Reverse mathematics: Classifying principles by the no randomized algorithm property.

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Summary

Introduction

NRA property

Classification

Conclusion
Plan

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Classification

Conclusion
What is Reverse Mathematics?

**Definition**
Program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

- Weak system ($\text{RCA}_0$)
- Prove equivalence of theorems and axioms over $\text{RCA}_0$
- Lattice of systems

**Applications**
- Soundness
- Heuristic for new proofs
Observation

Most theorems of “ordinary” mathematics

- live in weak systems.
- are equivalent to axioms over RCA₀

- Refine our structure of weak systems.
- Weaker than Ramsey theorem and König’s lemma.
Language of Second Order Arithmetic $L_2$

Numerical terms

\[ t ::= 0 \mid 1 \mid x \mid t_1 + t_2 \mid t_1 \cdot t_2 \]

Formulas

\[ f ::= \quad t_1 = t_2 \mid t_1 < t_2 \mid t_1 \in X \mid \forall x.f \]
\[ \mid \exists x.f \mid \forall X.f \mid \exists X.f \mid \neg f \mid f_1 \lor f_2 \]
Axioms of Second Order Arithmetic $\mathbb{Z}_2$

Basic axioms

\[
\begin{align*}
    n + 1 & \neq 0 & m + 1 = n + 1 \implies m = n \\
    m + 0 & = m & m + (n + 1) = (m + n) + 1 \\
    m \cdot 0 & = 0 & m \cdot (n + 1) = (m \cdot n) + m \\
    \neg m < 0 & & m < n + 1 \iff (m < n \lor m = n)
\end{align*}
\]

Induction axiom

\[
(0 \in X \land \forall n. (n \in X \implies n + 1 \in X)) \implies \forall n. (n \in X)
\]

Comprehension scheme

\[
\exists X. \forall n. (n \in X \iff \varphi(n))
\]

where $\varphi(n)$ is any formula of $L_2$ in which $X$ does not occur freely.
Subsystem of $\mathbb{Z}_2$

Definition (Subsystem of $\mathbb{Z}_2$)

System based of $L_2$ whose axioms are theorems of $\mathbb{Z}_2$
The system $\text{RCA}_0$

Basic axioms

$\Sigma^0_1$ Induction scheme

$$(\varphi(0) \land \forall n. (\varphi(n) \Rightarrow \varphi(n + 1))) \Rightarrow \forall n. \varphi(n)$$

where $\varphi(n)$ is any $\Sigma^0_1$ formula of $L_2$

$\Delta^0_1$ Comprehension scheme

$$\forall n (\varphi(n) \iff \psi(n)) \Rightarrow \exists X. \forall n. (x \in X \iff \varphi(n))$$

where $\varphi(n)$ is any $\Sigma^0_1$ formula of $L_2$ in which $X$ does not occur freely and $\psi(n)$ is any $\Pi^0_1$ formula of $L_2$. 
The “Big Five” subsystems

\[ \text{Pi11-CA} \]
\[ \text{ATR} \]
\[ \text{ACA} \]
\[ \text{WKL} \]
\[ \text{RCA} \]
Reverse mathematics zoo
\(\omega\)-structure

**Definition (\(\omega\)-structure)**

\[ M_S = (\omega, S, +, \times, <) \]

**Example (Minimal \(\omega\)-model of RCA_0)**

\(COMP\) is the \(\omega\)-structure where

\[ S = \{ X \in 2^\omega : X \text{ is computable} \} \]
Plan

Introduction

NRA property

Classification

Conclusion
No randomized algorithm property

Definition
Let $\vec{X}_i$ be a sequence of sets. $COMP(\vec{X}_i)$ is the $\omega$-structure where

$$S = \bigcup_{i \in \omega} \{ Y : Y \leq_T X_0 \oplus \cdots \oplus X_i \}.$$  

Question

Fix a system $P$ and pick a sequence $\vec{X}_i$ at random. What is the probability that $COMP(\vec{X}_i) \models P$?
No randomized algorithm property

Definition
A system $P$ has the no randomized algorithm property if when picking a sequence of sets $\vec{X}_i$, the probability that $COMP(\vec{X}_i) \models P$ is null.

Question
Which systems have the NRA property?
No randomized algorithm property

Why no randomized algorithm property?

• Consider a principle $P = \forall Y \exists Z \Phi(Y, Z)$.

• If $P$ has the NRA property, then for almost every sequence $\vec{X}_i$ there is a $Y \in COMP(\vec{X}_i)$ such that no probabilistic algorithm computes a $Z$ such that $\Phi(Y, Z)$. 
No randomized algorithm property

\( n\text{-RAN} \) (\( n\text{-randomness} \))

For every \( X \), there is a set \( Y \) which is \( n\text{-random} \) relative to \( X \).

\( n\text{-WWKL} \) (\( n\text{-weak weak König’s lemma} \))

Every subtree of \( 2^{<\omega} \) of positive measure computable in \( \emptyset^{(n-1)} \) has an infinite path.

**Theorem (Avigdod, Dean & Rute)**

For every standard \( n \),

\[
\text{RCA}_0 + B\Sigma_n \vdash n\text{-RAN} \iff n\text{-WWKL}
\]
Theorem

If a system $S$ has the NRA property

$$\forall n \quad \text{RCA}_0 \not\vdash n\text{-WWKL} \rightarrow S$$

Proof.

Pick the $\vec{X}_i$ at random. With probability 1, for all $i$, $X_{i+1}$ is $n$-random relative to the join of the $X_k$, $k < i$. Therefore, with probability 1, $COMP(\vec{X}_i)$ is a model of $n$-WWKL. \qed
Plan

Introduction

NRA property

Classification

Conclusion
No randomized algorithm property

Which systems have the NRA property?
We can take the zoo ...
... and classify it
A remark

A lot of (weak) principles have the NRA property

...
SADS (Stable ascending descending sequence)
Every linear order of order type $\omega + \omega^*$ has an infinite suborder of order type $\omega$ or $\omega^*$.

Theorem (Csima & Mileti)
SADS has the NRA property

Proof.
There is a computable linear order of order type $\omega + \omega^*$ such that the measure of oracles computing an infinite suborder of order type $\omega$ or $\omega^*$ is null.
**Ordering**

**CADS** *(Cohesive ascending descending sequence)*

Every linear order has a suborder of order type $\omega + \omega^*$ or $\omega$ or $\omega^*$.

**Theorem (Bienvenu, Patey & Shafer)**

**CADS has the NRA property**

**Proof.**

There is a computable linear order such that the measure of oracles computing an infinite suborder of order type $\omega + \omega^*$ or $\omega$ or $\omega^*$ is null.
\( \Pi^0_1 G \) (\( \Pi^0_1 \) genericity)

Any uniformly \( \Pi^0_1 \) collection of dense sets \( D_i \subseteq 2^{<\omega} \) has a \( G \) such that \( \forall i \exists s (G \upharpoonright s \in D_i) \).

**Theorem (Kurtz)**

*The upward closure of the weakly 2-generic degrees has measure 0.*

**Theorem (Bienvenu, Patey & Shafer)**

\( \Pi^0_1 G \) *has the NRA property*
First remark

... but there are non-trivial problems solved by randomness.
Genericity

1-GEN (1-genericity)
For any set $X$, there exists a set 1-generic relative to $X$.

Theorem (Kurtz)
Almost every set computes a 1-generic set.

Corollary
1-GEN does not have the NRA property.
Rainbow Ramsey Theorem

Definition ($k$-bounded function)
A coloring function $\mathbb{N}^n \to \mathbb{N}$ is $k$-bounded if
$\text{card}\{x \in \mathbb{N}^n : f(x) = c\} \leq k$ for every color $c$.

$\text{RRT}_k^n$ (Rainbow Ramsey Theorem)
For every $k$-bounded coloring function $f : \mathbb{N}^n \to \mathbb{N}$ there is an
infinite set $H$ such that $f \upharpoonright H^n$ is injective.
Rainbow Ramsey Theorem

Theorem (Csima & Mileti)
\[ \text{RCA}_0 \vdash 2\text{-RAN} \rightarrow \text{RRT}_2^2 \]

Theorem (Bienvenu, Patey & Shafer)
\[ \text{RRT}_2^3 \text{ has the NRA property.} \]

Proof.
There is a computable 2-bounded coloring \( c : [\mathbb{N}]^3 \rightarrow \mathbb{N} \) such that the measure of oracles computing an infinite rainbow for \( c \) is null.
Plan

Introduction

NRA property

Classification

Conclusion
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- The following principles have the NRA property: \( \Pi^0_1 G, \text{CADS}, \text{SEM}, \text{RRT}_2^3, \text{POS}, \text{STS}(2) \text{ RCOLOR}_2 \).

- Any principle below \( n\text{-WWKL} \) for some \( n \) does not have the NRA property.

- This suffices to classify the whole zoo.
Further research

- The NRA property: computing or not a solution with
  - randomness

- What about the ability to compute a solution with
  - randomness
  - other oracles (eg. 0′)
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Questions

Thank you for listening!