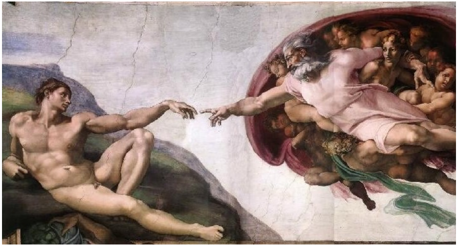
The weakness of Ramsey's theorem under omniscient reductions

Ludovic PATEY
UC Berkeley



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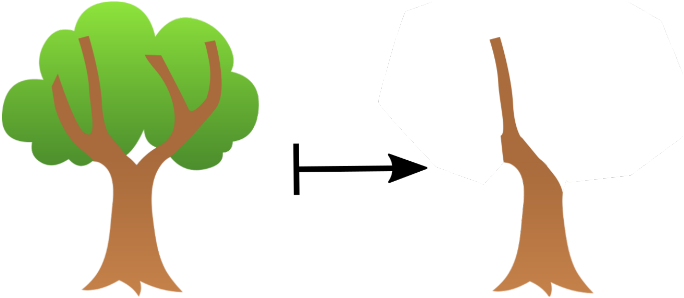
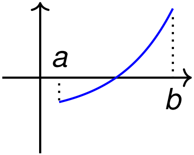
Many theorems can be seen as problems.

König's lemma
Every infinite, finitely branching tree admits an infinite path.



Some theorems are more effective than others.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
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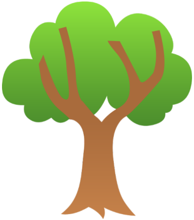


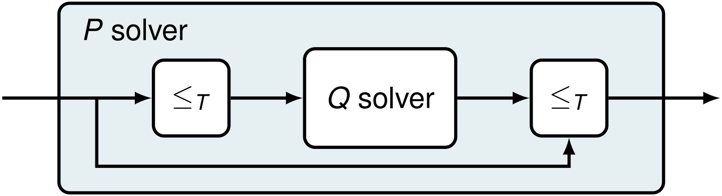
COMPUTABLE REDUCTION

$$\mathsf{P} \leq_c \mathsf{Q}$$

A problem P is computably reducible to a problem Q if for every P-instance X, there is a Q-instance $\hat{X} \leq_T X$ such that for every solution Y to $\hat{X}$, $Y \oplus X$ computes a solution to X.

COMPUTABLE REDUCTION

"Q is at least as hard as P"



RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

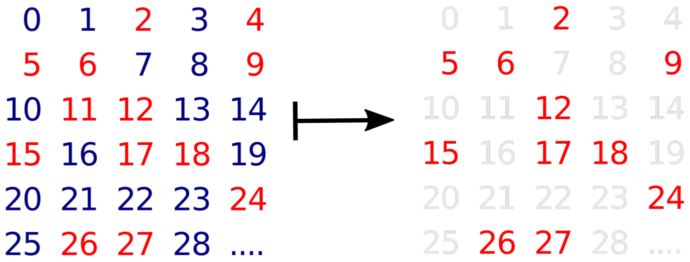
A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

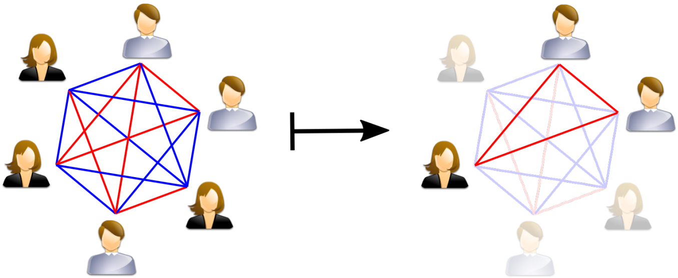
PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.



RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



AN EXAMPLE

A function $f : \mathbb{N} \to \mathbb{N}$ is hyperimmune if it is not dominated by any computable function.

A problem P preserves m among n hyperimmunities if for every n-tuple of hyperimmune functions $f_0, \dots, f_{n-1}$ and every computable P-instance X, there is a solution Y to X such that at least m among the f's are Y-hyperimmune.

AN EXAMPLE

$$\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_\ell$$

whenever $k > \ell \geq 2$ and $n \geq 2$.

(P.)

RT^2_ℓ preserves 2 among k hyperimmunities
but RT^2_k does not.

$$\mathsf{RT}^1_k =_c \mathsf{RT}^1_\ell$$

whenever $k, \ell \geq 1$.

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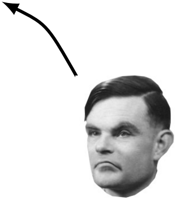


$$\mathsf{RT}^1_k =_c \mathsf{RT}^1_\ell$$

whenever $k, \ell \geq 1$.

Refining $\leq_c$

Weihrauch reduction
Consider the uniformity of reductions

Strong computable reduction
Removes access to the instance

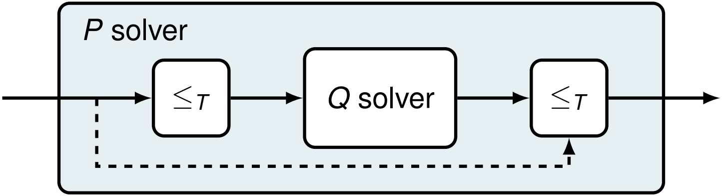
STRONG COMPUTABLE REDUCTION

$$\mathsf{P} \leq_{sc} \mathsf{Q}$$

A problem P is strongly computably reducible to Q if for every P-instance X, there is a Q-instance $\hat{X} \leq_T X$ such that every solution to $\hat{X}$ computes a solution to X.

STRONG COMPUTABLE REDUCTION

"Q is at least as hard as P"



A function $f : \mathbb{N} \to \mathbb{N}$ is hyperimmune if it is not dominated by any computable function.

A problem P strongly preserves m among n hyperimmunities if for every n-tuple of hyperimmune functions $f_0, \dots, f_{n-1}$ and every P-instance X, there is a solution Y to X such that at least m among the f's are Y-hyperimmune.

$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{RT}^1_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov)

RT^1_ℓ strongly preserves 2 among k hyperimmunities
but RT^1_k does not.

$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{RT}^1_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov)

The RT^1_k-instance witnessing it defeats all RT^1_ℓ-instances.

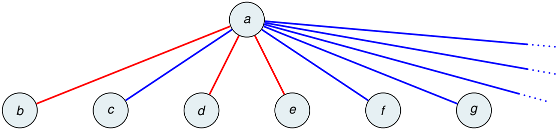
(Hirschfeldt, Jockusch, P.)

$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov, P., Solomon, Westrick)

SRT^2_k : Restriction of RT^2_k to stable colorings.



$$\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$$

whenever $k > \ell \geq 2$.

(Dzhafarov, P., Solomon, Westrick)

The RT^1_k-instance witnessing it defeats all SRT^2_ℓ-instances.

WKL : Restriction of König's lemma to binary trees.

$$\mathsf{WKL} \leq_c \mathsf{RT}^n_k$$

whenever $k \geq 2$ and $n \geq 3$.

(Jockusch)

$$\mathsf{WKL} \not\leq_c \mathsf{RT}^2_k$$

whenever $k \geq 1$.

(Liu)

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(Liu)

Definition

- A function f is a modulus of a set A if every function dominating f computes A.
- A set A is computably encodable if for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ computing A.

A is computably encodable $\Leftrightarrow$ A admits a modulus $\Leftrightarrow$ A is hyperarithmetic

(Solovay, Groszek and Slaman)

$$\mathrm{WKL} \not\leq_{sc} \mathrm{RT}^n_k$$

whenever $n, k \geq 1$.

(Hirschfeldt, Jockusch)

The WKL-instance witnessing it defeats all RT^n_k-instances.

WWKL : Restriction of WKL to trees of positive measure.

$$\mathsf{WWKL} \leq_c \mathsf{RT}^n_k$$

whenever $k \geq 2$ and $n \geq 3$.

(Jockusch)

$$\mathsf{WWKL} \not\leq_c \mathsf{RT}^2_k$$

whenever $k \geq 1$.

(Liu)

Definition

- A function f is a Π^0_1 modulus of a set $\mathcal{C} \subseteq \omega^\omega$ if $\mathcal{C}$ has a non-empty g-computably bounded $\Pi^{0,g}_1$ subset for every $g \geq f$.
- A set $\mathcal{C} \subseteq \omega^\omega$ is Π^0_1 encodable if for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ such that $\mathcal{C}$ admits a non-empty Y-computably bounded $\Pi^{0,Y}_1$ subset.

$$\mathcal{C} \text{ is } \Pi^0_1 \text{ encodable} \Leftrightarrow \mathcal{C} \text{ admits a } \Pi^0_1 \text{ modulus}$$
$$\Leftrightarrow \mathcal{C} \text{ has a non-empty } \Sigma^1_1 \text{ subset}$$

(Monin, P.)

$$\mathsf{WWKL} \not\leq_{sc} \mathsf{RT}^n_k$$

whenever $n, k \geq 1$.

(Monin, P.)

The WWKL-instance witnessing it defeats all RT^n_k-instances.

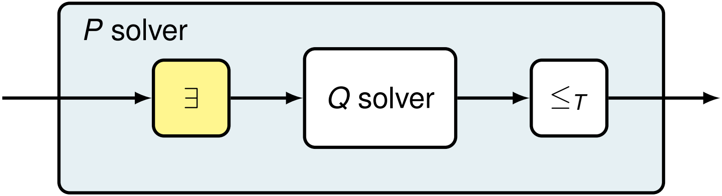
STRONG OMNISCIENT COMPUTABLE REDUCTION

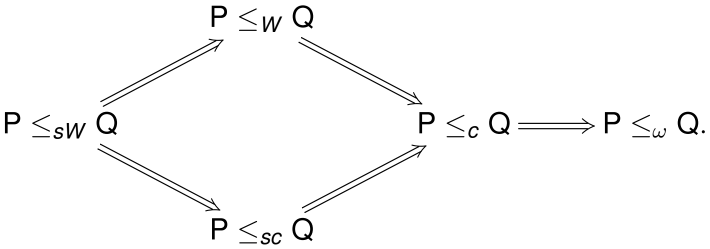
$$\mathrm{P} \leq_{soc} \mathrm{Q}$$

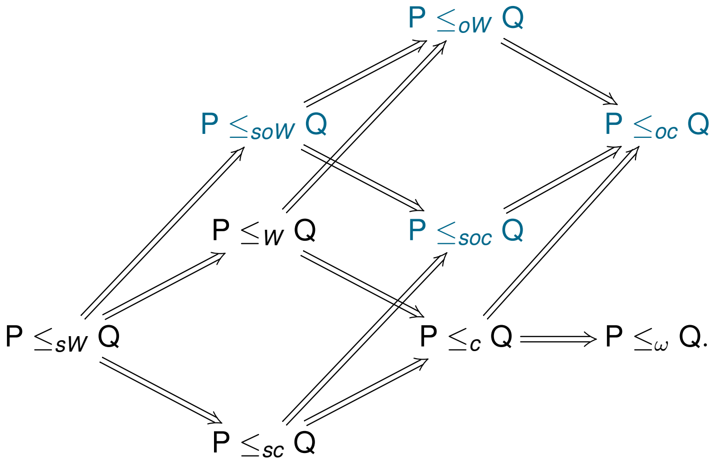
A problem P is strongly omnisciently computably reducible to Q if for every P-instance X, there is an arbitrary Q-instance $\hat{X}$ such that every solution to $\hat{X}$ computes a solution to X.

STRONG OMNISCIENT COMPUTABLE REDUCTION

"Q is at least as hard as P"




$P \leq_W Q$
$P \leq_{sW} Q$
$P \leq_c Q \Longrightarrow P \leq_\omega Q.$
$P \leq_{sc} Q$


$P \leq_{oW} Q$
$P \leq_{soW} Q$
$P \leq_{oc} Q$
$P \leq_{W} Q$
$P \leq_{soc} Q$
$P \leq_{sW} Q$
$P \leq_{c} Q \Longrightarrow P \leq_{\omega} Q.$
$P \leq_{sc} Q$

STRONG OMNISCIENT COMPUTABLE REDUCTIONS

Whenever $k > \ell \geq 1$

- $\mathsf{RT}^1_k \not\leq_{soc} \mathsf{RT}^1_\ell$ (Hirschfeldt, Jockusch, P.)
- $\mathsf{RT}^1_k \not\leq_{soc} \mathsf{SRT}^2_\ell$ (Dzhafarov, P., Solomon, Westrick)
- $\mathsf{WKL} \not\leq_{soc} \mathsf{RT}^n_k$ (Hirschfeldt, Jockusch)
- $\mathsf{WWKL} \not\leq_{soc} \mathsf{RT}^n_k$ (Monin, P.)

OMNISCIENT COMPUTABLE REDUCTIONS

- ACA $\not\leq_{oc}$ RT^1_k (Dzhafarov)
- WWKL $\not\leq_{oc}$ RT^1_k (Liu.)
- WWKL $\not\leq_{oc}$ FS (P.)
- RT^2_2 $\not\leq_{oc}$ FS (P.)

DIFFERENCES WITH $\leq_{sc}$

$\mathsf{SRT}^2_3 \not\leq_{sc} \mathsf{RT}^2_2$

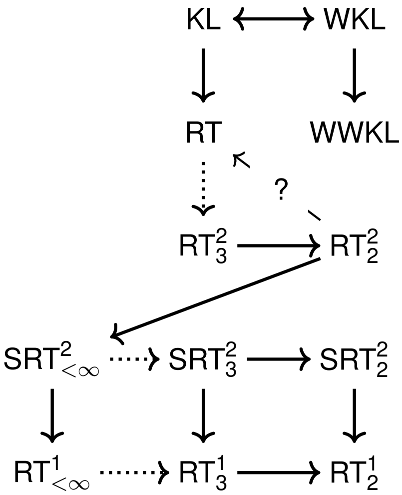
(P.)

$\mathsf{SRT}^2_{<\infty} \leq_{soc} \mathsf{RT}^2_2$

(Monin, P.)

Proof sketch : $g(x, y) = 1$ iff $f(x, y) = \lim_s f(y, s)$

DIAGRAM UNDER $\leq_{soc}$



QUESTIONS

Is $\mathsf{RT} \leq_{soc} \mathsf{RT}^2_2$?

Is $\mathsf{RT}^n_{k+1} \leq_{soc} \mathsf{RT}^n_k$? No for $n = 1$.

Is $\mathsf{RT}^{n+1}_k \leq_{soc} \mathsf{RT}^n_k$? No for $n = 1$.

Revisiting the big question

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

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Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

REVERSE MATHEMATICS

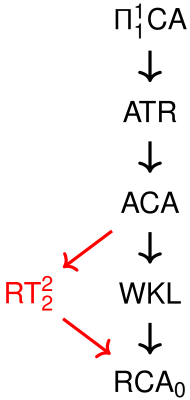
Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Π^1_1CA
↓
ATR
↓
ACA
↓
WKL
↓
RCA_0

REVERSE MATHEMATICS

Mathematics are computationally very structured



Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

COHESIVE SETS

An infinite set C is cohesive for a sequence of sets $R_0, R_1, \ldots$ if for every i, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

An infinite set C is p-cohesive if it is cohesive for the primitive recursive sets.

COH Every sequence of sets has a cohesive set.

$$\mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} + \mathsf{SRT}^2_2$$

Fix an instance $f : [\mathbb{N}]^2 \to 2$ of RT^2_2

Define $R_x = \{y : f(x, y) = 1\}$

Let C be cohesive for $R_0, R_1, \ldots$

$f : [C]^2 \to 2$ is an instance of SRT^2_2

THE BIG QUESTION

Does $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{COH}$?

THE BIG QUESTION

$$\text{Does } \mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{COH}\,?$$

Theorem (Chang, Slaman, Yang)
Nope.

REVISITING THE BIG QUESTION

Hirschfeldt : “We want a computability-theoretic answer”

An L_2-structure $\mathcal{M} = \langle M, S, 0, 1, +, \cdot \rangle$ is an ω-structure if M is the set of standard numbers, equipped with the standard operations

Does $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{COH}$
on ω-structures ?

REVISITING THE BIG QUESTION

Dzhafarov : "One step is already complicated"

$$\text{Is COH} \leq_c \text{SRT}^2_2\,?$$

REVISITING THE BIG QUESTION

P : "This is about the combinatorics of singletons"

$$\text{Is COH} \leq_{oc} \text{RT}^1_2\,?$$

Is there a set X, such that
every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$
computes a p-cohesive set ?

A set X is high if $X' \geq_T \emptyset''$.

Is there a set X, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ is high ?

If yes, then COH $\leq_{oc}$ RT^1_2.

If no, well, this is still interesting *per se*.

A set S is P-jump-encodable if there is an instance of P such that the jump of every solution computes S.

Are the RT^1_2-jump-encodable sets precisely the $\emptyset'$-computable ones ?

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