The strength of the thin set theorems in reverse mathematics

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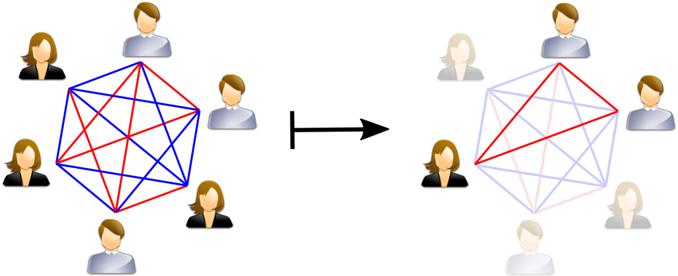
RAMSEY'S THEOREM

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.



REVERSE MATHEMATICS

Q is at least as hard as P if

$$\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$$

in a very weak theory RCA_0
capturing computable mathematics

(Harvey Friedman, 1974)

Turing ideal $\mathcal{M}$

- $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Examples

- $\{X : X$ is computable $\}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

Let $\mathcal{M}$ be a Turing ideal and P, Q be problems.

Satisfaction

$$\mathcal{M} \models \mathsf{P}$$

if every P-instance in $\mathcal{M}$ has a solution in $\mathcal{M}$.

Computable entailment

$$\mathsf{P} \models_c \mathsf{Q}$$

if every Turing ideal satisfying P satisfies Q.

Fix two problems P and Q.

How to prove that $\mathsf{P} \not\models_c \mathsf{Q}$?

Build a Turing ideal $\mathcal{M}$ such that

- $\mathcal{M} \models \mathsf{P}$
- $\mathcal{M} \not\models \mathsf{Q}$

A weakness property is a collection of sets closed downwards under the Turing reducibility.

Examples

- $\{X : X \text{ is low}\}$
- $\{X : A \not\leq_T X\}$ for some set A
- $\{X : X \text{ is hyperimmune-free}\}$

Fix a weakness property $\mathcal{W}$.

A problem P preserves $\mathcal{W}$ if for every $Z \in \mathcal{W}$,
every Z-computable P-instance X
has a solution Y such that $Y \oplus Z \in \mathcal{W}$

Lemma
If P preserves $\mathcal{W}$ but Q does not, then $\mathsf{P} \not\models_c \mathsf{Q}$

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k}$$

Using k colors

ACA: $(\forall X)(\exists Y) Y = X'$

$$\mathsf{ACA} \models_c \mathsf{RT}^n_k$$

whenever $n \geq 1$.

(Jockusch, 1972)

$$\mathsf{RT}^n_k \models_c \mathsf{ACA}$$

whenever $n \geq 3$.

(Jockusch, 1972)

$$\mathsf{RT}^2_k \not\models_c \mathsf{ACA}$$

whenever $n, k \geq 1$.

(Seetapun, 1995)

A problem P avoids cones if it preserves $\mathcal{W} = \{X : A \not\leq_T X\}$ for every set A.

- RT^2_k avoids cones
- ACA does not avoid cones

WKL: Every infinite binary tree has an infinite path.

$$\mathsf{WKL} \not\models_c \mathsf{RT}^2_k$$

whenever $k \geq 2$.

(Jockusch, 1972)

$$\mathsf{RT}^2_k \not\models_c \mathsf{WKL}$$

whenever $k \geq 1$.

(Liu, 2012)

$$\mathsf{RT}^2_k \not\models_c \mathsf{WKL}$$

whenever $k \geq 1$.

(Liu, 2012)

A problem P avoids PA if it preserves $\mathcal{W} = \{X : X \text{ is not PA }\}$.

- RT^2_k avoids PA
- WKL does not avoid PA

$$\begin{array}{ccc} \mathsf{RT}_2^{n\geq 3} & \longleftrightarrow & \mathsf{ACA} \\ \downarrow & & \downarrow \\ \mathsf{RT}_2^2 & & \mathsf{WKL} \end{array}$$

Over computable entailment

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^n_k$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

THIN SET THEOREM

$$\mathsf{TS}^{n}_{k} \qquad \mathsf{RT}^{n}_{k,k-1}$$

$$\mathsf{TS}^n_k \not\models_c \mathsf{ACA}$$

for k is large enough.

(Wang, 2014)

A problem P avoids cones if it preserves $\mathcal{W} = \{X : A \not\leq_T X\}$ for every set A.

- TS^n_k avoids cones when k is sufficiently large
- ACA does not avoid cones

$$\mathsf{TS}^{ns+1}_{k^s} \models_c \mathsf{TS}^{n+1}_{k}$$

$$\mathsf{TS}^{n+2}_{2^n} \models_c \mathsf{ACA}$$

(Wang, Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)

Tuples	Cone avoidance	Computes $\emptyset'$
TS^2_k	$k \geq 2$	never
TS^3_k	$k \geq 3$	$k = 2$
TS^4_k	$k \geq 6$	$k \leq 4$

(Jockusch, Wang, Dorais, Dzhafarov, Hirst, Mileti, Shafer, Cholak, P.)

Does TS^4_5 admit cone avoidance?

$$\mathsf{TS}^2_{k+1} \not\models_c \mathsf{TS}^2_k$$

whenever $k \geq 2$.

(P, 2016)

A problem P preserves k hyperimmunities if it preserves $\mathcal{W} = \{X : \forall i < k,\ A_i \text{ is X-hyperimmune } \}$ for every k-tuple of sets $A_0, \dots, A_{k-1}$.

- TS^2_{k+1} preserves k hyperimmunities
- TS^2_k does not preserve k hyperimmunities

$$\mathsf{TS}^n_\ell \not\models_c \mathsf{TS}^2_k$$

whenever $\ell \gg k, n$.

(P., 2016)

A problem P preserves k hyperimmunities if it preserves $\mathcal{W} = \{X : \forall i < k,\ A_i \text{ is X-hyperimmune }\}$ for every k-tuple of sets $A_0, \dots, A_{k-1}$.

When $\ell \gg k, n$, TS^n_ℓ preserves k hyperimmunities

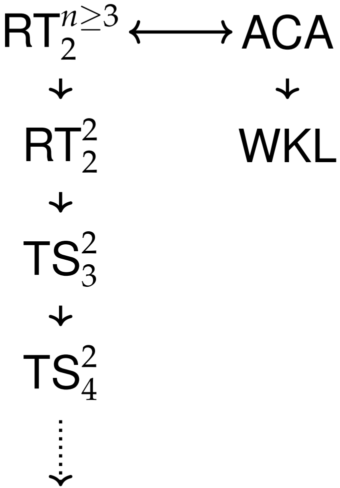
$$\mathsf{TS}^n_k \not\models_c \mathsf{WKL}$$

for k is large enough.

(P., 2016)

A problem P avoids PA if it preserves $\mathcal{W} = \{X : X \text{ is not PA }\}$.

- TS^n_k avoids PA when k is sufficiently large
- WKL does not avoid PA



Over computable entailment

TS^n

For every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, there is an infinite set H such that

$$f[H]^n \neq \mathbb{N}.$$

FS^n

For every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, there is an infinite set H such that

$$\forall c \in H, c \notin f[H \setminus \{c\}]^n.$$

$$\mathsf{TS}^n_k \models_c \mathsf{TS}^n$$

$$\mathsf{FS}^n \models_c \mathsf{TS}^n$$

Is FS^n strictly stronger than TS^n?

$\mathsf{TS}^2_3 \not\models_c \mathsf{FS}^2$

(P., 2017)

A formula $\varphi(\vec{U})$ is essential if for every $x \in \omega$, there are some sets $\vec{V} > x$ such that $\varphi(\vec{V})$ holds.

A function $f : \omega \to \omega$ is freely X-hyperimmune if for every $n \in \omega$, every essential $\Sigma^{0,X}_1$ formula $\varphi(\vec{U})$ and every function $g : \mathcal{P}_+(|\vec{U}|) \to \omega$, $\varphi(\vec{V})$ holds for some sets $\vec{V}$ such that

$$(\forall y \in \bigcup_i V_i) f(y) = g(\{i : y \in V_i\})$$

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