On universal instances of principles in reverse mathematics

Ludovic PATEY
PPS, Paris 7

July 04, 2014

SUMMARY

Shape of our theorems

Consider "ordinary" theorems

- (König's lemma) *Every* infinite, finitely branching tree *has* an infinite path.
- (Ramsey's theorem) *Every* coloring of tuples into finitely many colors *has* an infinite monochromatic subset.
- (Atomic model theorem) *Every* complete atomic theory *has* an atomic model.
- ...

OBSERVATION

Many theorems are of the form

$$(\forall X)(\exists Y)\Phi(X, Y)$$

where Φ is an arithmetical formula.

SHAPE OF OUR THEOREMS

Theorems usually come with a natural class of *instances*.

- In König's lemma, the infinite, finitely branching trees
- In Ramsey's theorem, the colorings of tuples into finitely many colors
- In AMT, the complete atomic theories

Given an instance X, a Y such that $\Phi(X, Y)$ holds is called a *solution* (of X).

EFFECTIVENESS

- Theorems are not all effective.
- Some theorems have computable instances with no computable solution.

Theorem (Kreisel)

There exists an infinite computable binary tree with no infinite computable path.

Effectiveness

Question

Are there instances that are harder to solve than any other instance?

We need to give a precise definition of “harder”.

EFFECTIVENESS

Definition
A instance I is harder than another instance J if every solution to I computes a solution to J.

A computable instance that is harder than every computable instance is called a *universal instance*.

UNIVERSAL INSTANCE

Which principles admit a universal instance ?

SUMMARY

Definitions

Definition (Tree)

- *A tree is a subset of* $\mathbb{N}^{<\mathbb{N}}$ *downward closed under* $\preceq$.
- *A tree* T *is finitely branching if for every* $\sigma \in T$, *there are finitely many* n *such that* $\sigma n \in T$.
- *A tree is binary if it is a subset of* $2^{<\mathbb{N}}$.

Definition (Path)

A path through a tree T *is a set* $X \in \mathbb{N}^{\mathbb{N}}$ *such that* $X \upharpoonright n \in T$ *for each* n. $[T]$ *is the collection of paths of* T.

König's lemma

Definition (König's lemma)

Every infinite, finitely branching tree has a path.

Definition (Weak König's lemma)

Every infinite binary tree has a path.

Weak König's lemma

Theorem (Solovay)

Weak König's lemma admits a universal instance.

Definition

A function f is d.n.c. relative to X if $(\forall e)[f(e) \neq \Phi_e^X(e)]$.

Proof.

- For every infinite, computable, binary tree T, every $\{0,1\}$-valued d.n.c. function computes a path through T.
- There exists a computable binary tree whose paths are exactly the $\{0,1\}$-valued d.n.c. functions.

□

König's lemma

Theorem (Jockusch & al.)

König's lemma admits a universal instance.

Proof.

- For every infinite, computable, finitely branching tree T, there exists an infinite $\emptyset'$-computable binary tree U whose paths have the same degrees as the degrees of the paths through T.
- Relativize previous theorem.

□

WEAK WEAK KÖNIG'S LEMMA

Definition

A binary tree T has positive measure if

$$\lim_n \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n} > 0$$

Definition (Weak weak König's lemma)

Every binary tree of positive measure has a path.

Weak Weak König's lemma

Theorem (Kucera)

Weak weak König's lemma admits a universal instance.

Definition

A Martin-Löf random is a set X such that $K(X \upharpoonright n) \geq n - c$ for some constant c, where K is prefix-free Kolmogorov complexity.

Proof.

- For every computable binary tree T of positive measure, every Martin-Löf random is, up to prefix, a path through T.
- There exists a computable binary tree of positive measure whose paths are all Martin-Löf randoms.

□

Cohesiveness

Definition

Given a sequence of sets $R_0, R_1, \ldots$, a set C is $\vec{R}$-cohesive if $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$ for each $i \in \mathbb{N}$.

Definition (Cohesiveness)

Every countable sequence of sets $\vec{R}$ admits an $\vec{R}$-cohesive set.

COHESIVENESS

Theorem (Jockusch & Stephan)

Cohesiveness admits a universal instance.

Proof.

- For every uniformly computable sequence of sets $\vec{R}$, every set P whose jump is of PA degree relative to $\emptyset'$ computes an $\vec{R}$-cohesive set.
- There exists a uniformly computable sequence of sets $\vec{R}$ such that the jump of every $\vec{R}$-cohesive set is of PA degree relative to $\emptyset'$.

□

RAINBOW RAMSEY THEOREM FOR PAIRS

Definition

A coloring function $f : [\mathbb{N}]^n \to \mathbb{N}$ is k-bounded if for each color i, $|f^{-1}(i)| \leq k$. An infinite set H is a rainbow for f if f is injective over $[H]^n$.

Definition (Rainbow Ramsey theorem for pairs)

Every 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ has a rainbow.

RAINBOW RAMSEY THEOREM FOR PAIRS

Theorem (J.S. Miller)

The rainbow Ramsey theorem for pairs admits a universal instance.

Proof.

- For every computable 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$, every function d.n.c. relative to $\emptyset'$ computes a rainbow for f.
- There exists a computable 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ such that every rainbow for f computes a function d.n.c. relative to $\emptyset'$.

□

OTHER PRINCIPLES

There exist a few other principles admitting a universal instance.

- Finite Intersection Property (Downey & al.)
- Ramsey-type weak weak König's lemma (Bienvenu & al.)

SUMMARY

A SIMPLE METHOD

Fix a principle P.

- Prove that every computable instance of P has a solution satisfying some property (e.g. Δ^0_2, low, ...)
- Prove that for every set X satisfying this property, there exists a computable instance I of P such that X does not compute a solution for I.
- Then P does not admit a universal instance.

GENERAL METHOD

Definition (Computable reducibility)

A principle P is computably reducible to Q ($P \leq_c Q$) if for every instance I of P, there exists an I-computable instance J of Q such that for every solution X of J, $X \oplus I$ computes a solution to I.

Many proofs of implications between principles in reverse mathematics is in fact a computable reduction.

Computable reducibility $\simeq$ Non-uniform Weihrauch reducibility

GENERAL METHOD

Fix two principles P and Q.

- Prove that every computable instance of P has a solution satisfying some property.
- Prove that for every set X satisfying this property, there exists a computable instance I of Q such that X does not compute a solution for I.
- Then no principle R such that $Q \leq_c R \leq_c P$ admits a universal instance.

Ascending Descending sequence

Definition (Ascending descending sequence)

Every infinite linear order has an infinite ascending or descending sequence.

Definition (Stable ascending descending sequence)

Every linear order of order type $\omega + \omega^*$ *has an infinite ascending or descending sequence.*

Ascending descending sequence

Theorem (Hirschfeldt & al.)

Fix a principle P such that SADS $\leq_c$ P. If every computable instance of P admits a low solution, then P admits no universal instance.

Proof.

For every low set X, there exists a computable linear order of order type $\omega + \omega^*$ having no X-computable infinite ascending or descending sequence. □

Corollary

SADS and SCAC (every stable partial order has an infinite chain or antichain) admit no universal instance.

Definition (Atomic model theorem)

Every complete atomic theory has an atomic model.

Theorem (Conidis & al.)

The following statements a computably equivalent:

- *The atomic model theorem*
- *For every Δ^0_2 function f, there exists a function g which is not dominated by f.*

$AMT \simeq$ non-uniform hyperimmunity relative to $\emptyset'$.

Theorem (Martin)

Fix a principle P such that AMT $\leq_c$ P. If every computable instance of P admits a Δ^0_2 low$_2$ solution, then P admits no universal instance.

Proof.

For any Δ^0_2 set X, a function is high relative to X iff it computes a function dominating every X-computable function. □

Corollary

AMT, but also SADS and SCAC admit no universal instance.

Definition

Given a coloring $f : [\mathbb{N}]^n \to k$, a set H is homogeneous for f if there exists a color $i < k$ such that $f([H]^n) = k$.

Definition (Ramsey theorem for tuples)

Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite homogeneous set.

We write RT^n_k to denote Ramsey's theorem restricted to colorings over n-tuples with k colors and SRT^n_k to denote the restriction of RT^n_k to stable colorings.

Theorem (Mileti)

Fix a principle P such that $SRT^n_2 \leq_c P$. *If every computable instance of P admits a* low_2 *over* $\emptyset^{(n-2)}$ *solution then P admits no universal instance.*

Proof.

By a finite injury priority construction. □

Corollary

For every $n \geq 2$, RT^n_2 *and* SRT^n_2 *admit no universal instance.*

Theorem (Patey)

Fix a principle P such that SADS $\leq_c$ *P. If every computable instance of P admits a* low_2 *solution then P admits no universal instance.*

Corollary

CAC, SCAC, ADS, SADS admit no universal instance.

Definition

Given a function $f : [\mathbb{N}]^n \to \mathbb{N}$, *an infinite set* H *is thin for* f *if* $f([H]^n) \neq \mathbb{N}$ *(avoids at least one color).*

Definition (Thin set theorem)

Every function $f : [\mathbb{N}]^n \to \mathbb{N}$ *has an infinite set thin for* f.

We write $TS(n)$ to denote thin set theorem restricted to colorings over n-tuples and $STS(n)$ to denote the restriction of $STS(n)$ to stable colorings.

Theorem (Patey)

Fix a principle P such that $STS(n) \leq_c P$. *If every computable instance of P admits a low*$_2$ *over* $\emptyset^{(n-2)}$ *solution then P admits no universal instance.*

Corollary

For every $n \geq 2$, $TS(n)$, $STS(n)$, RT^n_2, SRT^n_2, $FS(n)$ *(Free set) admit no universal instance.*

Stable Ramsey theorem for pairs

Theorem (Mileti)

Fix a principle P such that $SRT^n_2 \leq_c P$. If every computable instance of P admits an incomplete Δ^0_2 solution then P admits no universal instance.

Proof.

By a finite injury priority construction. □

Corollary

SRT^2_2 admits no universal instance.

Stable rainbow Ramsey theorem for pairs

Definition

A 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ is rainbow-stable if for every x, there is a y such that $f(x, s) = f(y, s)$ for cofinitely many s.

Definition (Stable rainbow Ramsey theorem for pairs)

Every rainbow-stable 2-bounded function $f : [\mathbb{N}]^2 \to \mathbb{N}$ admits a rainbow.

$SRRT^2_2$ is computably equivalent to the statement "for every Δ^0_2 function f, there exists a function g such that $f(x) \neq g(x)$ for each x."

Stable rainbow Ramsey theorem for pairs

Theorem (Patey)

Fix a principle P such that $SRRT^n_2 \leq_c P$. If every computable instance of P admits an incomplete Δ^0_2 solution then P admits no universal instance.

Proof.

By a finite injury priority construction. □

Corollary

$SRRT^2_2$, SRT^2_2, $STS(2)$, SEM (stable Erdös Moser theorem) admit no universal instance.

Erdős Moser case

Definition (Erdős Moser theorem)

Every infinite tournament admits an infinite transitive subtournament.

Theorem (Patey)

- *There exists a* low_2 *degree bounding EM.*
- $[\mathsf{STS}(2) \vee \mathsf{COH}] \leq_c \mathsf{EM}$

Question

Does the Erdős Moser theorem admit a universal instance ?

CONCLUSION

- Few Ramseyan principles admit a universal instance.
- Some principles equivalent to the "Big Five" do not admit a universal instance.
- It is currently unknown whether Erdős Moser theorem admits a universal instance.

References

Chris J Conidis.
Classifying model-theoretic properties.
Journal of Symbolic Logic, pages 885–905, 2008.

Barbara F Csima, Denis R Hirschfeldt, Julia F Knight, and Robert I Soare.
Bounding prime models.
Journal of Symbolic Logic, pages 1117–1142, 2004.

Denis R. Hirschfeldt and Richard A. Shore.
Combinatorial principles weaker than Ramsey's theorem for pairs.
Journal of Symbolic Logic, 72(1):171–206, 2007.

Joseph Roy Mileti.
Partition theorems and computability theory.
PhD thesis, 2004.

Ludovic Patey.
Somewhere over the rainbow Ramsey theorem for pairs.
Ongoing project.

QUESTIONS

Thank you for listening !