Computability-theoretic aspects of Milliken's tree theorem

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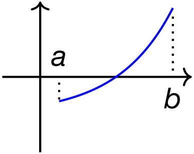
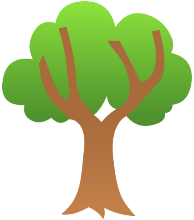
Consider mathematical problems

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



Reverse mathematics

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

RCA_0

Robinson's arithmetics

$$m + 1 \neq 0$$
$$m + 1 = n + 1 \to m = n$$
$$\neg(m < 0)$$
$$m < n + 1 \leftrightarrow (m < n \vee m = n)$$

$$m + 0 = m$$
$$m + (n + 1) = (m + n) + 1$$
$$m \times 0 = 0$$
$$m \times (n + 1) = (m \times n) + m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is a Σ^0_1 formula

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula where X appears freely, and ψ is a Π^0_1 formula.

Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

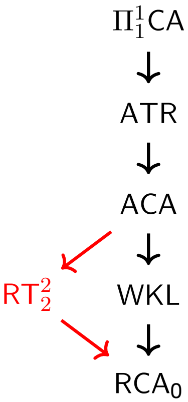
$$\Pi^1_1\mathrm{CA} \to \mathrm{ATR} \to \mathrm{ACA} \to \mathrm{WKL} \to \mathrm{RCA}_0$$

Reverse mathematics

Mathematics are computationally very structured

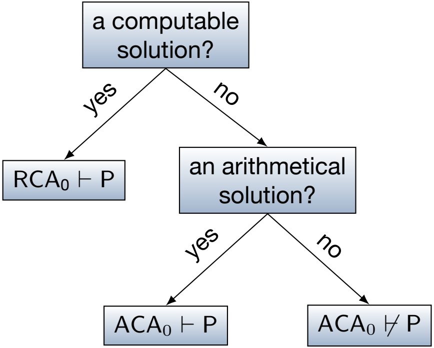
Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

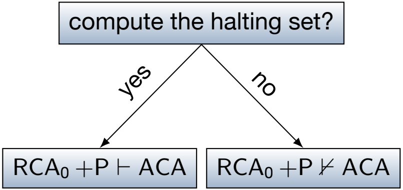


Lower and upper bounds

Does every computable instance of P admit



Is there a computable instance such that every solution



What sets can problems encode?

Defi

A problem P admits cone avoidance if for every non-computable set C, every computable instance of P has a solution which does not compute C.

If P admits cone avoidance, then $\mathrm{RCA}_0 + \mathrm{P} \nvdash \mathrm{ACA}_0$.

What functions can problems dominate?

A function f is hyperimmune if it is not dominated by any computable function.

Defi

A problem P admits preservation of 1 hyperimmunity if for every hyperimmune function f and every instance of P, there is a solution Y such that f is Y-hyperimmune.

Thm (Downey, Greenberg, Harrison-Trainor, P, Turetsky)

Cone avoidance and preservation of 1 hyperimmunity are equivalent.

Not equivalent in the unrelativized version!

- Fix a non-zero set Y of hyperimmune-free degree.
  Let $\mathsf{P}_1 : \emptyset \mapsto \{Y\}$.
- Fix a hyperimmune f below a Δ^1_1-random.
  Let $\mathsf{P}_2 : \emptyset \mapsto \{g : g \geq f\}$.

The strength of Ramsey's theorem

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

Thm (Ramsey's theorem)

RT^n_k: For every k-coloring of $[\mathbb{N}]^n$, there exists an infinite homogeneous set.

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

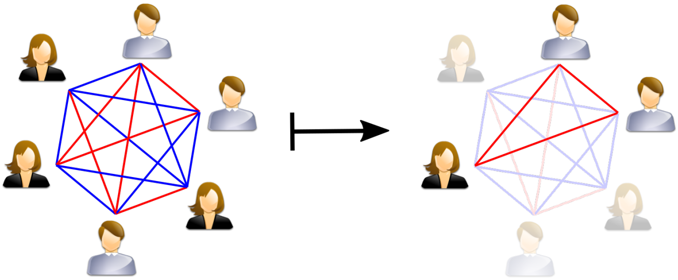
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Lower and upper bounds

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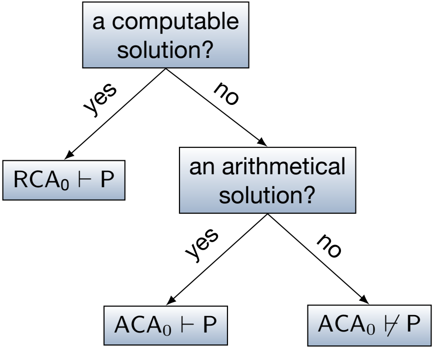

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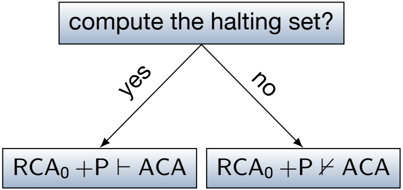


Thm

Every computable instance of RT^1_2 has a computable solution.

Thm (Jockusch)

For $n \geq 2$, every computable instance of RT^n_2 has an arithmetical solution.

Thm (Specker)

For $n \geq 2$, there is a computable instance of RT^n_2 with no computable solution.

Thm (Jockusch)

For $n \geq 3$, there is a computable instance of RT^n_2 whose solutions compute the halting set.

Thm (Seetapun)

RT^2_2 admits cone avoidance.

The encodability power of RT^n_k comes from the

sparsity

of its homogeneous sets.

Ramsey's theorem

Over n-tuples

$$\mathsf{RT}^n_k$$

Using k colors

Ramsey's theorem

Over n-tuples

$$\mathrm{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Thm (Wang)

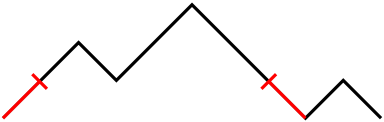
$\mathsf{RT}^n_{k,\ell}$ admits cone avoidance for sufficiently large ℓ.

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

There is a computable instance of $\mathsf{RT}^n_{k,\ell}$ whose solutions compute the halting set whenever $\ell < 2^{n-2}$.

Catalan numbers

C_n is the number of trails of length $2n$.

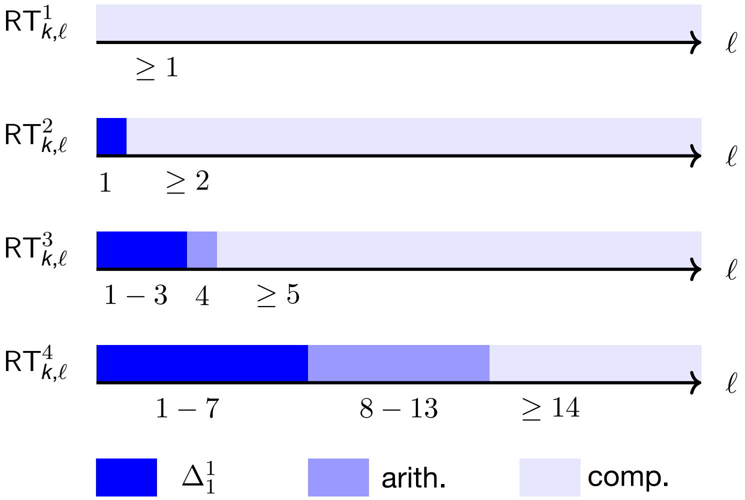


$$C_0 = 1 \qquad \text{and} \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

Thm (Cholak, P.)

$\mathsf{RT}^n_{k,\ell}$ admits cone avoidance if and only if $\ell \geq C_{n-1}$.

$\mathsf{RT}^n_{k,\ell}$-encodable sets



The strength of Milliken's tree theorem

Strings

$\omega^{<\omega}$ is the set of all finite sequences of integers

$\sigma \preceq \tau$ means σ is a prefix of τ

$\sigma \wedge \tau$ is the longest common prefix of σ and τ

$\sigma\tau$ is the concatenation of σ and τ

Trees

A tree is a prefix-closed $\wedge$-closed subset of $\omega^{<\omega}$

The level of $\sigma \in T$ is $|\{\tau \in T : \tau \prec \sigma\}|$

The height of T is the least ordinal α larger than the level of every $\sigma \in T$

For $n \in \mathbb{N}$, $T(n)$ denotes the set of all $\sigma \in T$ at level n

A node $\sigma \in T$ is k-branching in T if it has exactly k many children in T.

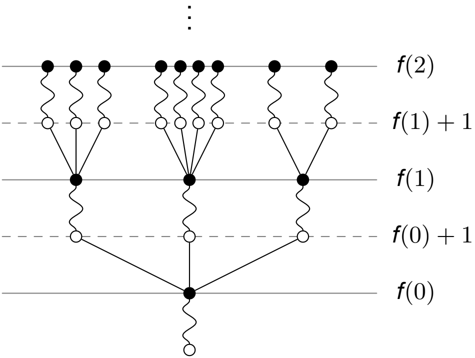
Strong subtrees

A tree S of height α is a strong subtree of a tree T if it satisfies the following two properties:

- there exists a function $f : \alpha \to \omega$, called a level function, such that for all $n < \alpha$, if $\sigma \in S(n)$ then $\sigma \in T(f(n))$;
- for all k, a node in S which is not at level $\alpha - 1$ in S is k-branching in S if and only if it is k-branching in T.

$\mathcal{S}_\alpha(T)$ is the collection of all strong subtrees of T of height α.

Strong subtrees



A strong subtree S of a tree T, with level function f.
The circles represent nodes in T; the solid circles in S.

Milliken's tree theorem

$\mathcal{S}_\alpha(T)$ is the collection of all strong subtrees of T of height α.

A k-coloring of $\mathcal{S}_n(T)$ is a map $f : \mathcal{S}_n(T) \to k$

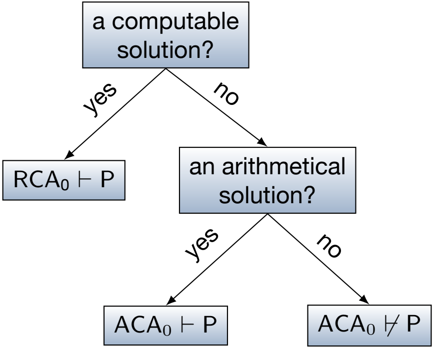
A tree T is homogeneous for f if $|f(\mathcal{S}_n(T))| = 1$.

Thm (Milliken's tree theorem)

MT^n_k: For every finitely branching tree T with no leaves, every k-coloring of $\mathcal{S}_n(T)$, admits a homogeneous strong subtree $S \in \mathcal{S}_\omega(T)$.

Lower and upper bounds

Does every computable instance of P admit



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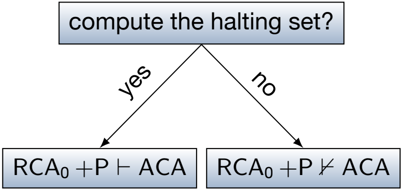


Since Milliken's tree theorem generalizes Ramsey's theorem:

Thm (Specker)

For $n \geq 2$, there is a computable instance of MT^n_2 with no computable solution.

Thm (Jockusch)

For $n \geq 3$, there is a computable instance of MT^n_2 whose solutions compute the halting set.

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

Every computable instance of MT^1_2 has a computable solution.

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

For $n \geq 2$, every computable instance of MT^n_2 has an arithmetical solution.

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

MT^2_2 admits cone avoidance.

Product of trees

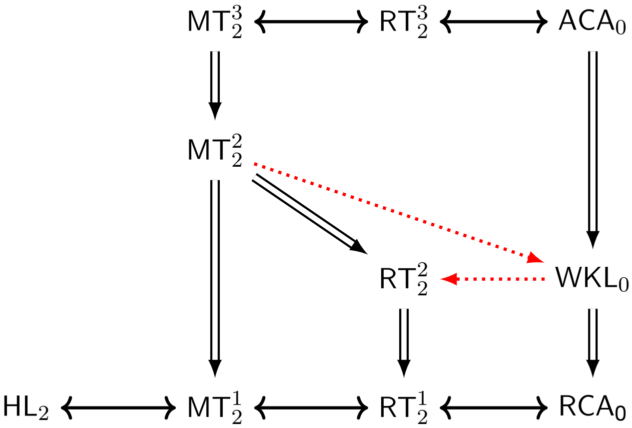
$\mathcal{S}_\alpha(T_0, \ldots, T_{d-1})$ is the collection of all d-tuples of strong subtrees of $T_0, \ldots, T_{d-1}$ of height α, respectively, as witnessed by the same level function.

Thm (Product version of Milliken's tree theorem)

PMT^n_k: For every d-tuple of finitely branching trees $T_0, \ldots, T_{d-1}$ with no leaves, every k-coloring of $\mathcal{S}_n(T_0, \ldots, T_{d-1})$, admits a homogeneous tuple of strong subtrees $\langle S_0, \ldots, S_{d-1} \rangle \in \mathcal{S}_\omega(T_0, \ldots, T_{d-1})$.

Halpern-Lauchli's theorem (HL_k) is PMT^1_k.

Summary



Applications of Milliken's tree theorem

Devlin's theorem

A Joyce tree T of size n is a finite 2-branching tree with n leaves, whose nodes are labelled by $\{1, \dots, 2n-1\}$, such that a node has lower label than its children

$\mathsf{DT}^n_{k,\ell}$: For every coloring $f : [\mathbb{Q}]^n \to k$, there is a subcopy $(U, <_{\mathbb{Q}})$ of $(\mathbb{Q}, <_{\mathbb{Q}})$ such that $|f[U]^n| \leq \ell$

Thm (Devlin's theorem)

$\forall k\, \mathsf{DT}^n_{k,\ell}$ holds if and only if ℓ is at least the number of Joyce trees of size n.

Let ℓ_n be the tight bounds of $\forall k\, \mathsf{DT}^n_{k,\ell_n}$

$\ell_1 = 1$; $\ell_2 = 2$; $\ell_3 = 16$; $\ell_4 = 272$; ...

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

Every computable instance of $\forall k\, \mathsf{DT}^1_{k,1}$ admits a computable solution.

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

There is a computable instance of $\forall k\, \mathsf{DT}^2_{k,3}$ such that every solution computes the halting set.

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

$\forall k\, \mathsf{DT}^2_{k,4}$ admits cone avoidance.

Rado graph theorem

The Rado graph G is the Fraissé limit of the age of finite graphs

Given a finite graph F, $\binom{G}{F}$ is the set of copies of F in G

$\mathrm{RG}^F_{k,\ell}$: For every coloring $f : \binom{G}{F} \to k$, there is Rado subgraph $\hat{G}$ of G such that $|f\binom{\hat{G}}{F}| \leq \ell$

Thm (Rado graph theorem)

For every finite graph F, there is some number of colors ℓ such that $\forall k\, \mathrm{RG}^F_{k,\ell}$ holds.

Let ℓ_F be the tight bound of $\forall k\, \mathrm{RG}^F_{k,\ell_F}$

$\ell_n = \ell_{K_n}$ where K_n is the complete graph of size n

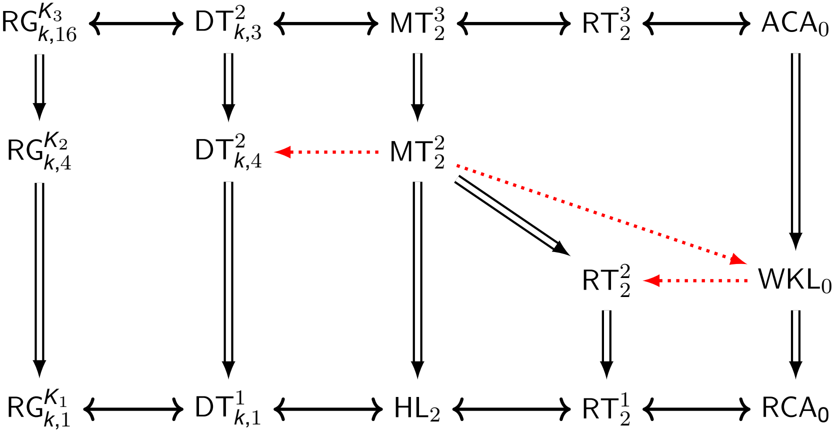
Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

For every finite graph F of size 2, $\forall k\, \mathrm{RG}^F_{k,\ell_F}$ admits cone avoidance.

Thm (Angles d'Auriac, Cholak, Dzhafarov, P. and Monin)

For every finite graph F of size 3, there is a computable instance of $\forall k\, \mathrm{DT}^F_{k,\ell_F}$ such that every solution computes the halting set.

Summary


$RG^{K_3}_{k,16}$
$DT^2_{k,3}$
MT^3_2
RT^3_2
ACA_0
$RG^{K_2}_{k,4}$
$DT^2_{k,4}$
MT^2_2
RT^2_2
WKL_0
$RG^{K_1}_{k,1}$
$DT^1_{k,1}$
HL_2
RT^1_2
RCA_0

Conclusion

Milliken's tree theorem satisfies the same threshold phenomenon as Ramsey's theorem at height 3.

Devlin's theorem for pairs is equivalent to ACA_0, while the Rado graph theorem for size 2 is strictly weaker.

Milliken's tree theorem for pairs implies neither Devlin's theorem for pairs, nor the Rado graph theorem for size 2.

References

Peter Cholak and Ludovic Patey.
Thin set theorems and cone avoidance.
Trans. Amer. Math. Soc., 373(4):2743–2773, 2020.

Paul-Elliot Angles d'Auriac, Peter A. Cholak, Damir D. Dzhafarov, Ludovic Patey, and Benoit Monin.
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