Ramsey's theorem and compactness

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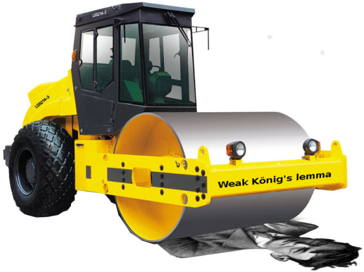

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THEOREMS AS PROBLEMS

Many theorems P are of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of P as a class of problems.

- An X such that $\Phi(X)$ holds is an instance.
- A Y such that $\Psi(X, Y)$ holds is a solution to X.

THEOREMS AS PROBLEMS

Examples:

- (König's lemma)
  Every infinite, finitely branching tree has an infinite path.
- (Ramsey's theorem)
  Every k-coloring has an infinite monochromatic subset.
- (The atomic model theorem)
  Every complete atomic theory has an atomic model.
- ...

TURING IDEALS

A Turing ideal is a collection of sets $\mathcal{M}$ closed under

- the Turing reduction: $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- the effective join: $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Example:

- $\{X : X \text{ is computable}\}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

COMPARE THEOREMS

A Turing ideal $\mathcal{M}$ satisfies a theorem P (written $\mathcal{M} \models \mathsf{P}$) if every P-instance in $\mathcal{M}$ has a solution in $\mathcal{M}$.

A theorem P computably entails a theorem Q (written $\mathsf{P} \vdash_c \mathsf{Q}$) if every Turing ideal satisfying P satisfies Q.

SEPARATING THEOREMS

Fix two theorems P and Q.

How to prove that $\mathsf{P} \not\vdash \mathsf{Q}$?

Build a Turing ideal $\mathcal{M}$ such that

- $\mathcal{M} \models \mathsf{P}$
- $\mathcal{M} \not\models \mathsf{Q}$

Separating theorems

Pick a Q-instance I with no I-computable solution.

Start with $\mathcal{M}_0 = \{Z : Z \leq_T I\}$.

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

SEPARATING THEOREMS

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1. pick some P-instance $X \in \mathcal{M}_n$

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2. choose a solution Y to X

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1. pick some P-instance $X \in \mathcal{M}_n$
2. choose a solution Y to X
3. let $\mathcal{M}_{n+1} = \{Z : Z \leq_T Y \oplus U\}$.

SEPARATING THEOREMS

Beware, while adding sets to $\mathcal{M}$,
we may add a solution to the Q-instance!

SEPARATING THEOREMS

An avoidance property is a collection of sets closed upwards under the Turing reducibility.

Examples

- $\{X : A \leq_T X\}$ for some set A
- $\{X : X \text{ is of PA degree}\}$
- $\{X : X \text{ computes a Martin-Löf random}\}$

SEPARATING THEOREMS

Fix a property $\mathcal{P}$.

A statement P avoids $\mathcal{P}$ if for every $Z \notin \mathcal{P}$, every Z-computable P-instance X has a solution Y such that $Y \oplus Z \notin \mathcal{P}$

Lemma

If P *avoids* $\mathcal{P}$ *but* Q *does not, then* $\mathsf{P} \not\vdash \mathsf{Q}$

Ramsey's theorem

RAMSEY'S THEOREM

Fix a coloring $f : [\mathbb{N}]^n \to k$. A set H is f-homogeneous if there exists a color $i < k$ such that $f([H]^n) = i$.

Ramsey's theorem

Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set.

CONE AVOIDANCE

A theorem P avoids cones if it avoids $\{A_0, A_1, \dots\}$ for every countable sequence of non-computable sets $A_0, A_1, \dots$.

- RT^3_2 does not avoid $\{\emptyset'\}$ (Jockusch, 1972)
- RT^2_2 avoids cones (Seetapun, 1995)

AVOIDANCE VS STRONG AVOIDANCE

Avoidance

$\equiv$

effective weakness

STRONG AVOIDANCE

Fix a property $\mathcal{P}$.

A statement P strongly avoids $\mathcal{P}$ if for every $Z \notin \mathcal{P}$, every P-instance X has a solution Y such that $Y \oplus Z \notin \mathcal{P}$

Avoidance vs strong avoidance

Strong avoidance

$\equiv$

combinatorial weakness

STRONG CONE AVOIDANCE

A theorem P strongly avoids cones if it strongly avoids $\{A_0, A_1, \dots\}$ for every countable sequence of non-computable sets $A_0, A_1, \dots$.

- RT^2_2 does not strongly avoid $\{\emptyset'\}$ (Jockusch, 1972)
- RT^1_2 strongly avoids cones (Dzhafarov and J., 2009)

König's lemma

König's lemma

A tree is a subset of $\mathbb{N}^{<\mathbb{N}}$ downward-closed under the prefix relation.

A tree T is finitely branching if for every $\sigma \in T$, there are finitely many n's such that $\sigma n \in T$.

König's lemma

Every infinite, finitely branching tree has an infinite path.

KÖNIG'S LEMMA

A tree is binary if it is a subset of $2^{<\mathbb{N}}$.

weak König's lemma

Every infinite, binary tree has an infinite path.

König's lemma

A binary tree T has positive measure if

$$\lim_s \frac{|\{\sigma \in T : |\sigma| = s|}{2^s} > 0$$

weak weak König's lemma

Every binary tree of positive measure has an infinite path.

AVOIDANCE

- KL does not avoid $\{\emptyset'\}$ (J., Lewis, Remmel, 1991)
- WKL avoids cones (J. and Soare, 1972)

- WKL does not avoid PA degrees (Solovay)
- WWKL avoids PA degrees (Kučera, 1985)

SUMMARY

$$\mathrm{RT}_2^{n\geq 3} \longleftrightarrow \mathrm{KL}$$

$$\mathrm{RT}_2^{n\geq 3} \rightarrow \mathrm{RT}_2^2 \qquad \mathrm{KL} \rightarrow \mathrm{WKL} \rightarrow \mathrm{WWKL}$$

Ramsey vs König

A function is hyperimmune if it is not dominated by any computable function.

- RT^2_2 does not avoid hyp. functions (Jockusch, 1972)
- WKL avoids hyp. functions (J. and Soare, 1972)

- RT^2_2 avoids PA degrees (Liu, 2012)
- RT^1_2 strongly avoids PA degrees (Liu, 2012)

CONSTANT-BOUND ENUMERATIONS

A k-enumeration of a class $\mathcal{C} \subseteq \mathbb{N}^{\mathbb{N}}$ is a sequence $E_0, E_1, \ldots$ such that for each $n \in \mathbb{N}$,

- E_n contains k strings of length n
- $\mathcal{C} \cap [E_n] \neq \emptyset$

A constant-bound enumeration of $\mathcal{C}$ is a k-enum for some $k \in \omega$.

C.B-ENUM AVOIDANCE

A theorem P (strongly) avoids c.b-enums if it (strongly) avoids the c.b-enum's of $\mathcal{C}$ for every class $\mathcal{C} \subseteq 2^{\mathbb{N}}$.

- WWKL does not avoid c.b-enums (Liu, 2015)
- RT^2_2 avoids c.b-enums (Liu, 2015)
- RT^1_2 strongly avoids c.b-enums (Liu, 2015)

C.b-enum avoidance

If a theorem P avoids c.b-enums then

- P avoids cones
- P avoids PA degrees

  Any c.b-enum of $\mathcal{C} = \{X : X \text{ is a completion of PA}\}$ computes a member of $\mathcal{C}$.

  $\mathsf{RT}^2_2 \wedge \mathsf{WWKL} \not\vdash_c \mathsf{WKL}$

Which theorems avoid c.b-enums?

RAMSEY'S THEOREM

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Thin set theorem

$$\mathsf{TS}^{n}_{k} \qquad \mathsf{RT}^{n}_{k,k-1}$$

ALLOWING MORE COLORS

For every n and sufficiently large k's

- TS^n_k strongly avoids cones (Wang, 2014)
- TS^n_k strongly avoids c.b-enums (P.)

- The free set theorem avoids c.b-enums (P.)
- The rainbow Ramsey theorem avoids c.b-enums (P.)

Can RT^2_2 avoid arbitrary paths?

PATH AVOIDANCE

A theorem P avoids paths if it avoids $\mathcal{C}$ for every closed class $\mathcal{C} \subseteq \mathbb{N}^{\mathbb{N}}$.

- Cohesiveness avoids paths (P.)
- The atomic model theorem avoids paths (P.)

PATH AVOIDANCE

Given a class $\mathcal{C} \subseteq \mathbb{N}^{\mathbb{N}}$, $\deg(\mathcal{C}) = \{\deg(X) : X \in \mathcal{C}\}$.

Simpson's embedding lemma

For every Π^0_1 class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ and every Σ^0_3 class $\mathcal{D} \subseteq \mathbb{N}^{\mathbb{N}}$, there is a Π^0_1 class $\mathcal{E} \subseteq 2^{\mathbb{N}}$ such that

$$\deg(\mathcal{E}) = \deg(\mathcal{C}) \cup \deg(\mathcal{D})$$

PATH AVOIDANCE

If for some P-instance X with no X-computable solution

$$\mathcal{D}_X = \{Y : Y \text{ is a solution to } X\}$$

is Σ^0_3, then P does not avoid paths.

- RT^2_2 does not avoid paths (P.)
- RT^1_2 does not strongly avoid paths (P.)

Can RT^2_2 avoid 1-enums?

1-ENUM AVOIDANCE

A theorem P (strongly) avoids 1-enums if it (strongly) avoids the 1-enum's of $\mathcal{C}$ for every class $\mathcal{C} \subseteq 2^{\mathbb{N}}$.

Every c.b-enum of a Π^0_1 class computes a 1-enum.

- RT^2_2 avoids 1-enums of Π^0_1 classes (Liu, 2015)
- rainbow Ramsey's theorem for pairs avoids 1-enums (P.)

1-ENUM AVOIDANCE

Theorem (P.)

There is a class $\mathcal{C} \subseteq 2^{\mathbb{N}}$

- *with no computable 1-enum*
- *with a computable 2-enum* $(\sigma_0, \tau_0), (\sigma_1, \tau_1), \dots$
- *such that* $\{n : \mathcal{C} \cap [\sigma_n] \neq \emptyset\}$ *is* Δ^0_2.

- RT^2_2 does not avoid 1-enums (P.)

Can RT^2_2 simultaneously avoid countably many c.b-enums?

SIMULTANEOUS C.B-ENUM AVOIDANCE

A theorem P simultaneously avoids c.b-enums if it avoids the c.b-enum's of all the $\mathcal{C}$'s for every countable sequence of classes $\mathcal{C}_0, \mathcal{C}_1, \dots \subseteq 2^{\mathbb{N}}$.

If P avoids c.b-enums, then it simultaneously avoids c.b-enums for every increasing countable sequences of classes.

- the Erdős-Moser theorem simu. avoids c.b-enums (P.)
- TS^2_{k+1} simultaneously avoids k c.b-enums (P.)
- TS^2_k does not simultaneously avoid k c.b-enums (P.)

CONCLUSION

- Ramsey's theorem for pairs is effectively weak, but not combinatorially.
- The free set, thin set, Erdös moser and rainbow Ramsey theorems are combinatorially weak.
- Many Ramsey-type theorems have the ability to compute paths through binary trees with no computable paths.

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Questions

Thank you for listening !