The complexity of satisfaction problems in reverse mathematics

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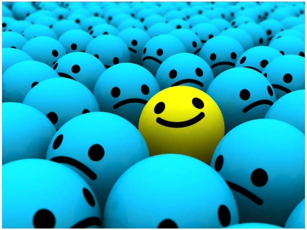

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Summary

SOME TERMINOLOGY

Definition

1. A *literal* is either x (positive literal) or $\neg x$ (negative literal) for a variable x.
2. A *k-clause* is a disjunction of k literals.
3. A finite set of clauses $\{\varphi_1, \dots, \varphi_n\}$ over a set of variables V is *satisfiable* if there exists an assignment $\rho : V \to \{\mathtt{T}, \mathtt{F}\}$ such that each clause is true under the assignment.

Satisfiability in complexity theory

Definition

ISAT : Given a *finite* set of clauses, is it satisfiable ?

Theorem (Cook (1971), Levin (1973))

ISAT *is NP-complete*

SATISFIABILITY IN COMPLEXITY THEORY

Definition

1. A *Boolean relation* R is a subset of $\{\mathtt{T}, \mathtt{F}\}^n$
2. Given a set S of Boolean relations, an S-formula is a formula $R(x_0, \dots, x_n)$ for some $R \in S$.

Definition

Fix a finite set S of Boolean relations. $\mathsf{ISAT}(\mathsf{S})$: Given a finite set of S-formulas, is it satisfiable ?

SATISFIABILITY IN COMPLEXITY THEORY

Definition

A clause is

1. *bijunctive* if it contains at most 2 literals
2. *horn* if it contains at most one positive literal
3. *co-horn* if it contains at most one negative literal

Note:

$$(\neg x_0 \vee \ldots \neg x_n \vee y) \equiv (x_0 \wedge \cdots \wedge x_n \rightarrow y)$$

SATISFIABILITY IN COMPLEXITY THEORY

Definition

A formula φ is

1. *0-valid* if $\varphi(\texttt{F}, \ldots, \texttt{F})$ is true.
2. *1-valid* if $\varphi(\texttt{T}, \ldots, \texttt{T})$ is true.
3. *bijunctive* if it is a conjunction of bijunctive clauses.
4. *horn* if it is a conjunction of horn clauses.
5. *co-horn* if it is a conjunction of co-horn clauses.
6. *affine* if it is a conjunction of formulas of the form $x_1 \oplus \cdots \oplus x_n = i$ for $i \in \{0, 1\}$ where $\oplus$ is the exclusive or.

Satisfiability in complexity theory

Given a formula φ and a canonical ordering of the variables, define $[\varphi]$ to be the corresponding relation, i.e. the set of assignments satisfying it.

Example

$$[(x \vee y) \wedge x] = \{10, 11\}$$

Define in a natural way 0-valid, 1-valid, ... relations

SATISFIABILITY IN COMPLEXITY THEORY

Theorem (Schaefer's dichotomy (1978))

Let S be a finite set of Boolean relations. If S satisfies one of the conditions (a) – (f) below, then ISAT(S) *is in P.*
Otherwise, ISAT(S) *is NP-complete.*

(a) *Every relation in S is 0-valid.*
(b) *Every relation in S is 1-valid.*
(c) *Every relation in S is horn*
(d) *Every relation in S is co-horn*
(e) *Every relation in S is affine.*
(f) *Every relation in S is bijunctive.*

What is reverse mathematics ?

Definition

Reverse mathematics is program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

- Weak system (RCA_0)
- Prove equivalence of theorems and axioms over RCA_0

Applications

- Deeper undestanding
- Search for more elementary proofs

REVERSE MATHEMATICS

- RCA_0 contains
  - basic Peano axioms
  - the comprehension scheme restricted to Δ^0_1 formulas
  - the induction scheme restricted to Σ^0_1 formulas
- RCA_0 captures "computational mathematics".

Reverse mathematics

Observation

Most theorems of “ordinary” mathematics

- live in weak systems.
- are equivalent to axioms over RCA_0

SATISFACTION IN REVERSE MATHEMATICS

Definition

An infinite set C of formulas is *finitely satisfiable* if every finite subset of C has a satisfying assignment.

Definition

SAT : Every finitely satisfiable set C of formulas has a satisfying assignment.

Satisfaction in reverse mathematics

Definition

WKL: Every infinite binary tree has an infinite path.

Theorem (Simpson)

$\mathsf{RCA}_0 \vdash \mathsf{SAT} \leftrightarrow \mathsf{WKL}$

Summary

A few definitions

Definition
Let S be a finite set of relations. $\mathsf{SAT}(\mathsf{S})$: Every finitely satisfiable set C of S-formulas has a satisfying assignment.

Is there a similar dichotomy theorem ?

A few definitions

Definition

A relation R is *i-default* for $i = 0, 1$ if for every assignment $\vec{r} \in R$ and every position $j < |\vec{r}|$, the assignemnt

$$\vec{s}(k) = \begin{cases} i & \text{if } k = j \\ \vec{r}(k) & \text{otherwise} \end{cases}$$

is also in R.

Example: If $01101 \in R$ for a 0-default relation R, then $01100, 01001, 00101, 01000, 00100, 00001, 00000 \in R$.

In particular every i-default relation is i-valid.

A FIRST DICHOTOMY THEOREM

Theorem

If S satisfies one of the conditions $(a) - (d)$ below, then $\mathsf{SAT}(\mathsf{S})$ is provable in RCA_0. Otherwise $\mathsf{SAT}(\mathsf{S})$ is equivalent to WKL_0 over RCA_0.

(a) *Every relation in S is 0-valid.*

(b) *Every relation in S is 1-valid.*

(c) *If $R \in S$ is not 0-default then $R = [x](= \{1\})$.*

(d) *If $R \in S$ is not 1-default then $R = [\neg x](= \{0\})$.*

RAMSEY-TYPE VERSION OF SATISFACTION

Question

What if we only ask for an assignment of infinitely many variables ?

Summary

Ramsey-type version of satisfaction

Definition

A set $H \subseteq \mathbb{N} \times \{\mathtt{T}, \mathtt{F}\}$ is *homogeneous* for a set of formulas C if every finite subset of C has a satisfying assignment ν such that $\nu(i) = t$ for each $(i, t) \in H$.

Definition

1. LRSAT : For every infinite set X and every finitely satisfiable set of formulas, there exists an infinite homogeneous subset of $X \times \{\mathtt{T}, \mathtt{F}\}$.
2. Fix a finite set S of Boolean relations.
   $\mathsf{LRSAT}(\mathsf{S})$: For every infinite set X and every finitely satisfiable set of S-formulas, there exists an infinite homogeneous subset of $X \times \{\mathtt{T}, \mathtt{F}\}$.

A dichotomy theorem

Theorem

Either $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(\mathsf{S})$ *or* $\mathsf{LRSAT}(\mathsf{S})$ *is equivalent to one of the following principles over* RCA_0*:*

1. LRSAT
2. $\mathsf{LRSAT}([\mathsf{x} \neq \mathsf{y}])$
3. $\mathsf{LRSAT}(\mathit{Affine})$
4. $\mathsf{LRSAT}(\mathit{Bijunctive})$

THE PROOF

How to prove dichotomy theorems
on Boolean satisfaction problems ?

... use universal algebra

A FIRST GAP

Theorem

If S satisfies one of (a)-(d) below then $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(\mathsf{S})$.
Otherwise $\mathsf{RCA}_0 \vdash \mathsf{LRSAT}(\mathsf{S}) \to \mathsf{LRSAT}([\mathsf{x} \neq \mathsf{y}])$.

(a) *Every relation in S is 0-valid.*

(b) *Every relation in S is 1-valid.*

(c) *Every relation in S is horn.*

(d) *Every relation in S is co-horn.*

THE CO-CLONES COWARDS

Definition

For any set S of relations, the *co-clone* of S is the closure of S by existential quantification, equality and conjunction. We denote it by $\langle S \rangle$.

Lemma

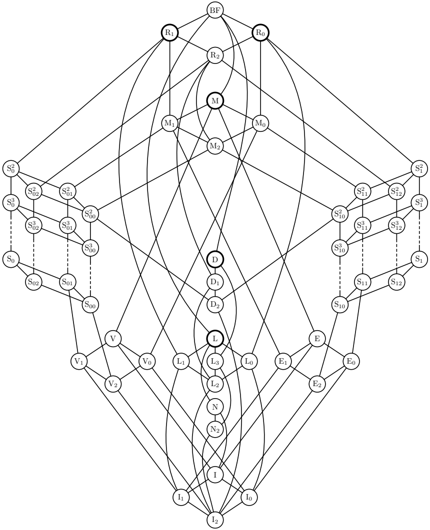
If $\mathsf{RCA_0} \vdash \mathsf{LRSAT(S)} \rightarrow \mathsf{LRSAT}([\mathsf{x} \neq \mathsf{y}])$ *then*
$\mathsf{RCA_0} \vdash \mathsf{LRSAT(S)} \leftrightarrow \mathsf{LRSAT}(\langle \mathsf{S} \rangle)$

CO-CLONES, POLYMORPHISMS AND CLONES

Definition
An m-ary function f is a *polymorphism* of a relation $R \subseteq \{0,1\}^n$ if for every m-tuple $\langle v_1, \ldots, v_m \rangle$ of vectors of R, $\vec{f}(v_1, \ldots, v_m) \in R$ where $\vec{f}$ is the coordinate-wise application of the function f.

Co-clones are characterized by their polymorphisms.

Post's lattice



POST'S LATTICE AND DICHOTOMIES

All remains is making a case analysis over Post's lattice.

CONCLUSION

- Full and Ramsey-type satisfaction principles admit dichotomy theorems
- The dichotomy differs from complexity theory
- It is unknown whether the systems LRSAT, $\mathsf{LRSAT}([x \neq y])$, $\mathsf{LRSAT}(\text{Affine})$ and $\mathsf{LRSAT}(\text{Bijunctive})$ are strictly different over RCA_0.

References

Eric Allender, Michael Bauland, Neil Immerman, Henning Schnoor, and Heribert Vollmer.
The complexity of satisfiability problems: Refining schaefers theorem.
In Mathematical Foundations of Computer Science 2005, pages 71–82. Springer, 2005.

Ludovic Patey.
The complexity of satisfaction problems in reverse mathematics.
In Language, Life, Limits, pages 333–342. Springer, 2014.

Thomas J. Schaefer.
The complexity of satisfiability problems.
In Proceedings of the tenth annual ACM symposium on Theory of computing, pages 216–226, 1978.

QUESTIONS

Thank you for listening !