Canonical notions of forcing in reverse mathematics

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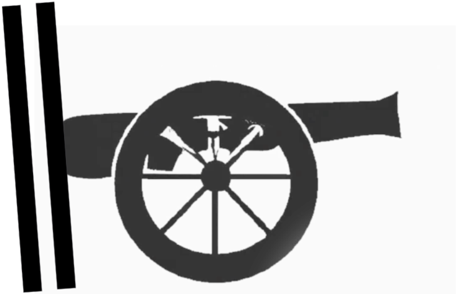


Computability 101

A set $A \subseteq \omega$ is computable if there is a computer program which, on input n, decides whether $n \in A$ or not.

A set $A \subseteq \omega$ is computable in B if there is a computer program in an language augmented with the characteristic function of B which, on input n, decides whether $n \in A$ or not.

$$A \leq_T B$$

A is computable in B

$\leq_T$ is a preorder

Turing equivalence
$A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$

Turing degree
$\deg_T(A) = \{B : B \equiv_T A\}$

Turing degrees
$(\mathcal{D}, \leq)$ is a partial order

The Turing degrees are

- countable
- robust

They represent
computational powers

$$\Phi_e(x)\downarrow$$

The e-th program halts on input x.

$$\Phi_e(x)[t]\downarrow$$

The e-th program halts on input x
in less than t steps.

$$\Phi_e^A(x)\downarrow$$

The e-th program with oracle A halts on input x.

$$\Phi_e^A(x)[t]\downarrow$$

The e-th program with oracle A halts on input x in less than t steps.

Halting set

$$\emptyset' = \{e : \Phi_e(e) \downarrow\}$$

Thm (Turing)

The halting set is not computable.

Turing jump

$$A' = \{e : \Phi_e^A(e)\downarrow\}$$

Thm

- $A <_T A'$
- $A \equiv_T B \to A' \equiv_T B'$

Arithmetic hierarchy

$$\Sigma^0_n \qquad \exists x_1 \forall x_2 \dots Q x_n \, R(x_1, \dots, x_n)$$

$$\Pi^0_n \qquad \forall x_1 \exists x_2 \dots Q x_n \, R(x_1, \dots, x_n)$$

where R has only bounded quantifiers.

- A set is Σ^0_n (Π^0_n) if it is definable by a Σ^0_n (Π^0_n) formula
- A set is Δ^0_n if it is both Σ^0_n and Π^0_n.

Computability $\equiv$ Definability

Thm (Post)

A set is c.e. iff it is Σ^0_1 and computable iff it is Δ^0_1.

Thm (Post)

A set is $\emptyset^{(n)}$-c.e. iff it is Σ^0_{n+1} and $\emptyset^{(n)}$-computable iff it is Δ^0_{n+1}.

Have we found the right techniques?

- Would martians come up with the same proof?
- Do we loose in generality with our constructions?

Example : weak 1-genericity

- A set $D \subseteq 2^{<\omega}$ is dense if for every $\sigma \in 2^{<\omega}$ there is a $\tau \succeq \sigma$ in D.
- A real R meets D if $\sigma \in D$ for some $\sigma \prec R$.
- A real R is weakly 1-generic if it meets every dense Σ^0_1 set.

Example : weak 1-genericity

List all the Σ^0_1 sets $W_0, W_1, W_2, \dots \subseteq 2^{<\omega}$

Build a real with the finite extension method $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots$

Let $f : \omega \to \omega$ be an increasing time function.

- Search for an extension σ_{s+1} of σ_s in some unsatisfied W_e such that σ_s has no extension in $W_i[f(|\sigma_{s+1}|)]$ for any unsatisfied W_i with $i < e$

Thm (Kurtz)

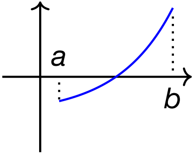
Every weakly 1-generic real computes a function f which makes this construction produce a weakly 1-generic real.

The construction is without loss of generality

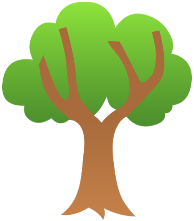
- The construction is natural
- The resulting object carries its own construction

Consider mathematical problems

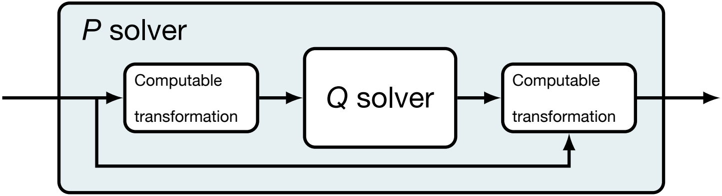
Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



Computable reduction



$$P \leq_c Q$$

Every P-instance I computes a Q-instance J such that for every solution X to J, $X \oplus I$ computes a solution to I.

Observations

When proving that P $\not\leq_c$ Q, we usually

- construct a computable instance of P with complex solutions
- construct for every computable instance of Q a simple solution
- use a notion of forcing to build solutions to Q-instances

Observations

The notion of forcing for Q
does not depend on P

- Q seems to have a canonical notion of forcing
- Separation proofs can be obtained without loss of generality using this notion of forcing

Examples

For WKL, forcing with Π^0_1 classes :

- ACA $\not\leq_c$ WKL (cone avoidance)
- $\mathrm{RT}^2_2 \not\leq_c$ WKL (ω hyperimmunities)
- ...

For ADS, forcing with split pairs :

- ACA $\not\leq_c$ ADS (cone avoidance)
- CAC $\not\leq_c$ ADS (Towsner)
- $\mathrm{RT}^2_2 \not\leq_c$ ADS (dep. hyperimmunity)
- DNC $\not\leq_c$ ADS (non-DNC degree)

For DNC, forcing with bushy trees

Towards a framework

Weakness property

- A weakness property is a class $\mathcal{W} \subseteq 2^\omega$ which is closed downward under Turing reducibility.
- A problem P computably satisfies a weakness property $\mathcal{W}$ if every computable instance of P has a solution in $\mathcal{W}$.

Example : Given a set A, let $\mathcal{W}_A = \{X : X \not\geq_T A\}$.
Then WKL computably satisfies $\mathcal{W}_A$ for every $A \not\leq_T \emptyset$.

Weakness property

If Q computably satisfies $\mathcal{W}$ but P does not, then

$$\mathsf{P} \not\leq_c \mathsf{Q}$$

P-forcing

Fix a problem P.

- A P-forcing is a forcing family $\mathbb{P} = (\mathbb{P}_I : I \in \text{dom}\,\mathsf{P})$ such that for every P-instance I, every sufficiently generic filter yields a solution to I.

- A P-forcing $\mathbb{P}$ computably satisfies a weakness property $\mathcal{W}$ if every computable $I \in \text{dom}(\mathsf{P})$, every sufficiently generic filter yields an element in $\mathcal{W}$.

Example : Given a set A, let $\mathcal{W}_A = \{X : X \not\geq_T A\}$.
Forcing with Π^0_1 classes computably satisfies $\mathcal{W}_A$ for every $A \not\leq_T \emptyset$.

Fix a class $\mathfrak{W}$ of weakness properties.

Defi

A P-forcing $\mathbb{P}$ is canonical for $\mathfrak{W}$ if for every $\mathcal{W} \in \mathfrak{W}$ such that P computably satisfies $\mathcal{W}$, then so does $\mathbb{P}$.

What class $\mathfrak{W}$ to consider?

Weakness properties

Effectiveness properties:

Lowness ($\mathcal{W} = \{X : X' \leq_T \emptyset'\}$)

Arithmetical hierarchy
($\mathcal{W} = \{X : X \text{ is arithmetical }\}$)

Genericity properties:

Cone avoidance
($\mathcal{W}_A = \{X : X \not\geq_T A\}$ for $A \not\leq_T \emptyset$)

Preservation of hyperimmunity
($\mathcal{W}_f = \{X : f \text{ is } X\text{-hyperimmune}\}$)

Preservation of non-Σ^0_1 definitions
($\mathcal{W}_A = \{X : A \not\in \Sigma^{0,X}_1\}$ for $A \not\in \Sigma^0_1$)

Closed set avoidance

A closed set avoidance property is a property of the form

$$\mathcal{W}_{\mathcal{C}} = \{X : \mathcal{C} \text{ has no } X\text{-computable member}\}$$

for some closed set $\mathcal{C} \subseteq \omega^\omega$ in the Baire space.

- Cone avoidance: $\mathcal{C}_A = \{A\}$
- Preservation of hyperimmunity: $\mathcal{C}_f = \{g \in \omega^\omega : g \geq f\}$
- Non-DNC degree $\mathcal{C} = \{g \in \omega^\omega : \exists n(g(n) = \Phi_n(n))\}$

First jump part

- First order part of P: first-order consequences of P in Reverse Mathematics

$$\{T \in \mathcal{L}_{PA} : \mathrm{RCA}_0 + \mathsf{P} \vdash T\}$$

- First order part of P: first-order problems reducible to it in Weihrauch degrees

$$\{\mathsf{Q} \subseteq \omega^\omega \rightrightarrows \omega : \mathsf{Q} \leq_W \mathsf{P}\}$$

- First jump part of P : closed sets computably avoided by the problem

$$\{\text{ closed } \mathcal{C} \subseteq \omega^\omega : \mathsf{P} \text{ computably satisfies } \mathcal{W}_\mathcal{C}\}$$

What problems admit a canonical forcing?

Trivial examples

Cohen genericity

Lem (Folklore)

If $\mathcal{C} \subseteq \omega^\omega$ is a closed set with no computable member, then $\mathcal{C}$ has no *G*-computable member for every sufficiently Cohen generic.

Proof: Given a Cohen condition $\sigma \in 2^{<\omega}$ forcing totality of a functional Φ_e, there is a $\tau \succeq \sigma$ such that $[\Phi_e^\tau] \cap \mathcal{C} = \emptyset$.

The Atomic Model Theorem (AMT) admits a canonical notion of forcing for closed set avoidance properties.

Highness

Lem (Folklore)

If $\mathcal{C} \subseteq \omega^\omega$ is a closed set with no computable member and *A* $\in 2^\omega$, then $\mathcal{C}$ has no *G*-computable member for some *G* such that $G' \geq_T A$.

Proof: Use forcing conditions (h, n), where $h \subseteq \omega^2 \to 2$ is a finite Δ^0_2 approximation, and *n* fixes the first *n* columns to *A*.

Cohesiveness (COH) and highness admit a canonical notion of forcing for closed set avoidance properties.

A non-trivial example

Weak König's lemma

WKL: Every infinite binary tree has an infinite path

Let $\mathcal{C}$ be the closed set of all completions of PA.

Then WKL does not computably preserve $\mathcal{W}_{\mathcal{C}}$.

Thm

The WKL-forcing with non-empty Π^0_1 classes is canonical for closed set avoidance properties.

Weak König's lemma

Thm

The WKL-forcing with non-empty Π^0_1 classes is canonical for closed set avoidance properties.

- Fix a closed set $\mathcal{C} \subseteq \omega^\omega$ and a functional Φ_e.
- Try to prove that the set of Π^0_1 classes forcing $\Phi_e^G \not\in \mathcal{C}$ is dense.
- If it fails, show that WKL does not computably preserve $\mathcal{W}_\mathcal{C}$.

Weak König's lemma

Fix a closed set $\mathcal{C} \subseteq \omega^\omega$, a non-empty Π^0_1 class $\mathcal{D}$ and Φ_e.

Success if

- there is a $\sigma \in 2^{<\omega}$ such that $[\sigma] \cap \mathcal{D} \neq \emptyset$ and $[\Phi_e^\sigma] \cap \mathcal{C} = \emptyset$.
- or $\{X \in \mathcal{D} : \Phi_e^X(n) \uparrow\} \neq \emptyset$ for some n.

Otherwise $\{\Phi_e^X : X \in \mathcal{D}\}$ is an effectively compact subset of $\mathcal{C}$.
Every PA degree computes a member of $\mathcal{C}$.

Weak König's lemma

Thm

WKL computably preserves $\mathcal{W}_{\mathcal{C}}$ iff $\mathcal{C}$ has no non-empty effectively compact subset.

- Cone avoidance : $\mathcal{C} = \{A\}$ if $A \not\leq_T \emptyset$
- Preservation of hyperimmunity: $\mathcal{C}_f = \{g \in \omega^\omega : g \geq f\}$
- DNC : The Π^0_1 class of $\{0, 1\}$-valued DNC is a non-empty effectively compact subset

Consequences

The proofs of canonicity yield forcing-free criteria of preservations.

The existence of a canonical notion of forcing yields uniform procedures.

Ascending Descending Sequence

SADS: Every linear order of type $\omega + \omega^*$ has an infinite ascending or descending sequence.

Let $L = (\omega, <_L)$ be an instance of SADS with ω-part U and ω^*-part V.

Forcing conditions : (σ_0, σ_1) such that

- $\sigma_0, \sigma_1 \in \omega^{<\omega}$ are $<_{\mathbb{N}}$-ascending
- $\sigma_0 \subseteq U$ is $<_L$-ascending
- $\sigma_1 \subseteq V$ is $<_L$-descending

Ascending Descending Sequence

Thm

The SADS-forcing is canonical for closed set avoidance properties.

- Fix a closed set $\mathcal{C} \subseteq \omega^\omega$ and two functionals Φ_{e_0}, Φ_{e_1}.
- Try to prove that the set of conditions (σ_0, σ_1) forcing $\Phi_{e_0}^{G_0} \not\in \mathcal{C} \vee \Phi_{e_1}^{G_1} \not\in \mathcal{C}$ is dense.
- If it fails, show that SADS does not computably preserve $\mathcal{W}_\mathcal{C}$.

Ascending Descending Sequence

Thm

The SADS-forcing is canonical for closed set avoidance properties.

A split pair is a pair (τ_0, τ_1) such that

- $\tau_0, \tau_1 \in \omega^{<\omega}$ are $<_{\mathbb{N}}$-ascending
- τ_0 is $<_L$-ascending, τ_1 is $<_L$-descending
- $\max_L \tau_0 <_L \min_L \tau_1$

Ascending Descending Sequence

Fix a closed set $\mathcal{C} \subseteq \omega^\omega$, a condition (σ_0, σ_1) and Φ_{e_0}, Φ_{e_1}.

Success if

- there is a $\tau_0 \succeq \sigma_0$ such that (τ_0, σ_1) is a condition, and $[\Phi_{e_0}^{\tau_0}] \cap \mathcal{C} = \emptyset$.
- there is a $\tau_1 \succeq \sigma_1$ such that (σ_0, τ_1) is a condition, and $[\Phi_{e_1}^{\tau_1}] \cap \mathcal{C} = \emptyset$.
- or there is no split pair (τ_0, τ_1) with $\tau_0 \succeq \sigma_0$, $\tau_1 \succeq \sigma_1$ and such that $\{0, \dots, n\} \subseteq \operatorname{dom} \Phi_{e_0}^{\tau_0} \cap \operatorname{dom} \Phi_{e_1}^{\tau_1}$ for some n.

Otherwise, we can computably enumerate split pairs (τ_0^s, τ_1^s) such that $\Phi_{e_0}^{\tau_0^s}$ and $\Phi_{e_1}^{\tau_1^s}$ are defined on $\{0, \dots, s\}$.
$\{\max \tau_0^s : s \in \omega\}$ is a computable instance of SADS such that every solution computes member of $\mathcal{C}$.

Second-jump parts

Closed set jump avoidance

A closed set jump avoidance property is a property of the form

$$\mathcal{J}_{\mathcal{C}} = \{X : \mathcal{C} \text{ has no } X'\text{-computable member}\}$$

for some closed set $\mathcal{C} \subseteq \omega^\omega$ in the Baire space.

Let $\mathcal{C}$ be the closed set of all completions of PA relative to 0'.
Then COH does not computably preserve $\mathcal{J}_{\mathcal{C}}$.

Cohesiveness

Let $R_0, R_1, R_2, \dots$ be an instance of COH

Let $R_\sigma = \bigcap_{\sigma(i)=1} R_i \bigcap_{\sigma(i)=0} \overline{R}_i$

Forcing conditions: $(F, \sigma, \mathcal{D})$ such that

- F is a finite set, $\sigma \in 2^{<\omega}$
- $\mathcal{D}$ is a non-empty $\Pi^{0,\emptyset'}_1$ subclass of $[\sigma]$

$(E, \tau, \mathcal{E}) \leq (F, \sigma, \mathcal{D})$ if

- (E, R_τ) Mathias extends (F, R_σ).
- $\sigma \prec \tau$ and $\mathcal{E} \subseteq \mathcal{D}$.

Cohesiveness

Thm

The COH-forcing is canonical for closed set jump avoidance properties.

Thm

COH computably preserves $\mathcal{J}_\mathcal{C}$ iff $\mathcal{C}$ has no non-empty $\emptyset'$-effectively compact subset.

- If *A* is not Δ^0_2, every computable instance of COH admits a solution *G* such that *A* is not $\Delta^0_2(G)$.

Open questions

DNC functions

A function f is DNC if for every e, $f(e) \neq \Phi_e(e)$.

A tree $T \subseteq \omega^{<\omega}$ is k-bushy above $\sigma \in \omega^{<\omega}$ if every element of T is comparable with T, and for every $\tau \in T$ which extends σ and is not a leaf, τ has at least k immediate extensions in T.

A set $B \subseteq \omega^{<\omega}$ is k-small above σ if there is no finite tree k-bushy above σ whose leaves belong to B.

DNC functions

Bushy tree forcing : (σ, B) where

- $\sigma \in \omega^{<\omega}$
- B is k-small above σ for some k.

Question

Is bushy tree DNC*-forcing canonical for closed set avoidance properties?*

Intuition

Lem

Let X be a set. TFAE

- X computes a DNC function
- X computes a function g such that if $|W_e| \leq n$, then $g(e, n) \notin W_e$.

Lem

Suppose B is a k-small c.e. set above σ. Then the set

$$\{n : B \text{ is not } k\text{-small above } \sigma n\}$$

is c.e. of size at most $k - 1$.

Conclusion

Natural combinatorial problems seem to have canonical notions of forcing.

The proofs of canonicity yield forcing-free criteria of preservations.

The right notion of forcing for DNC functions is not fully understood.

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