Reverse Mathematics and a Weak Ramsey-Type König’s Lemma

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Summary

1 Introduction
   - Subsystems of $\mathbb{Z}_2$
   - The system $\text{RCA}_0$
   - König’s Lemmas
   - Ramsey’s Theorems
   - Ramsey-Type Weak König’s Lemmas
   - Diagonally Non-Computable functions

2 The system $\text{WRKL}$

3 Conclusion
Plan

1. Introduction
   - Subsystems of $\mathbb{Z}_2$
   - The system $\text{RCA}_0$
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   - Ramsey’s Theorems
   - Ramsey-Type Weak König’s Lemmas
   - Diagonally Non-Computable functions

2. The system $\text{WRKL}$

3. Conclusion
What are Reverse Mathematics?

**Definition**
Program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

- Weak system ($\text{RCA}_0$)
- Prove equivalence of theorems and axioms over $\text{RCA}_0$
- Lattice of systems

**Applications**
- Soundness
- Heuristic for new proofs
Observation

Most theorems of ”ordinary” mathematics

- live in weak systems.
- are equivalent to axioms over $\text{RCA}_0$

- Refine our structure of weak systems.
- Weaker than Ramsey theorem and König’s lemma.
Language of Second Order Arithmetic $L_2$

Numerical terms

$t ::= 0 | 1 | x | t_1 + t_2 | t_1 \cdot t_2$

Formulas

$f ::= t_1 = t_2 | t_1 < t_2 | t_1 \in X | \forall x. f \\
| \exists x. f | \forall X. f | \exists X. f | \neg f | f_1 \lor f_2$
Axioms of Second Order Arithmetic $\mathbb{Z}_2$

**Basic axioms**

- $n + 1 \neq 0$
- $m + 0 = m$
- $m \cdot 0 = 0$
- $\neg m < 0$
- $m + 1 = n + 1 \Rightarrow m = n$
- $m + (n + 1) = (m + n) + 1$
- $m \cdot (n + 1) = (m \cdot n) + m$
- $m < n + 1 \Leftrightarrow (m < n \lor m = n)$

**Induction axiom**

$$(0 \in X \land \forall n.(n \in X \Rightarrow n + 1 \in X)) \Rightarrow \forall n.(n \in X)$$

**Comprehension scheme**

$$\exists X. \forall n.(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of $L_2$ in which $X$ does not occur freely.
Subsystem of $\mathbb{Z}_2$

Definition (Subsystem of $\mathbb{Z}_2$)
System based of $L_2$ whose axioms are theorems of $\mathbb{Z}_2$
Definition (\(\Sigma^0_1\), \(\Pi^0_1\) and \(\Delta^0_1\) formulas)

- \(\Sigma^0_1\) : \(\exists n. \phi\)
- \(\Pi^0_1\) : \(\forall n. \phi\)
- \(\Delta^0_1\) : \(\Sigma^0_1\) and \(\Pi^0_1\)

where \(\phi\) is a \(L_2\)-formula containing only bounded quantifiers.

Theorem (Post’s theorem)

A set \(A\) is computably enumerable (resp. computable) in \(B_1, B_2, \ldots\) iff it is definable by a \(\Sigma^0_1\) formula (resp. \(\Delta^0_1\) formula) with parameters \(B_1, B_2, \ldots\).
The system \( \mathbf{RCA}_0 \)

**Basic axioms**

**\( \Sigma^0_1 \) Induction axiom**

\[
(\varphi(0) \land \forall n. (\varphi(n) \Rightarrow \varphi(n + 1))) \Rightarrow \forall n. \varphi(n)
\]

where \( \varphi(n) \) is any \( \Sigma^0_1 \) formula of \( L_2 \)

**\( \Delta^0_1 \) Comprehension axiom**

\[
\forall n (\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X. \forall n. (x \in X \Leftrightarrow \varphi(n))
\]

where \( \varphi(n) \) is any \( \Sigma^0_1 \) formula of \( L_2 \) in which \( X \) does not occur freely and \( \psi(n) \) is any \( \Pi^0_1 \) formula of \( L_2 \).
**Definition (Tree)**

A set $T$ is a tree iff it is closed under prefixes:

$$\forall \sigma \in T, \tau < \sigma \Rightarrow \tau \in T$$

**Definition (Path)**

$P$ is a path in a tree $T$ iff all prefixes of $P$ are in $T$.

$$\forall \sigma \in P, \sigma \in T$$

**Definition (Measure of a tree)**

$$\mu(T) \overset{def}{=} \lim_{n \to \infty} \frac{\text{card} \{ \sigma \in T : |\sigma| = n \}}{2^n}$$
König’s Lemmas

König’s lemma
Every finitely branching infinite tree has a path.

Definition ($\text{ACA}_0$)
$\text{RCA}_0 +$ König’s lemma

Definition ($\text{WKL}_0$)
$\text{RCA}_0 +$ Every infinite subtree of $2^{<\omega}$ has a path.

Definition ($\text{WWKL}_0$)
$\text{RCA}_0 +$ Every subtree of $2^{<\omega}$ of positive measure has a path.
König’s Lemmas

Theorem (Simpson et al.)

\( \text{RCA}_0 \subsetneq \text{WWKL}_0 \subsetneq \text{WKL}_0 \subsetneq \text{ACA}_0 \)
Notation

\([\mathbb{N}]^n\) is the collection of subsets of \(\omega\) of size \(n\)

Definition (System \(\text{RT}\))

\(\text{RCA}_0 + \) “given \(n\) and \(k \in \omega\), for every function (called a coloring) \(f \in \{0, \ldots, k - 1\}^{[\mathbb{N}]^n}\), there is an infinite set \(H \subseteq \omega\) which is given one color by \(f\)”.

Definition (System \(\text{RT}^n_k\))

Restriction of \(\text{RT}\) to a fixed \(n\) and \(k\).
Theorem (Simpson)

(i) For each \( n \geq 3 \) and \( k \geq 2 \), \( \text{RCA}_0 \vdash \text{RT}^n_k \iff \text{ACA}_0 \).

(ii) \( \text{RT} \) is not provable in \( \text{ACA}_0 \).

Theorem

- \( \text{RCA}_0 \vdash \text{RT}^2_1 \)
- \( \text{RCA}_0 \vdash \text{RT}^2_2 \subset \text{ACA}_0 \) \( (1995) \)
- \( \text{RT}^2_2 \not\subseteq \text{WKL}_0 \) \( (2001) \)
- \( \text{RT}^2_2 \not\supset \text{WKL}_0 \) \( (2011) \)
Definition (Homogeneous set)

A set $H$ is *homogeneous for $\sigma \in 2^{<\omega}$ with color $c \in \{0, 1\}$* if $\sigma(x) = c$ for each $x \in H$ s.t. $x < |\sigma|$. $H$ is *homogeneous for a path trough $T$* if $\exists c \in \{0, 1\}$ s.t. $H$ is homogeneous for $\sigma$ with color $c$ for arbitrarily long $c \in T$.

Definition (System RKL)

$\text{RCA}_0 + \text{“each binary tree } T \text{ has an infinite set which is homogeneous for a path through } T.\text{”}$
Theorem (Flood)

(i) $\text{RKL} < \text{RT}_2^2$
(ii) $\text{RKL} < \text{WKL}_0$
(iii) $\text{DNC} \leq \text{RKL}$

Definition (System \text{WRKL})

\text{WRKL} is obtained from \text{RKL} by considering only trees of positive measure.
Diagonally Non-Computable functions

What is it?
Function which gives non-trivial meta-informations about functions.

Advantages
- Uniform framework
- Easier separation between principles
- Better understanding
Diagonally Non-Computable functions

$X$ is a set, $f$ a computable function and $e$ a Turing index.

**DNC**

$\forall X, \exists f \in \omega^\omega$ such that $\forall e, f(e) \neq \Phi^X_e(e)$.

**DNC$_k$**

$\forall X, \exists f \in k^\omega$ such that $\forall e, f(e) \neq \Phi^X_e(e)$.

**DNC$_h$ (where $h$ is a computable function)**

$\forall X$, there exists a $h$-bounded total function $f \in \omega^\omega$ such that for $\forall e$, $f(e) \neq \Phi^X_e(e)$.

**FPF**

$\forall X, \exists f \in \omega^\omega$ such that $\forall e, \Phi^X_{f(e)} \neq \Phi^X_e$. 
Diagonally Non-Computable functions

**Theorem (Jockusch, Lerman, Soare & Solovay)**

\[
\text{RCA}_0 \vdash \text{DNC} = \text{FPF}
\]

**Theorem (Jockusch)**

For all \( k \geq 2 \) and \( f \in \text{DNC}_{k+1} \), there exists a functionnal \( \Gamma \) such that \( \Gamma^f \in \text{DNC}_k \). However the reduction is not uniform.

**Theorem (Jockusch)**

\[
\text{RCA}_0 \vdash \text{WKL}_0 = \text{DNC}_2
\]
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The system **WRKL**

**Theorem**

\[ \text{RCA}_0 \vdash \text{DNC} = \text{WRKL} \]

**Intuition**

- **WWKL}_0 \iff \text{existence of a Martin-Löf Random}
- **WRKL \iff \text{existence of an infinite subset of a Martin-Löf Random}

**Theorem (Kjos-Hanssen, Greenberg & Miller)**

The following are equivalent:

(i) \(A\) computes a DNC function.

(ii) \(A\) computes an infinite subset of a Martin L"of random.
The system WRKL

- Flood proved $\text{DNC} \leq \text{WRKL}$.
- We proved that $\text{DNC} \geq \text{WRKL}$.

Lemma

Let $S$ be a c.e. set of cardinality at most $n$. Using a DNC function we can uniformly compute a value outside $S$.

Lemma

There are computable functions $g$ and $h \in \omega^\omega$ such that for each binary tree $T$ of measure $\mu(T) > 2^{-m}$,

$$\text{card} \left\{ n \in \omega : \mu(T \cap \Gamma^0_n) \leq 2^{-g(m)} \right\} < h(m)$$
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### Summary

<table>
<thead>
<tr>
<th>Tree</th>
<th>fin. branch.</th>
<th>bounded</th>
<th>$h$-bounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hom. set $h$-bounded</td>
<td>$\text{ACA}_0$</td>
<td>$\text{WKL}_0$</td>
<td>$\text{WKL}_0$</td>
</tr>
<tr>
<td>Hom. set with color 0</td>
<td>$\text{WRKL}_h$</td>
<td>$\text{DNC}$</td>
<td>$\text{WRKL}_h$</td>
</tr>
<tr>
<td>Hom. set</td>
<td>$\text{RKL}$</td>
<td>$\text{DNC}$</td>
<td>$\text{DNC}$</td>
</tr>
</tbody>
</table>

**Table:** Paths and homogeneous sets existence for classes of trees
Figure: Summary of classes considered

- $ACA_0$
- $DNC_2$
- $RKL$
- $WWKL_0$
- $WRKL$
- $WRKL^+$
- $RCA_0$
- $WKL_0$
Further research

Separation questions
- DNC \( \neq \) RKL ?
- WWKL\(_0\) \( \neq \) RKL ?

Characterization questions
- RKL
- WRKL\(_h\)

More natural proof
- RKL \( \neq \) WKŁ\(_0\)
References

Stephen Flood. Reverse mathematics and a ramsey-type könig’s lemma. 2011.

Questions

Thank you for listening!