Canonical notions of forcing in computability theory

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Have we found the right techniques?

- Would martians come up with the same proof?
- Do we loose in generality with our constructions?

Example : weak 1-genericity

- A set $D \subseteq 2^{<\omega}$ is dense if for every $\sigma \in 2^{<\omega}$ there is a $\tau \succeq \sigma$ in *D*.
- A real *R* meets *D* if $\sigma \in D$ for some $\sigma \prec R$.
- A real *R* is weakly 1-generic if it meets every dense Σ^0_1 set.

Example : weak 1-genericity

List all the Σ^0_1 sets $W_0, W_1, W_2, \dots \subseteq 2^{<\omega}$

Build a real with the finite extension method $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots$

Let $f : \omega \to \omega$ be an increasing time function.

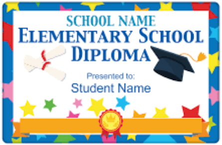
- Search for an extension σ_{s+1} of σ_s in some unsatisfied W_e such that σ_s has no extension in $W_i[f(|\sigma_{s+1}|)]$ for any unsatisfied W_i with $i < e$

Thm (Kurtz)

Every weakly 1-generic real computes a function f which makes this construction produce a weakly 1-generic real.

Example : minimal degree

- A Turing degree $\mathbf{d} > \mathbf{0}$ is minimal if there is no degree in between $\mathbf{0}$ and $\mathbf{d}$.
- Two strings τ_0, τ_1 are a Ψ-splitting if Ψ^{τ_0} and Ψ^{τ_1} are incompatible.
- $T \subseteq 2^{<\omega}$ is a delayed Ψ-splitting tree if T is a tree and whenever $\sigma_0, \sigma_1 \in T$ are incompatible, any $\tau_0, \tau_1 \in T$ properly extending σ_0 and σ_1 respectively are a Ψ-splitting.



Example : minimal degree

- A set $T \subseteq 2^{<\omega}$ is a c.e. tree if there is a computable enumeration of finite sets $\{T_n\}_{n\geq 0}$ such that $|T_0| = 1$ and if $\sigma \in T_{s+1} \setminus T_s$, then σ extends a leaf of T_s.

Thm (Lewis)

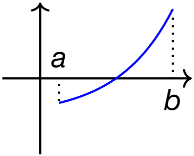
A set G is of minimal degree iff G is incomputable, and whenever Ψ^G is total and incomputable, then Ψ^G lies on a delayed Ψ-splitting c.e. tree.

Both constructions are without loss of generality

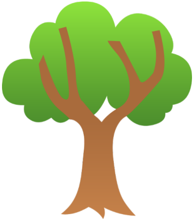
- The constructions are natural
- The resulting objects carry their own construction

Consider mathematical problems

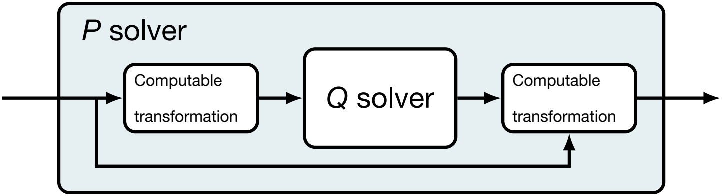
Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



Computable reduction



$$\mathsf{P} \leq_c \mathsf{Q}$$

Every P-instance I computes a Q-instance J such that for every solution X to J, $X \oplus I$ computes a solution to I.

Observations

When proving that P $\not\leq_c$ Q, we usually

- construct a computable instance of P with complex solutions
- construct for every computable instance of Q a simple solution
- use a notion of forcing to build solutions to Q-instances

Observations

The notion of forcing for Q does not depend on P

- Q seems to have a canonical notion of forcing
- Separation proofs can be obtained without loss of generality using this notion of forcing

Examples

For WKL, forcing with Π^0_1 classes :

- ACA $\not\leq_c$ WKL (cone avoidance)
- $\mathsf{RT}^2_2 \not\leq_c$ WKL (ω hyperimmunities)
- ...

For ADS, forcing with split pairs :

- ACA $\not\leq_c$ ADS (cone avoidance)
- CAC $\not\leq_c$ ADS (Towsner)
- $\mathsf{RT}^2_2 \not\leq_c$ ADS (dep. hyperimmunity)
- DNC $\not\leq_c$ ADS (non-DNC degree)

For DNC, forcing with bushy trees

Towards a framework

Weakness property

- A weakness property is a class $\mathcal{W} \subseteq 2^\omega$ which is closed downward under Turing reducibility.
- A problem P computably satisfies a weakness property $\mathcal{W}$ if every computable instance of P has a solution in $\mathcal{W}$.

Example : Given a set A, let $\mathcal{W}_A = \{X : X \not\geq_T A\}$.
Then WKL computably satisfies $\mathcal{W}_A$ for every $A \not\leq_T \emptyset$.

Weakness property

If Q computably satisfies $\mathcal{W}$ but P does not, then

$$\mathsf{P} \not\leq_c \mathsf{Q}$$

P-forcing

Fix a problem P.

- A P-forcing is a forcing family $\mathbb{P} = (\mathbb{P}_I : I \in \operatorname{dom} \mathsf{P})$ such that for every P-instance I, every sufficiently generic filter yields a solution to I.
- A P-forcing $\mathbb{P}$ computably satisfies a weakness property $\mathcal{W}$ if every computable $I \in \operatorname{dom}(\mathsf{P})$, every sufficiently generic filter yields an element in $\mathcal{W}$.

Example : Given a set A, let $\mathcal{W}_A = \{X : X \not\geq_T A\}$.
Forcing with Π^0_1 classes computably satisfies $\mathcal{W}_A$ for every $A \not\leq_T \emptyset$.

Fix a class $\mathfrak{W}$ of weakness properties.

Defi

A P-forcing $\mathbb{P}$ is canonical for $\mathfrak{W}$ if for every $\mathcal{W} \in \mathfrak{W}$ such that P computably satisfies $\mathcal{W}$, then so does $\mathbb{P}$.

What class $\mathfrak{W}$ to consider?

Weakness properties

Effectiveness properties:

Lowness ($\mathcal{W} = \{X : X' \leq_T \emptyset'\}$)

Arithmetical hierarchy
($\mathcal{W} = \{X : X \text{ is arithmetical }\}$)

Genericity properties:

Cone avoidance
($\mathcal{W}_A = \{X : X \not\geq_T A\}$ for $A \not\leq_T \emptyset$)

Preservation of hyperimmunity
($\mathcal{W}_f = \{X : f \text{ is } X\text{-hyperimmune}\}$)

Preservation of non-Σ^0_1 definitions
($\mathcal{W}_A = \{X : A \not\in \Sigma^{0,X}_1\}$ for $A \not\in \Sigma^0_1$)

Closed set avoidance

A closed set avoidance property is a property of the form

$$\mathcal{W}_{\mathcal{C}} = \{X : \mathcal{C} \text{ has no } X\text{-computable member}\}$$

for some closed set $\mathcal{C} \subseteq \omega^\omega$ in the Baire space.

- Cone avoidance: $\mathcal{C}_A = \{A\}$
- Preservation of hyperimmunity: $\mathcal{C}_f = \{g \in \omega^\omega : g \geq f\}$
- Non-DNC degree $\mathcal{C} = \{g \in \omega^\omega : \exists n(g(n) = \Phi_n(n))\}$

What problems admit a canonical forcing?

Cohen genericity

Lem (Folklore)

If $\mathcal{C} \subseteq \omega^\omega$ is a closed set with no computable member, then $\mathcal{C}$ has no G-computable member for every sufficiently Cohen generic.

Proof: Given a Cohen condition $\sigma \in 2^{<\omega}$ forcing totality of a functional Φ_e, there is a $\tau \succeq \sigma$ such that $[\Phi_e^\tau] \cap \mathcal{C} = \emptyset$.

The Atomic Model Theorem (AMT) admits a canonical notion of forcing for closed set avoidance properties.

Highness

Lem (Folklore)

If $\mathcal{C} \subseteq \omega^\omega$ is a closed set with no computable member and $A \in 2^\omega$, then $\mathcal{C}$ has no G-computable member for some G such that $G' \geq_T A$.

Proof: Use forcing conditions (h, n), where $h \subseteq \omega^2 \to 2$ is a finite Δ^0_2 approximation, and n fixes the first n columns to A.

Cohesiveness (COH) and highness admit a canonical notion of forcing for closed set avoidance properties.

Weak König's lemma

WKL: Every infinite binary tree has an infinite path

Let $\mathcal{C}$ be the closed set of all completions of PA.

Then WKL does not computably preserve $\mathcal{W}_{\mathcal{C}}$.

Thm

The WKL-forcing with non-empty Π^0_1 classes is canonical for closed set avoidance properties.

Weak König's lemma

Thm

The WKL-forcing with non-empty Π^0_1 classes is canonical for closed set avoidance properties.

- Fix a closed set $\mathcal{C} \subseteq \omega^\omega$ and a functional Φ_e.
- Try to prove that the set of Π^0_1 classes forcing $\Phi_e^G \notin \mathcal{C}$ is dense.
- If it fails, show that WKL does not computably preserve $\mathcal{W}_\mathcal{C}$.

Weak König's lemma

Fix a closed set $\mathcal{C} \subseteq \omega^\omega$, a non-empty Π^0_1 class $\mathcal{D}$ and Φ_e.

Success if

- there is a $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathcal{D} \neq \emptyset$ and $[\Phi_e^\sigma] \cap \mathcal{C} = \emptyset$.
- or $\{X \in \mathcal{D} : \Phi_e^X(n) \uparrow\} \neq \emptyset$ for some n.

Otherwise $\{\Phi_e^X : X \in \mathcal{D}\}$ is an effectively compact subset of $\mathcal{C}$.
Every PA degree computes a member of $\mathcal{C}$.

Weak König's lemma

Thm

WKL computably preserves $\mathcal{W}_{\mathcal{C}}$ iff $\mathcal{C}$ has no non-empty effectively compact subset.

- Cone avoidance : $\mathcal{C} = \{A\}$ if $A \not\leq_T \emptyset$
- Preservation of hyperimmunity: $\mathcal{C}_f = \{g \in \omega^\omega : g \geq f\}$
- DNC : The Π^0_1 class of $\{0, 1\}$-valued DNC is a non-empty effectively compact subset

Ascending Descending Sequence

SADS: Every linear order of type $\omega + \omega^*$ has an infinite ascending or descending sequence.

Let $L = (\omega, <_L)$ be an instance of SADS with *omega*-part U and ω^*-part V.

Forcing conditions : (σ_0, σ_1) such that

- $\sigma_0, \sigma_1 \in \omega^{<\omega}$ are $<_{\mathbb{N}}$-ascending
- $\sigma_0 \subseteq U$ is $<_L$-ascending
- $\sigma_1 \subseteq V$ is $<_L$-descending

Ascending Descending Sequence

Thm

The SADS-forcing is canonical for closed set avoidance properties.

A split pair is a pair (τ_0, τ_1) such that

- $\tau_0, \tau_1 \in \omega^{<\omega}$ are $<_{\mathbb{N}}$-ascending
- τ_0 is $<_L$-ascending, τ_1 is $<_L$-descending
- $\max_L \tau_0 <_L \min_L \tau_1$

Closed set jump avoidance

A closed set jump avoidance property is a property of the form

$$\mathcal{J}_{\mathcal{C}} = \{X : \mathcal{C} \text{ has no } X'\text{-computable member}\}$$

for some closed set $\mathcal{C} \subseteq \omega^\omega$ in the Baire space.

Let $\mathcal{C}$ be the closed set of all completions of PA relative to 0'.
Then COH does not computably preserve $\mathcal{J}_{\mathcal{C}}$.

Cohesiveness

Let $R_0, R_1, R_2, \dots$ be an instance of COH

Let $R_\sigma = \bigcap_{\sigma(i)=1} R_i \bigcap_{\sigma(i)=0} \overline{R}_i$

Forcing conditions: $(F, \sigma, \mathcal{D})$ such that

- F is a finite set, $\sigma \in 2^{<\omega}$
- $\mathcal{D}$ is a non-empty $\Pi_1^{0,\emptyset'}$ subclass of $[\sigma]$

$(E, \tau, \mathcal{E}) \leq (F, \sigma, \mathcal{D})$ if

- (E, R_τ) Mathias extends (F, R_σ).
- $\sigma \prec \tau$ and $\mathcal{E} \subseteq \mathcal{D}$.

Cohesiveness

Thm

The COH-forcing is canonical for closed set jump avoidance properties.

Thm

COH computably preserves $\mathcal{J}_{\mathcal{C}}$ iff $\mathcal{C}$ has no non-empty $\emptyset'$-effectively compact subset.

- If A is not Δ^0_2, every computable instance of COH admits a solution G such that A is not $\Delta^0_2(G)$.

Open questions

DNC functions

A function f is DNC if for every e, $f(e) \neq \Phi_e(e)$.

A tree $T \subseteq \omega^{<\omega}$ is k-bushy above $\sigma \in \omega^{<\omega}$ if every element of T is comparable with T, and for every $\tau \in T$ which extends σ and is not a leaf, τ has at least k immediate extensions in T.

A set $B \subseteq \omega^{<\omega}$ is k-small above σ if there is no finite tree k-bushy above σ whose leaves belong to B.

DNC functions

Bushy tree forcing : (σ, B) where

- $\sigma \in \omega^{<\omega}$
- *B* is *k*-small above σ for some *k*.

Question

Is bushy tree DNC*-forcing canonical for closed set avoidance properties?*

Intuition

Lem

Let X be a set. TFAE

- X computes a DNC function
- X computes a function g such that if $|W_e| \leq n$, then $g(e, n) \notin W_e$.

Lem

Suppose B is a k-small c.e. set above σ. Then the set

$$\{n : B \text{ is not } k\text{-small above } \sigma n\}$$

is c.e. of size at most $k - 1$.

Conclusion

Natural combinatorial problems seem to have canonical notions of forcing.

The proofs of canonicity yield forcing-free criteria of preservations.

The right notion of forcing for DNC functions is not fully understood.

References

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