On combinatorial weaknesses of Ramseyan principles

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SUMMARY

What is reverse mathematics ?

Definition

Reverse mathematics is program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

- Weak system (RCA_0)
- Prove equivalence of theorems and axioms over RCA_0

Applications

- Deeper undestanding
- Search for more elementary proofs

WHAT IS RCA_0 ?

- basic Peano axioms
- the comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X.\forall n.(x \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula and $\psi(n)$ is any Π^0_1 formula.

- the induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula

ω-STRUCTURES

Definition

An *ω-structure* is a tuple $(\omega, \mathcal{S}, <, +, \times)$ where $\mathcal{S}$ is a collection of reals.

An ω-structure is characterized by its second order part $\mathcal{S}$.

ω-MODELS OF RCA_0

Definition

A *Turing ideal* if a collection $\mathcal{S}$ such that

1. If $X \in \mathcal{S}$ and $Y \leq_T X$ then $Y \in \mathcal{S}$
2. If $X, Y \in \mathcal{S}$ then $X \oplus Y \in \mathcal{S}$

Theorem (Friedman 1975)

An ω-structure is a model of RCA_0 iff its second order part is a Turing ideal.

RCA_0

There is a minimal ω-model of RCA_0 with second order part

$$\mathcal{S} = \{X : X \text{ is computable }\}$$

RCA_0 captures “computational mathematics”.

SHAPE OF OUR STATEMENTS

Most of principles studied in reverse mathematics are of the form

$$(\forall X)(\exists Y)\Phi(X, Y)$$

where Φ is an arithmetical formula.

Think about $(\forall X)(\exists Y)\Phi(X, Y)$ as a problem.

- The set X is called an *instance*.
- Every Y such that $\Phi(X, Y)$ holds is called a *solution* (of X).

BUILDING ω-MODELS

Consider the statement RT^n_k:
Every function $f : [\omega]^n \to k$ has an infinite f-homogeneous set H (i.e. $|f([H]^n)| = 1$).

You want to build an ω-model of $\mathsf{RCA}_0 + \mathsf{RT}^n_k$.

$$\equiv$$

You want to build a Turing ideal $\mathcal{S}$ such that if $f \in \mathcal{S}$ is a code for a function $[\omega]^n \to k$, there exists $H \in \mathcal{S}$ which is an infinite f-homogeneous set.

BUILDING ω-MODELS

1. Start with $\mathcal{S}_0 = \{X : X \text{ is computable from } \emptyset\}$
2. At stage i, $\mathcal{S}_i = \{X : X \text{ is computable from } Z_i\}$.
   Take the ith infinite function $f \in \mathcal{S}_i$.
   Choose an infinite f-homogeneous set H and set
   $\mathcal{S}_{i+1} = \{X : X \text{ is computable from } Z_i \oplus H\}$.
3. Iterate step 2.

The ω-structure with second order part $\bigcup_i \mathcal{S}_i$ is model of $\mathsf{RCA}_0 + \mathsf{RT}^n_k$.

NON-IMPLICATION

Consider the statement

ACA: *Every set has a jump, i.e.* $(\forall X)(\exists Y)[Y = \{e : \Phi_e(e)^X \downarrow\}]$.

You want to show that RT^2_2 does not imply ACA over RCA_0.

You want to build a Turing ideal $\mathcal{S}$ such that

1. if $f \in \mathcal{S}$ is a code for a function $[\omega]^2 \to 2$, there exists $H \in \mathcal{S}$ which is an infinite f-homogeneous set.
2. there exists a set $X \in \mathcal{S}$ such that $X' \notin \mathcal{S}$

NON-IMPLICATION

Suppose you have the following property:

For every $Z \not\geq_T \emptyset'$ and every infinite Z-computable function $f : [\omega]^2 \to 2$, there exists an infinite f-homogeneous set H such that $H \oplus Z \not\geq_T \emptyset'$.

Then you can create a model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ not model of ACA.

BUILDING ω-MODELS

1. Start with $\mathcal{S}_0 = \{X : X \text{ is computable from } \emptyset\}$
2. At stage i, $\mathcal{S}_i = \{X : X \text{ is computable from } Z\}$ with $Z \not\geq_T \emptyset'$.
   Take the ith infinite function $f \in \mathcal{S}_s$.
   Choose an infinite f-homogeneous set H such that
   $Z \oplus H \not\geq_T \emptyset'$ and set
   $\mathcal{S}_{i+1} = \{X : X \text{ is computable from } Z \oplus H\}$.
3. Iterate step 2.

The ω-structure with second order part $\bigcup_i \mathcal{S}_i$ is model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ but $\emptyset' \notin \bigcup_i \mathcal{S}_i$.

Avoidance

- Is RT^2_2 able to *avoid* $\emptyset'$?
- What classes of sets can a principle avoid ?

We need to define formally the notion of avoidance.

AVOIDANCE

Definition

Fix a principle P.

1. P admits *$\mathcal{C}$-avoidance* for a class of reals $\mathcal{C}$ upward closed (by Turing reducibility) if for every $X \notin \mathcal{C}$, there exists a solution Y of X such that $Y \oplus X \notin \mathcal{C}$.
2. P admits *$\mathcal{C}$-avoidance* for an arbitrary class $\mathcal{C}$ if it admits $\mathcal{D}$-avoidance where $\mathcal{D}$ is the upward-closure of $\mathcal{C}$.

AVOIDANCE

Lemma

If a principle P admits $\{\emptyset'\}$-avoidance then there exists an ω-model of $\mathsf{RCA}_0 + P$ *not model of* ACA.

SUMMARY

CONE AVOIDANCE

Definition

A principle admits *cone avoidance* if it admits $\{A_0, A_1, \dots\}$-avoidance for every countable sequence of non-computable sets $A_0, A_1, \dots$.

In particular, if P has cone avoidance, then $\mathsf{RCA}_0 \nvdash P \to \mathsf{ACA}$.

Cone avoidance

Theorem (Jockusch, 1972)

RT^3_2 *does not admit cone avoidance.*

(In fact, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \leftrightarrow \mathsf{ACA}$ for every $n \geq 3$)

Theorem (Seetapun, 1995)

RT^2_2 *admits cone avoidance.*

AVOIDANCE VS STRONG AVOIDANCE

Avoidance expresses the *effective* weakness of a principle.

What if the instance is not required to be computable ?

AVOIDANCE VS STRONG AVOIDANCE

Definition

Fix a principle P.

1. P admits *strong $\mathcal{C}$-avoidance* for an upward-closed class $\mathcal{C}$ if for every X (in $\mathcal{C}$ or not) and every $Z \notin \mathcal{C}$, there exists a solution Y of X such that $Y \oplus Z \notin \mathcal{C}$.

2. P admits *strong $\mathcal{C}$-avoidance* for an arbitrary class $\mathcal{C}$ if it admits strong $\mathcal{D}$-avoidance where $\mathcal{D}$ is the upward-closure of $\mathcal{C}$.

AVOIDANCE VS STRONG AVOIDANCE

Strong avoidance expresses the *combinatorial* weakness of a principle.

Which Ramseyan principles admit strong cone avoidance ?

STRONG CONE AVOIDANCE

Theorem (Dzhafarov and Jockusch, 2009)
RT^1_2 *admits strong cone avoidance.*

Theorem (Jockusch, 1972)
RT^2_2 *does not admit strong cone avoidance.*

STRONG CONE AVOIDANCE

When slightly relaxing the constraints...

Definition

$\mathsf{ART}^n_{k,d}$: Every function $f : [\omega]^n \to k$ has an infinite set H such that $|f([H]^n)| \leq d$.

Theorem (Wang, 2013)

$\mathsf{ART}^n_{<\infty,d_n}$ admits strong cone avoidance for every $n \geq 1$ and sufficiently large d_n.

In particular $\mathsf{ART}^2_{<\infty,2}$ admits strong cone avoidance.

STRONG CONE AVOIDANCE

Various consequences of Ramsey theorem have been proven to admit strong cone avoidance.

- Free sets, thin sets, rainbow Ramsey theorem (Wang, 2013)
- Erdös Moser (Patey)

Other consequences do not

- Ascending descending sequence (Wang)

WKL and PA degrees

Definition
WKL: Every infinite binary tree has an infinite path.

Theorem (Jockusch and Soare, 1972)
There exists a universal instance of WKL, *i.e. there exists an infinite computable binary tree such that every infinite path computes a path in every infinite computable binary tree.*

WKL and PA degrees

The computable tree whose paths are $\{0,1\}$-valued completions of the partial function $e \mapsto \Phi_e(e)$ is universal.

Definition

A principle P admits (strong) PA avoidance if its admits (strong) $\{X : \Phi_e(e) \downarrow \rightarrow X(e) = \Phi_e(e)\}$-avoidance.

In particular if a principle admits PA avoidance, then $\mathsf{RCA}_0 \nvdash P \rightarrow \mathsf{WKL}$.

PA avoidance

As $\mathsf{RCA}_0 \vdash \mathsf{RT}^3_2 \to \mathsf{ACA} \to \mathsf{WKL}$

- RT^3_2 does not admit PA avoidance.
- RT^2_2 does not admit strong PA avoidance.

Theorem (Liu, 2012)

- RT^2_2 *admits PA avoidance.*
- RT^1_2 *admits strong PA avoidance.*

PA AVOIDANCE

Theorem (Patey)

The principle "For every Π^0_1 class of functions $[\omega]^2 \to 2$, there exists an infinite set homogeneous for one of the functions" admits PA avoidance.

Strong PA avoidance

Still slightly relaxing the constraints...

Theorem (Patey)

$\mathsf{ART}^n_{<\infty,d_n}$ *admits strong PA avoidance for every* $n \geq 1$ *and sufficiently large* d_n*.*

In particular $\mathsf{ART}^2_{<\infty,2}$ admits strong PA avoidance.

STRONG PA AVOIDANCE

Various consequences of Ramsey's theorem
admit strong PA avoidance

- Rainbow Ramsey theorem for pairs (Wang, 2013)
- Free sets, thin sets,
  rainbow Ramsey theorem, Erdös Moser (Patey)

PATH AVOIDANCE

Question

Can RT^2_2 *avoid computing a path in any infinite binary tree with no computable member ?*

....no

PATH AVOIDANCE

Theorem (Patey)

There exists a infinite (non-computable) binary tree with no computable member, together with a computable function $f : [\omega]^2 \to 2$ such that every infinite f-homogeneous set computes an infinite path in the tree.

Also the case for

- stable thin set for pairs
- stable ascending descending sequence
- rainbow Ramsey theorem for triples

CONCLUSION

- RT^2_2 and ascending descending sequence are effectively weak but not combinatorially weak.
- Free sets, thin sets, Erdös moser and rainbow Ramsey theorem are combinatorially weak.
- Many Ramseyan principles have the ability to compute paths in binary trees with no computable paths.

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QUESTIONS

Thank you for listening !