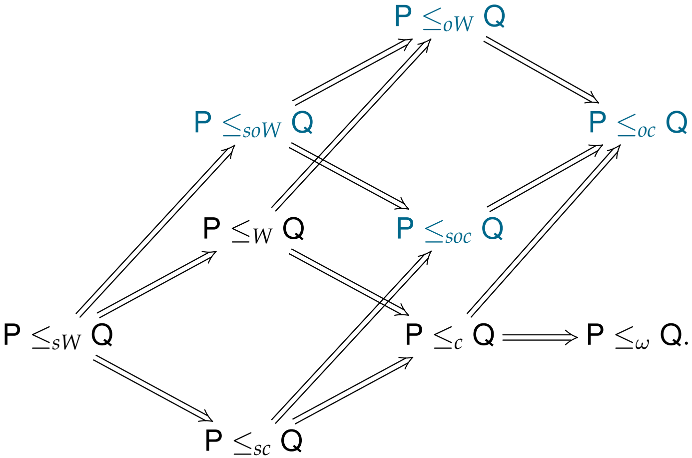
The WWKL-instance witnessing it defeats all RT^n_k-instances.

STRONG OMNISCIENT COMPUTABLE REDUCTION

"Q is at least as hard as P"



$$\mathsf{P} \leq_{soc} \mathsf{Q}$$


$\mathsf{P} \leq_{oW} \mathsf{Q}$
$\mathsf{P} \leq_{soW} \mathsf{Q}$
$\mathsf{P} \leq_{oc} \mathsf{Q}$
$\mathsf{P} \leq_{W} \mathsf{Q}$
$\mathsf{P} \leq_{soc} \mathsf{Q}$
$\mathsf{P} \leq_{sW} \mathsf{Q}$
$\mathsf{P} \leq_{c} \mathsf{Q}$
$\mathsf{P} \leq_{\omega} \mathsf{Q}.$
$\mathsf{P} \leq_{sc} \mathsf{Q}$

STRONG OMNISCIENT COMPUTABLE REDUCTIONS

Whenever $k > \ell \geq 1$

- $\mathsf{RT}^1_k \not\leq_{soc} \mathsf{RT}^1_\ell$ (Hirschfeldt, Jockusch, P.)
- $\mathsf{RT}^1_k \not\leq_{soc} \mathsf{SRT}^2_\ell$ (Dzhafarov, P., Solomon, Westrick)
- $\mathsf{WKL} \not\leq_{soc} \mathsf{RT}^n_k$ (Hirschfeldt, Jockusch)
- $\mathsf{WWKL} \not\leq_{soc} \mathsf{RT}^n_k$ (Monin, P.)

Omniscient computable reductions

- $\mathsf{ACA} \not\leq_{oc} \mathsf{RT}^1_k$ (Dzhafarov)
- $\mathsf{WWKL} \not\leq_{oc} \mathsf{RT}^1_k$ (Liu.)
- $\mathsf{WWKL} \not\leq_{oc} \mathsf{FS}$ (P.)
- $\mathsf{RT}^2_2 \not\leq_{oc} \mathsf{FS}$ (P.)

DIFFERENCES WITH $\leq_{sc}$

$$\mathsf{SRT}^2_3 \not\leq_{sc} \mathsf{RT}^2_2$$

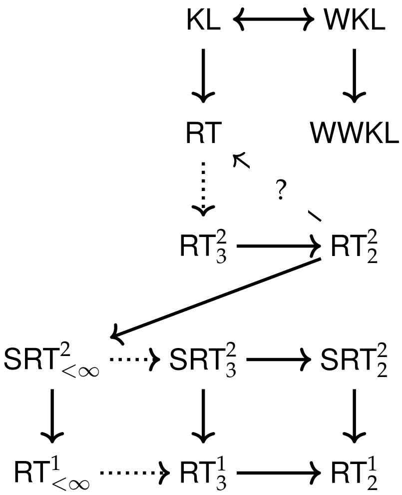
(P.)

$$\mathsf{SRT}^2_{<\infty} \leq_{soc} \mathsf{RT}^2_2$$

(Monin, P.)

Proof sketch : $g(x, y) = 1$ iff $f(x, y) = \lim_s f(y, s)$

DIAGRAM UNDER $\leq_{soc}$



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