Reverse Mathematics and a Weak Ramsey-Type König's Lemma

Ludovic Patey
ludovic.patey@computability.fr

Laurent Bienvenu
laurent.bienvenu@computability.fr

LIAFA

September 4, 2012

Summary

1. Introduction
   - Subsystems of $\mathbf{Z_2}$
   - The system $\mathbf{RCA_0}$
   - König's Lemmas
   - Ramsey's Theorems
   - Ramsey-Type Weak König's Lemmas
   - Diagonally Non-Computable functions
2. The system **WRKL**
3. Conclusion

Plan

1. Introduction
   - Subsystems of $\mathbf{Z_2}$
   - The system $\mathbf{RCA_0}$
   - König's Lemmas
   - Ramsey's Theorems
   - Ramsey-Type Weak König's Lemmas
   - Diagonally Non-Computable functions
2. The system **WRKL**
3. Conclusion

What are Reverse Mathematics ?

Definition

Program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

- Weak system ($\mathbf{RCA_0}$)
- Prove equivalence of theorems and axioms over $\mathbf{RCA_0}$
- Lattice of systems

Applications

- Soundness
- Heuristic for new proofs

Observation

Most theorems of "ordinary" mathematics

- live in weak systems.
- are equivalent to axioms over $\mathbf{RCA_0}$

- Refine our structure of weak systems.
- Weaker than Ramsey theorem and König's lemma.

Language of Second Order Arithmetic L_2

Numerical terms

$$t ::= 0 \mid 1 \mid x \mid t_1 + t_2 \mid t_1 \cdot t_2$$

Formulas

$$\begin{aligned} f ::= \quad & t_1 = t_2 \mid t_1 < t_2 \mid t_1 \in X \mid \forall x.f \\ & \mid \exists x.f \mid \forall X.f \mid \exists X.f \mid \neg f \mid f_1 \vee f_2 \end{aligned}$$

Axioms of Second Order Arithmetic $\mathbf{Z_2}$

Basic axioms

$$
\begin{array}{ll}
n+1 \neq 0 & m+1 = n+1 \Rightarrow m = n \\
m+0 = m & m+(n+1) = (m+n)+1 \\
m \cdot 0 = 0 & m \cdot (n+1) = (m \cdot n) + m \\
\neg m < 0 & m < n+1 \Leftrightarrow (m < n \vee m = n)
\end{array}
$$

Induction axiom

$$(0 \in X \wedge \forall n.(n \in X \Rightarrow n+1 \in X)) \Rightarrow \forall n.(n \in X)$$

Comprehension scheme

$$\exists X. \forall n.(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Subsystem of $\mathbf{Z_2}$

Definition (Subsystem of $\mathbf{Z_2}$)

System based of L_2 whose axioms are theorems of $\mathbf{Z_2}$

The system $\mathbf{RCA_0}$

Definition (Σ^0_1, Π^0_1 and Δ^0_1 formulas)

- Σ^0_1 : $\exists n.\phi$
- Π^0_1 : $\forall n.\phi$
- Δ^0_1 : Σ^0_1 and Π^0_1

where ϕ is a L_2-formula containing only bounded quantifiers.

Theorem (Post's theorem)

A set A is computably enumerable (resp. computable) in $B_1, B_2, \dots$ iff it is definable by a Σ^0_1 formula (resp. Δ^0_1 formula) with parameters $B_1, B_2, \dots$.

The system $\mathbf{RCA_0}$

Basic axioms

Σ^0_1 Induction axiom

$$(\varphi(0) \wedge \forall n.(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

Δ^0_1 Comprehension axiom

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X.\forall n.(x \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

König's Lemmas

Definition (Tree)

A set T is a tree iff it is closed under prefixes:

$$\forall \sigma \in T, \tau \prec \sigma \Rightarrow \tau \in T$$

Definition (Path)

P is a path in a tree T iff all prefixes of P are in T.

$$\forall \sigma \in P, \sigma \in T$$

Definition (Measure of a tree)

$$\mu(T) \stackrel{def}{=} \lim_{n \to \infty} \frac{\texttt{card}\,\{\sigma \in T \ : \ |\sigma| = n\}}{2^n}$$

König's Lemmas

König's lemma

Every finitely branching infinite tree has a path.

Definition ($\mathbf{ACA_0}$)

$\mathbf{RCA_0}$ + König's lemma

Definition ($\mathbf{WKL_0}$)

$\mathbf{RCA_0}$ + Every infinite subtree of $2^{<\omega}$ has a path.

Definition ($\mathbf{WWKL_0}$)

$\mathbf{RCA_0}$ + Every subtree of $2^{<\omega}$ of positive measure has a path.

Theorem (Simpson et al.)

$$\mathbf{RCA_0} \subsetneq \mathbf{WWKL_0} \subsetneq \mathbf{WKL_0} \subsetneq \mathbf{ACA_0}$$

Ramsey's Theorems

Notation

$[\mathbb{N}]^n$ is the collection of subsets of ω of size n

Definition (System $\mathbf{RT}$)

$\mathbf{RCA_0}$ + "given n and $k \in \omega$, for every function (called a coloring) $f \in \{0, \dots, k-1\}^{[\mathbb{N}]^n}$, there is an infinite set $H \subseteq \omega$ which is given one color by f".

Definition (System $\mathbf{RT^n_k}$)

Restriction of $\mathbf{RT}$ to a fixed n and k.

Ramsey's Theorems

Theorem (Simpson)

(i) *For each* $n \geq 3$ *and* $k \geq 2$, $\mathbf{RCA_0} \vdash \mathbf{RT^n_k} \Leftrightarrow \mathbf{ACA_0}$.

(ii) $\mathbf{RT}$ *is not provable in* $\mathbf{ACA_0}$.

Theorem

- $\mathbf{RCA_0} \vdash \mathbf{RT^2_1}$
- $\mathbf{RCA_0} \vdash \mathbf{RT^2_2} < \mathbf{ACA_0}$ *(1995)*
- $\mathbf{RT^2_2} \not\Leftarrow \mathbf{WKL_0}$ *(2001)*
- $\mathbf{RT^2_2} \not\Rightarrow \mathbf{WKL_0}$ *(2011)*

Ramsey-Type Weak König's Lemmas

Definition (Homogeneous set)

A set H is *homogeneous for* $\sigma \in 2^{<\omega}$ *with color* $c \in \{0,1\}$ if $\sigma(x) = c$ for each $x \in H$ s.t. $x < |\sigma|$. H is *homogeneous for a path trough* T if $\exists c \in \{0,1\}$ s.t. H is homogeneous for σ with color c for arbitrarily long $c \in T$.

Definition (System RKL)

$\mathbf{RCA_0}$ + "each binary tree T has an infinite set which is homogeneous for a path through T."

Ramsey-Type Weak König's Lemmas

Theorem (Flood)

(i) $\mathbf{RKL} < \mathbf{RT}^2_2$

(ii) $\mathbf{RKL} < \mathbf{WKL_0}$

(iii) $\mathbf{DNC} \leq \mathbf{RKL}$

Definition (System WRKL)

WRKL is obtained from **RKL** by considering only trees of positive measure.

Diagonally Non-Computable functions

What is it ?

Function which gives non-trivial meta-informations about functions.

Advantages

- Uniform framework
- Easier separation between principles
- Better understanding

Diagonally Non-Computable functions

X is a set, f a computable function and e a Turing index.

DNC

$\forall X, \exists f \in \omega^\omega$ such that $\forall e$, $f(e) \neq \Phi_e^X(e)$.

$\mathbf{DNC_k}$

$\forall X, \exists f \in k^\omega$ such that $\forall e$, $f(e) \neq \Phi_e^X(e)$.

$\mathbf{DNC_h}$ (where h is a computable function)

$\forall X$, there exists a h-bounded total function $f \in \omega^\omega$ such that for $\forall e$, $f(e) \neq \Phi_e^X(e)$.

FPF

$\forall X, \exists f \in \omega^\omega$ such that $\forall e$, $\Phi_{f(e)}^X \neq \Phi_e^X$.

Diagonally Non-Computable functions

Theorem (Jockusch, Lerman, Soare & Solovay)

$\mathbf{RCA_0} \vdash \mathbf{DNC} = \mathbf{FPF}$

Theorem (Jockusch)

For all $k \geq 2$ and $f \in \mathbf{DNC_{k+1}}$, there exists a functionnal Γ such that $\Gamma^f \in \mathbf{DNC_k}$. However the reduction is not uniform.

Theorem (Jockusch)

$\mathbf{RCA_0} \vdash \mathbf{WKL_0} = \mathbf{DNC_2}$

Plan

1. Introduction
   - Subsystems of $\mathbf{Z_2}$
   - The system $\mathbf{RCA_0}$
   - König's Lemmas
   - Ramsey's Theorems
   - Ramsey-Type Weak König's Lemmas
   - Diagonally Non-Computable functions
2. The system **WRKL**
3. Conclusion

The system WRKL

Theorem

$\mathbf{RCA_0} \vdash \mathbf{DNC} = \mathbf{WRKL}$

Intuition

- $\mathbf{WWKL_0} \Leftrightarrow$ existence of a Martin-Löf Random
- $\mathbf{WRKL} \Leftrightarrow$ existence of an infinite subset of a Martin-Löf Random

Theorem (Kjos-Hanssen, Greenberg & Miller)

The following are equivalent:

(i) *A computes a DNC function.*

(ii) *A computes an infinite subset of a Martin Löf random.*

The system **WRKL**

- Flood proved **DNC** $\leq$ **WRKL**.
- We proved that **DNC** $\geq$ **WRKL**.

Lemma

Let S be a c.e. set of cardinality at most n. Using a DNC function we can uniformly compute a value outside S.

Lemma

There are computable functions g and $h \in \omega^\omega$ such that for each binary tree T of measure $\mu(T) > 2^{-m}$,

$$\texttt{card}\left\{n \in \omega \ : \ \mu(T \cap \Gamma_n^0) \leq 2^{-g(m)}\right\} < h(m)$$

Plan

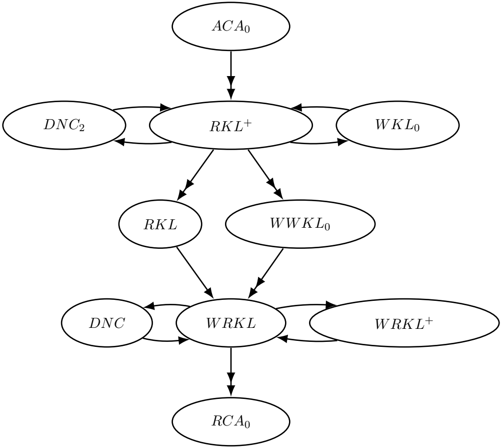
Summary

Tree	fin. branch.	bounded	h-bounded
Path			
Hom. set h-bounded	$\mathbf{ACA_0}$	$\mathbf{WKL_0}$	$\mathbf{WKL_0}$
Hom. set with color 0			
Hom. set			

Tree	2-bounded	pos. meas.
Path		$\mathbf{WWKL_0}$
Hom. set h-bounded	$\mathbf{WKL_0}$	$\mathbf{WRKL_h}$
Hom. set with color 0		**DNC**
Hom. set	**RKL**	

Table: Paths and homogeneous sets existence for classes of trees

Summary



Further research

Separation questions

- $\mathbf{DNC} \neq \mathbf{RKL}$?
- $\mathbf{WWKL_0} \neq \mathbf{RKL}$?

Characterization questions

- **RKL**
- $\mathbf{WRKL_h}$

More natural proof

- $\mathbf{RKL} \neq \mathbf{WKL_0}$

References

Stephen Flood.
Reverse mathematics and a ramsey-type könig's lemma.
2011.

S.G. Simpson, Association for Symbolic Logic, and Inc ebrary.
Subsystems of second order arithmetic, volume 42.
Springer Berlin, 1999.

Questions

Thank you for listening !