# Compactness avoidance 5

Compactness arguments form a central tool in mathematics in general and in topology in particular. From a reverse mathematical viewpoint, many ordinary theorems are equivalent to the Heine-Borel compactness theorem. Some other theorems contain weaker compactness arguments, and some are compactness-free. In this chapter, we study various levels of compactness, namely, weak König's lemma (PA degrees), weak weak König's lemma (random degrees), DNC degrees, and a Ramsey-type weak König's lemma. For the three former notions, we develop the tools to prove that some theorems lack compactness.

This chapter pushes further the correspondence between computability-theoretic features of a generic set and the existence of a forcing question with appropriate definability and combinatorial features. In particular, PA and DNC avoidance both result from the existence of a forcing question with the ability to find simultaneous answers to independent questions.



**Prerequisites**: Chapters 2 and 3

## 5.1 PA avoidance

PA degrees are one of the most important notions in computability-theory, both from a conceptual and a technical perspective. In particular, they form a natural Muchnik degree[1] of intermediate strength between $\mathbf{0}$ and $\mathbf{0}'$. In reverse mathematics, the existence of PA degrees is equivalent to the system $\mathsf{WKL}_0$, which informally corresponds to compactness arguments. Many theorems, such as the Heine-Borel compactness theorem, or Gödel's completeness theorem, are equivalent to $\mathsf{WKL}_0$. Thus, the notion of PA avoidance is not only a technical tool to separate a theorem from $\mathsf{WKL}_0$ in reverse mathematics, but it also reflects the lack of compactness in the proof of the theorem, which is an interesting result in its own right.

1: Muchnik degrees are a generalization of Turing degrees. Many natural computational phenomena are better expressed as families of Turing degrees rather than individual degrees.

**Definition 5.1.1.** A problem P admits *PA avoidance*[2] if for every pair of sets $Z$ and $D \leq_T Z$ such that $Z$ is not of PA degree over $D$, every $Z$-computable instance $X$ of P admits a solution $Y$ such that $Y \oplus Z$ is not of PA degree over $D$. ◊

2: Here again, the unrelativized formulation with $Z = D = \emptyset$ is far more natural, but does not behave well with artificial problems.

Recall that a *Scott ideal* is a Turing ideal $\mathcal{M}$ such that for every $X \in \mathcal{M}$, there is a set $Y \in \mathcal{M}$ of PA degree over $X$. Equivalently, a Scott ideal is a Turing ideal such that for every infinite binary tree $T \in \mathcal{M}$, there is an infinite path $P \in [T]$ in $\mathcal{M}$. In reverse mathematics, Turing ideals and Scott ideals are exactly the second-order parts of $\omega$-models of $\mathsf{RCA}_0$ and $\mathsf{WKL}_0$, respectively.

**Exercise 5.1.2.** Let P be a $\Pi^1_2$ problem which admits PA avoidance. Show the existence of an $\omega$-model of $\mathsf{RCA}_0$ + P which does not contain any set of PA degree. ⋆

Let us start with a concrete example of a proof of PA avoidance. As usual, Cohen forcing is the best behaving notion of forcing, as its partial order is computable. In all our proofs of PA avoidance, we shall use $\{0, 1\}$-valued DNC functions. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is diagonally non-computable (DNC) if for every $e \in \mathbb{N}$, $f(e) \neq \Phi_e(e)$. A degree is PA iff it computes a $\{0,1\}$-valued DNC function.

**Theorem 5.1.3**
*For every sufficiently Cohen generic set $G$, $G$ is not of PA degree.*

Proof. It suffices to prove the following lemma, where "$\Phi_e^G$ is not a DNC$_2$ function" is a shorthand for $\exists x \Phi_e^G(x)\uparrow \vee \exists x \Phi_e^G(x)\downarrow = \Phi_x(x)$. We shall assume as usual that every Turing functional is $\{0,1\}$-valued.

**Lemma 5.1.4.** For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^G$ not to be a DNC$_2$ function. ⋆

Proof. Fix a condition $\sigma$. Consider the following set[3]

$$U = \{(x, v) \in \mathbb{N} \times 2 : \exists \tau \geq \sigma \ \Phi_e^\tau(x)\downarrow = v\}$$

3: Notice that this set is the same as in Lemma 3.2.2.

Note that the set $U$ is $\Sigma_1^0$. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. Let $\tau \geq \sigma$ witness $(x, \Phi_x(x)) \in U$, that is, let $\tau \geq \sigma$ be such that $\Phi_e^\tau(x)\downarrow = \Phi_x(x)$. Then $\tau$ forces $\Phi_e^G$ not to be a DNC$_2$ function.
- Case 2: $(x, 0), (x, 1) \notin U$ for some $x \in \mathbb{N}$. We claim that $\sigma$ already forces $\Phi_e^G(x)\uparrow$.[4] Indeed, if for some $Z \in [\sigma]$, $\Phi_e^Z(x)\downarrow$, then by the use property, there is some $\tau \leq Z$ such that $\Phi_e^\tau(x)\downarrow$, and by choosing $\tau$ long enough, it would witness $(x, v) \in U$ for $v = \Phi_e^\tau(x)$, contradiction.
- Case 3: None of Case 1 and Case 2 holds. Then $U$ is a $\Sigma_1^0$ graph of a $\{0,1\}$-valued DNC function. This contradicts the fact that the degree $\mathbf{0}$ is not PA. ■

4: Note that we exploit the assumption that the functionals are $\{0,1\}$-valued to force divergence. Indeed, the contradiction comes from the fact that $v \in \{0,1\}$.

We are now ready to prove Theorem 5.1.3. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $\tau$ forcing $\Phi_e^G$ not to be a DNC$_2$ function. It follows from Lemma 5.1.4 that every $\mathcal{D}_e$ is dense, hence for every $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic set $G$, $G$ is not of PA degree. ■

**Exercise 5.1.5.** Adapt the proof of Theorem 3.2.4 to show that for any set $A$, there exists a set $G$ such that $G' \geq_T A$ and $G$ is not of PA degree. ⋆

On the other hand, one cannot adapt the proof of Theorem 3.2.6 to show that WKL admits PA avoidance. Indeed, the class of $\{0,1\}$-valued DNC functions is $\Pi_1^0$.

**Exercise 5.1.6.** Try to adapt the proof of Theorem 3.2.6 to show that any non-empty $\Pi_1^0$ class admits a member of non-PA degree. Identify the point of failure. ⋆

The main structural difference between the cone avoidance proof of Theorem 3.2.1 and the PA avoidance proof of Theorem 5.1.3 is in Case 2: Assuming the forcing question gives a negative answer independently to $p \mathrel{?\vdash} \Phi_e^G(x)\downarrow = 0$ and $p \mathrel{?\vdash} \Phi_e^G(x)\downarrow = 1$, we use the existence of a single extension (which in the proof of Theorem 5.1.3 is $p$ itself) forcing simultaneously $\neg(\Phi_e^G(x)\downarrow = 0)$ and $\neg(\Phi_e^G(x)\downarrow = 1)$. Assuming the functional is $\{0,1\}$-valued, then the extension forces divergence. This ability to give a single extension witnessing simultaneously two independent negative answers is the core feature of PA avoidance.

**Definition 5.1.7.** Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas $\Gamma$, a forcing question is $\Gamma$-*merging* if for every $p \in \mathbb{P}$ and every pair of $\Gamma$-formulas $\varphi_0(G), \varphi_1(G)$, if $p \operatorname{?}\!\vdash \varphi_0(G)$ and $p \operatorname{?}\!\vdash \varphi_1(G)$ both hold, then there is an extension $q \leq p$ forcing $\varphi_0(G) \wedge \varphi_1(G)$. ◊

Note that a forcing question for $\Sigma^0_n$ formulas induces a forcing question for $\Pi^0_n$ formulas by considering the complement. Thus, by extension, we say that a forcing question for $\Sigma^0_n$ formulas is $\Pi^0_n$-merging if, whenever $p \operatorname{?}\!\nvdash \varphi_0(G)$ and $p \operatorname{?}\!\nvdash \varphi_1(G)$, there is an extension forcing $\neg\varphi_0(G) \wedge \neg\varphi_1(G)$.

**Remark 5.1.8.** In Section 3.3, the forcing questions at the left-most position are $\Sigma^0_1$-merging, and the ones at the right-most position are $\Pi^0_1$-merging. We shall see examples of $\Pi^0_1$ forcing questions at intermediary positions. ⋆

We have the necessary ingredients to prove our abstract theorem on PA avoidance.

**Theorem 5.1.9**
*Let $(\mathbb{P}, \leq)$ be a notion of forcing with a $\Sigma^0_1$-preserving $\Pi^0_1$-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is not of PA degree.*

Proof. It suffices to prove the following lemma:

**Lemma 5.1.10.** For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi^G_e$ not to be a $\mathrm{DNC}_2$ function. ⋆

Proof. Consider the following set

$$U = \{(x, v) \in \mathbb{N} \times 2 : p \operatorname{?}\!\vdash \Phi^G_e(x)\downarrow = v\}$$

Since the forcing question is $\Sigma^0_1$-preserving, the set $U$ is $\Sigma^0_1$. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi^G_e(x)\downarrow = \Phi_x(x)$.
- Case 2: $(x, 0), (x, 1) \notin U$ for some $x \in \mathbb{N}$. Since the forcing question is $\Pi^0_1$-merging, there is an extension $q \leq p$ forcing $\neg(\Phi^G_e(x)\downarrow = 0) \wedge \neg(\Phi^G_e(x)\downarrow = 1)$, hence forcing $\Phi^G_e$ not to be a $\mathrm{DNC}_2$ function.
- Case 3: None of Case 1 and Case 2 holds. Then $U$ is a $\Sigma^0_1$ graph of a $\{0, 1\}$-valued DNC function. This contradicts the fact that **0** is not PA. ■

We are now ready to prove Theorem 5.1.9. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi^G_e$ not to be a $\mathrm{DNC}_2$ function. It follows from Lemma 5.1.10 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathcal{F}}$ is not of PA degree. This completes the proof of Theorem 5.1.9. ■

## 5.2 Weak merging

In some cases, such as with disjunctive notions of forcing with $\Sigma^0_1$-preserving disjunctive forcing questions, the forcing question is not $\Pi^0_1$-merging simply because given a pair of $\Pi^0_1$ formulas $\varphi_0(G)$ and $\varphi_1(G)$ the extension might force $\varphi_0(G_0)$ on the left side, and $\varphi_1(G_1)$ on the right side. If however one considers three $\Pi^0_1$ formulas, by the pigeonhole principle, two of them must be forced on the same side. We will later consider tree-like notions of forcing whose number of disjunctive clauses might increase over extension, thus requiring a larger number of formulas to find an extension forcing two of them simultaneously. This motivates the following definition.

**Definition 5.2.1.** Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas $\Gamma$, a forcing question is *weakly* $\Gamma$*-merging*[5] if for every $p \in \mathbb{P}$, there is some $k \in \mathbb{N}$ such that for every $k$-tuple of $\Gamma$-formulas $\varphi_0(G), \dots, \varphi_{k-1}(G)$, if $p \mathrel{?{\vdash}} \varphi_i(G)$ for each $i < k$, then there is an extension $q \leq p$ and two indices $i < j < k$ such that $q$ forces $\varphi_i(G) \wedge \varphi_j(G)$. ◇

5: Note that in the definition of a weakly $\Gamma$-merging forcing question, the parameter $k$ might depend on the condition $p$.

The following exercise shows that the forcing question of the Dzhafarov-Jockusch theorem is weakly $\Pi^0_1$-merging, with the appropriate adaptation to disjunctive forcing notions.

**Exercise 5.2.2.** Consider the question of forcing of Exercise 3.4.9. Let $\{\varphi^j_0(G), \varphi^j_1(G) : j < 3\}$ be a family of $\Sigma^0_1$ formulas. Show that if for each $j < 3$, $p \mathrel{?{\nvdash}} \varphi^j_0(G_0) \vee \varphi^j_1(G_1)$, then there is an extension $q \leq p$, a side $i < 2$ and two indices $a < b < 3$ such that $q$ forces $\neg\varphi^a_i(G_i) \wedge \neg\varphi^b_i(G_i)$. ★

As for every avoidance or preservation notion, the key diagonalization lemma is based on a 3-case analysis. The first case says that the Turing functional outputs some erroneous description of an object, while the second case ensures that the Turing functional is partial. The two first cases are not mutually exclusive. The third case, which consists of the negation of Case 1 and Case 2, cannot happen, because otherwise, there will be an effective description of some uncomputable object. For cone avoidance, preservation of 1 hyperimmunity, or preservation of 1 non-$\Sigma^0_1$ definition, the third case was trivial. Working with weakly merging forcing questions yields the first non-trivial case analysis. Let us first introduce some terminology.

A *valuation*[6] is a partial $\{0,1\}$-valued function $h \subseteq \mathbb{N} \to 2$. A valuation is finite if it has finite support, that is, $\operatorname{dom} h$ is finite. A valuation $h$ is *correct* if for every $n \in \operatorname{dom} h$, $\Phi_n(n)\downarrow \neq h(n)$. Two valuations $f$ and $h$ are *compatible* if for every $n \in \operatorname{dom} f \cap \operatorname{dom} h$, $f(n) = h(n)$.

6: The idea is the following: We considered so far only valuations with a singleton domain, thus there were at most 2 incompatible such valuations. Considering valuations with finite domain is a way to obtain more pairwise incompatible valuations.

**Lemma 5.2.3 (Liu [10]).** Let $U$ be a c.e. set of finite valuations. Either $U$ contains a correct valuation, or for every $k \in \mathbb{N}$, there are $k$ pairwise incompatible finite valuations outside of $U$. ★

Proof. Suppose $U$ contains no correct valuation, otherwise we are done. Let $S$ be the set of finite sets $F \subseteq \mathbb{N}$ such that for each $n \notin F$, either $\Phi_n(n)\downarrow$, or there is a valuation $h \in U$ such that $F \cup \{n\} \subseteq \operatorname{dom} h$ and for every $m \in \operatorname{dom} h \setminus (F \cup \{n\})$, $\Phi_m(m)\downarrow \neq h(m)$. Note that if $F \notin S$, this is witnessed by some $n \notin F$.

Claim 1: $\emptyset \notin S$. Indeed, otherwise, for each $n \in \mathbb{N}$, one of the two $\Sigma^0_1$ cases holds:

1. $\Phi_n(n)\downarrow$ ;
2. there is a finite valuation $h \in U$ such that $n \in \operatorname{dom} h$ and for every $m \neq n$, $\Phi_m(m)\downarrow \neq h(m)$.

Then one can compute a $\{0,1\}$-valued DNC function by waiting on input $n$ for either case to occur. Then output $1 - \Phi_n(n)$ in the former case, and $1 - h(n)$ in the latter case. Since $U$ contains no correct valuation, $h(n) = \Phi_n(n)$.

Claim 2: For any set $F \notin S$ and $w$ witnessing this fact, $F \cup \{w\} \notin S$. Indeed, otherwise, for each $n \notin F \cup \{w\}$, one of the two $\Sigma^0_1$ cases holds:

1. $\Phi_n(n)\downarrow$ ;
2. there is a finite valuation $h \in U$ such that $F \cup \{w, n\} \subseteq \operatorname{dom} h$ and for every $m \notin F \cup \{w, n\}$, $\Phi_m(m)\downarrow \neq h(m)$.

Here again, one can compute a $\{0,1\}$-valued DNC function by hardcoding the appropriate values on $F \cup \{w\}$, and for any $n \notin F \cup \{w\}$, waiting for either case to occur. In the first case, output $1 - \Phi_m(m)$, and in the second case, output $1 - h(n)$. We cannot have $\Phi_n(n)\downarrow \neq h(n)$, otherwise $h$ would be a counter-example to the fact that $w$ is a witness of $F \notin S$.

Using Claim 1 and Claim 2, one can define for any $k$ an infinite sequence $n_0, n_1, \dots$ such that for any $i \in \mathbb{N}$, $n_i$ witnesses that $\{n_j : j < i\} \notin S$. There are $2^{i+1}$ many pairwise incompatible valuations with domain $\{n_j : j \leq i\}$, and none of them can be in $U$, as it would contradict the fact that $n_i$ is a witness of $\{n_j : j < i\} \notin S$. ■

We can prove the following abstract PA avoidance theorem using Liu's lemma. [10]

**Theorem 5.2.4**
*Let $(\mathbb{P}, \leq)$ be a notion of forcing with a $\Sigma^0_1$-preserving weakly $\Pi^0_1$-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is not of PA degree.*

Proof. It suffices to prove the following diagonalization lemma.

**Lemma 5.2.5.** For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi^G_e$ not to be a $DNC_2$ function. ⋆

Proof. Let $k \in \mathbb{N}$ witness that the forcing question is weakly $\Pi^0_1$-merging for $p$. Consider the following set

$$U = \{h \text{ finite valuation } : p \,?\!\vdash \Phi^G_e \text{ is incompatible with } h\}$$

Note that being incompatible is a $\Sigma^0_1$ statement, so since the forcing question is $\Sigma^0_1$-preserving, the set $U$ is $\Sigma^0_1$. There are three cases:

- Case 1: $U$ contains a correct valuation $h$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi^G_e$ to be incompatible with $h$. In particular, $q$ forces $\Phi^G_e$ not to be a $DNC_2$ function.
- Case 2: there are $k$ pairwise incompatible finite valuations $h_0, \dots, h_{k-1}$ outside of $U$. Since the forcing question is $\Pi^0_1$-merging, there is an extension $q \leq p$ and two indices $a < b < k$ such that $q$ forces $\Phi^G_e$ to be compatible simultaneously with $h_a$ and $h_b$. Since $h_a$ and $h_b$ are incompatible, then $q$ forces $\Phi^G_e$ to be partial.
- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.2.3. ■

We are now ready to prove Theorem 5.2.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi^G_e$ not to be a $DNC_2$ function. It follows from

Lemma 5.2.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathcal{F}}$ is not of PA degree. This completes the proof of Theorem 5.2.4. ■

## 5.3 Ramsey-type WKL

Both the original proof and the modern proof of Seetapun's theorem involve $\Pi^0_1$ classes of instances of $\mathsf{RT}^1_2$, and thus make use of compactness. It is natural to ask whether this use is necessary. Liu's theorem states that Ramsey's theorem for pairs admits PA avoidance. However, PA avoidance only means that full compactness is not needed, but does not rule out the presence of some weak form of compactness. As it turns out, Ramsey's theorem for pairs implies a weak form of compactness called the Ramsey-type weak König's lemma (RWKL). Informally, RWKL states that for every non-empty $\Pi^0_1$ class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists some infinite set $H$ which is homogeneous for one of the members $X \in \mathcal{P}$ seen as an instance of $\mathsf{RT}^1_2$. However, the exact formulation requires more technicality not to imply the existence of $X$.

**Definition 5.3.1.** Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. A finite set $F \subseteq \mathbb{N}$ is *homogeneous* for $T$ if $\{\sigma \in T : F \subseteq \sigma \vee \sigma \subseteq F\}$ is infinite. An infinite set $H \subseteq \mathbb{N}$ is *homogeneous* for $T$ if every finite subset of it is homogeneous for $T$. ◊

By extension, we say that an infinite set $H$ is homogeneous for a $\Pi^0_1$ class $\mathcal{P}$ if it is homogeneous for a tree $T$ such that $\mathcal{P} = [T]$. The Ramsey-type weak König's lemma (RWKL)[7] is the statement "Every infinite binary tree admits an infinite homogeneous set."

7: The statement was originally introduced by Flood [28] under the name Ramsey-type König's lemma (RKL). It was later renamed for consistency.

**Proposition 5.3.2.** $\mathsf{RT}^2_2$ implies RWKL over $\mathsf{RCA}_0$. ★

Proof. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. Define $f : [\mathbb{N}]^2 \to 2$ by $f(x, y) = \sigma_y(x)$, where $\sigma_y$ is the left-most element of $T$ of length $y$. Any infinite homogeneous set for $f$ is homogeneous for $T$. ■

The remainder of this section is devoted to the proof that RWKL admits PA avoidance, hence is strictly weaker than $\mathsf{WKL}_0$.[8]

8: There exists an alternative simpler proof [29] of this theorem which exploits the fact that the class of $\{0, 1\}$-valued DNC functions is $\Pi^0_1$ and not simply closed in Cantor space. The proof given in this book, although more complex, is morally the "true" proof, in that its combinatorics extend to stronger theorems, such as Liu [30].

**Theorem 5.3.3 (Liu [10])**
*Let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be a non-empty $\Pi^0_1$ class. There is an infinite homogeneous set $H$ for $\mathcal{P}$ of non-PA degree.*

Proof. Let $\mathbb{P}$ be the notion of forcing whose conditions are tuples $(k, \vec{\sigma}, \mathcal{A})$ where

1. $k \in \mathbb{N}$ is the number of parts ;
2. $\vec{\sigma} = \langle \sigma_0, \dots, \sigma_{k-1} \rangle$ is a $k$-tuple of binary strings ;
3. $\mathcal{A} \subseteq k^{\omega}$ is a non-empty $\Pi^0_1$ class of $k$-partitions.

One can see a condition $p = (k, \vec{\sigma}, \mathcal{A})$ as a $k$-tuple of families of Mathias preconditions[9] $(\sigma_i, X^{-1}(i) \setminus \{0, \dots, |\sigma|\})$ for any $X \in \mathcal{A}$. We say that *part $i$ of $p$ is acceptable* if there exists some $X \in \mathcal{A}$ such that $X^{-1}(i)$ is infinite.

9: A *Mathias precondition* is a pair $(\sigma, X)$ such that $\forall x \in X\ x > |\sigma|$, but $X$ might be finite or empty.

The intended initial condition is $(2, \langle\emptyset,\emptyset\rangle, \mathscr{P})$. The *interpretation* of a condition $(k, \vec{\sigma}, \mathscr{A})$ is

$$[k, \vec{\sigma}, \mathscr{A}] = \{(G_0, \dots, G_{k-1}) : \exists X \in \mathscr{A}\ \forall i < k\ \sigma_i \subseteq G_i \subseteq \sigma_i \cup X^{-1}(i)\}$$

A condition $q = (\ell, \vec{\tau}, \mathscr{B})$ *extends* $p = (k, \vec{\sigma}, \mathscr{A})$ if $\ell \geq k$ and there is a map[10] $f : \ell \to k$ such that for every $Y \in \mathscr{B}$, there is some $X \in \mathscr{A}$ such that for every $i < \ell$, $(\tau_i, Y^{-1}(i))$ Mathias extends $(\sigma_i, X^{-1}(i))$, that is, $Y^{-1}(i) \subseteq X^{-1}(i)$ and $\sigma_i \subseteq \tau_i \subseteq \sigma_i \cup X^{-1}(i)$. We say that *part $i$ of $q$ refines part $f(i)$ of $p$.*

10: Over extension, some parts of a condition might be splitting. The map keeps track of which part refines which one. This map may not be unique, but it does not matter.

Given a condition $p = (k, \vec{\sigma}, \mathscr{A})$, we shall construct actually only two kinds of extensions:

- A condition $q = (\ell, \vec{\tau}, \mathscr{B})$ is a *part $i$ extension* of $p$ if $\ell = k$, the extension map $f$ is the identity function, and $\tau_j = \sigma_j$ for all $j \neq i$.
- A condition $q = (\ell, \vec{\tau}, \mathscr{B})$ is a *splitting extension* of $p$ if, letting $f$ be the map witnessing the extension, for every $i < \ell$, $\tau_i = \sigma_{f(i)}$.

Given a condition $p = (k, \vec{\sigma}, \mathscr{A})$, and some Turing index $e$, let $I_e(p) \subseteq k$ be the set of acceptable parts $i$ of $p$ which do not already force $\Phi_e^G$ not to be a $\mathrm{DNC}_2$ function.

**Lemma 5.3.4.** For every condition $p = (k, \vec{\sigma}, \mathscr{A})$ and every Turing index $e$ such that $I_e(p) \neq \emptyset$, there is an extension $q \leq p$ such that $I_e(q) \subsetneq I_e(p)$. ⋆

Proof. We will either find a part $i$ extension $q \leq p$ for some $i \in I_e(p)$ such that $q$ which will force $\Phi_e^G$ not to be a $\mathrm{DNC}_2$ function on part $i$, in which case $I_e(q) = I_e(p) \setminus \{i\}$, or a splitting extension forcing $\Phi_e^G$ not to be a $\mathrm{DNC}_2$ function on every part, in which case $I_e(q) = \emptyset$.

Recall the notion of valuation from Theorem 5.2.4. Consider the following set:[11]

11: The set $U$ plays the same role as in Lemma 5.2.5.

$$U = \left\{ h \text{ finite valuation } : \begin{array}{l} \forall X \in \mathscr{A}\ \exists i \in I_e(p)\ \exists \rho \subseteq X^{-1}(i) \\ \Phi_e^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

Note that by effective compactness, letting $T \subseteq k^{<\mathbb{N}}$ be a computable tree such that $[T] = \mathscr{A}$, the set $U$ can equivalently be defined as

$$U = \left\{ h \text{ finite valuation } : \begin{array}{l} \exists n \forall \tau \in T \cap k^n\ \exists i \in I_e(p)\ \exists \rho \subseteq \tau^{-1}(i) \\ \Phi_e^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

Thus, the set $U$ is $\Sigma_1^0$. There are three cases.

- Case 1: $U$ contains a correct valuation $h$. Fix some $X \in \mathscr{A}$, and let $i \in I_e(p)$ and $\rho \subseteq X^{-1}(i)$ be such that $\Phi_e^{\sigma_i \cup \rho}$ is incompatible with $h$. Letting $\mathscr{B} = \{Y \in \mathscr{A} : \rho \subseteq Y^{-1}(i)\}$, $\tau_i = \sigma_i \cup \rho$ and $\tau_j = \sigma_j$ otherwise, the condition $(k, \vec{\tau}, \mathscr{B})$ is a part $i$ extension of $p$ forcing $\Phi_e^G$ to be incompatible with $h$ on part $i$, hence forcing $\Phi_e^G$ not to be a $\mathrm{DNC}_2$ function on part $i$.
- Case 2: there are $k+1$ pairwise incompatible finite valuations $h_0, \dots, h_k$ outside of $U$. For each $s \leq k$, let $\mathscr{B}_s \subseteq k^{\mathbb{N}}$ be the $\Pi_1^0$ class of all $X \in \mathscr{A}$ such that for every $i \in I_e(p)$ and every $\rho \subseteq X^{-1}(i)$, $\Phi_e^{\sigma_i \cup \rho}$ is compatible with $h_s$. By assumption, $\mathscr{B}_s \neq \emptyset$ for every $s \leq k$. We say that $Y \in (k^{k+1})^{\omega}$ is the *refined partition* of $(X_0, \dots, X_k) \in \mathscr{B}_0 \times \dots \times \mathscr{B}_k$ if for every $\nu < k^{k+1}$ interpreted as a $k$-ary string of length $k+1$, $Y^{-1}(\nu) = \bigcap_{s \leq k} X_s^{-1}(\nu(s))$. Let $\mathscr{B} \subseteq (k^{k+1})^{\omega}$ be the class of all refined partitions

of members of $\mathscr{B}_0 \times \dots \times \mathscr{B}_k$. By the pigeonhole principle, for every $\nu \in k^{k+1}$, there is some $i_\nu \in k$ and some $s < t \le k$ such that $\nu(s) = \nu(t) = i_\nu$. Let $f : k^{k+1} \to k$ be the defined by $f(\nu) = i_\nu$. For each $\nu \in k^{k+1}$, let $\tau_\nu = \sigma_{f(\nu)}$. The condition $q = (k^{k+1}, \vec{\tau}, \mathscr{B})$ is a splitting extension of $p$. Moreover, every part $\nu$ of $q$ refining some part $i \in I_e(p)$ of $p$ forces $\Phi_e^G$ to be compatible with $h_s$ and $h_t$, for $s < t \le k$ such that $\nu(s) = \nu(t) = f(\nu)$. Since $h_s$ and $h_t$ are incompatible, such part $\nu$ of $q$ forces $\Phi_e^G$ to be partial, hence $\nu \notin I_e(q)$. Last, if part $\nu$ of $q$ refines some part $i \notin I_e(p)$ of $p$, then $\nu \notin I_e(q)$, so $I_e(q) = \emptyset$.

- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.2.3. ■

Consider an infinite, sufficiently generic decreasing sequence of conditions $p_0 \ge p_1 \ge \dots$ with $p_s = (k_s, \vec{\sigma}_s, \mathscr{A}_s)$, together with the refinement maps $f_s : k_{s+1} \to k_s$ witnessing the extensions. Note that each condition has an acceptable part, and if part $i$ of $p_{s+1}$ is acceptable, then so is part $f_s(i)$ of $p_s$. Thus, by König's lemma, there exists a sequence $P \in \omega^\omega$ such that for every $s$, part $P(s)$ of $p_s$ is acceptable, and part $P(s+1)$ of $p_{s+1}$ refines part $P(s)$ of $p_s$, that is, $f_s(P(s+1)) = P(s)$. This induces a set $G_P$ defined by $G = \bigcup_s \sigma_{s,P(s)}$. By genericity of the sequence, $G_P$ is infinite. Moreover, by Lemma 5.3.4, $G_P$ is not of PA degree. This completes the proof of Theorem 5.3.3. ■

## 5.4 Liu's theorem

Liu's theorem states that Ramsey's theorem for pairs admits PA avoidance. Recall that the modern proof of Seetapun's theorem (Theorem 3.4.10) was divided into a proof of cone avoidance of COH and a proof of strong cone avoidance of $\mathsf{RT}^1_2$. The proof of Liu's theorem follows the same structure.

Recall that an infinite set $C$ is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$. The cohesiveness principle (COH) is the problem whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets.

**Exercise 5.4.1.** Combine Exercise 3.4.3 and Exercise 5.1.5 to prove that COH admits PA avoidance. ⋆

**Exercise 5.4.2.** Recall the notion of computable Mathias forcing from Exercise 3.2.8. Given a condition $(\sigma, X)$ and a $\Sigma^0_1$ formula $\varphi(G)$, let $(\sigma, X) \mathrel{?\vdash} \varphi(G)$ hold if there is some $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds.

1. Show that this is a $\Sigma^0_1$-preserving, $\Pi^0_1$-merging forcing question.
2. Deduce that COH admits PA avoidance. ⋆

Our last step consists in proving that $\mathsf{RT}^1_2$ admits strong PA avoidance.[12]

**Theorem 5.4.3 (Liu [10])**
*For every set $A$, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ of non-PA degree.*[13]

12: The original proof of Liu's theorem was also using the decomposition into COH and $\mathsf{RT}^1_2$. However, it directly proved that $\mathsf{RT}^1_2$ admits strong PA avoidance without using PA avoidance of RWKL. Proving first PA avoidance of RWKL enables to reduce the complexity of each forcing, by separating the compactness from the disjunction issues.

13: From many viewpoints, the proof of this theorem will be similar to the proof of Theorem 3.4.5. It is strongly advised to have a good understanding of the latter proof.

Proof. Fix $A$. As in Theorem 3.4.5, we shall build two sets $G_0, G_1$ simultaneously, with $G_0 \subseteq A$ and $G_1 \subseteq \overline{A}$. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

The two sets will be constructed through a variant of Mathias forcing, whose *conditions* are triples $(\sigma_0, \sigma_1, X)$ where

1. $(\sigma_i, X)$ is a Mathias condition for each $i < 2$ ;
2. $\sigma_i \subseteq A_i$ ;
3. $X$ is not of PA degree[14].

14: This is the only difference with the notion of forcing of Theorem 3.4.5.

The *interpretation* $[\sigma_0, \sigma_1, X]$ of a condition $(\sigma_0, \sigma_1, X)$ is the class

$$[\sigma_0, \sigma_1, X] = \{(G_0, G_1) : \forall i < 2\ \sigma_i \preceq G_i \subseteq \sigma_i \cup X\}$$

A condition $(\tau_0, \tau_1, Y)$ *extends* $(\sigma_0, \sigma_1, X)$ if $(\tau_i, Y)$ Mathias extends $(\sigma_i, X)$ for each $i < 2$. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$. Note that $(G_{\mathcal{F},0}, G_{\mathcal{F},1}) \in \bigcap\{[\sigma_0, \sigma_1, X] : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$.

The goal is therefore to build two infinite sets $G_0, G_1$, satisfying the following requirements for every $e_0, e_1 \in \mathbb{N}$:

$$\mathcal{R}_{e_0,e_1} : \Phi_{e_0}^{G_0} \text{ is not DNC}_2 \vee \Phi_{e_1}^{G_1} \text{ is not DNC}_2$$

If every requirement is satisfied, then a pairing argument shows that either $G_0$, or $G_1$ is not of PA degree. We make the following assumption:

$$\text{There is no infinite set } H \subseteq A \text{ or } H \subseteq \overline{A} \text{ of non-PA degree.} \qquad \text{(H1)}$$

Under this assumption, one can prove that if $\mathcal{F}$ is sufficiently generic, then both $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ are infinite.

**Lemma 5.4.4.** Suppose (H1). Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$. There is an extension $(\tau_0, \tau_1, Y)$ of $p$ and some $n > |\sigma_i|$ such that $n \in \tau_i$. ⋆

Proof. If $X \cap A^i$ is empty, then $X \subseteq A^{1-i}$, but $X$ is of non-PA degree, which contradicts (H1). Thus, there is $n \in X \cap A^i$. Let $\tau_i = \sigma_i \cup \{n\}$, and $\tau_{1-i} = \sigma_{1-i}$. Then, $(\tau_0, \tau_1, X \setminus \{0, \ldots, n-1\})$ is an extension of $p$ such that $n \in \tau_i$. ■

We will now prove the core lemma.

**Lemma 5.4.5.** Let $p = (\sigma_0, \sigma_1, X)$ be a condition, and $e_0, e_1 \in \mathbb{N}$. There is an extension $(\tau_0, \tau_1, Y)$ of $p$ forcing $\mathcal{R}_{e_0,e_1}$. ⋆

15: The set $U$ is a combination of the forcing question of Theorem 3.4.5, but working with valuations due to the disjunctive nature of the forcing question.

Proof. Consider the following set[15]

$$U = \left\{ h \text{ finite valuation } : \begin{array}{l} \forall Z_0 \sqcup Z_1 = X\ \exists i < 2\ \exists \rho \subseteq Z_i \\ \Phi_{e_i}^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

Here again, the previous set is $\Sigma^0_1(X)$, as it can be equivalently defined as

$$\left\{ h \text{ finite valuation } : \begin{array}{l} \exists \ell \in \mathbb{N} \forall Z_0 \sqcup Z_1 = X \upharpoonright_\ell\ \exists i < 2\ \exists \rho \subseteq Z_i \\ \Phi_{e_i}^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

There are three cases:

- Case 1: $U$ contains a correct valuation $h$. Letting $Z_0 = A_0 \cap X$ and $Z_1 = A_1 \cap X$, there is some $i < 2$ and some $\rho \subseteq Z_i$ such that $\Phi_{e_i}^{\sigma_i \cup \rho}$ is incompatible with $h$. Letting $\tau_i = \sigma_i \cup \rho$ and $\tau_{1-i} = \sigma_{1-i}$, the condition $(\tau_0, \tau_1, X \setminus \{0, \dots, \max \rho\})$ is an extension of $p$ forcing $\Phi_{e_i}^{G_i}$ to be incompatible with $h$, hence not being a $\mathsf{DNC}_2$ function.
- Case 2: there are 3 pairwise incompatible finite valuations $h_0, h_1, h_2$ outside of $U$. For each $s < 3$, let $\mathscr{P}_s \subseteq 2^{\mathbb{N}}$ be the $\Pi^0_1$ class of all $Y_s$ such that, letting $Y_{s,0} = Y_s$ and $Y_{s,1} = \overline{Y}_s$, for every $i < 2$ and every $\rho \subseteq Y_{s,i} \cap X$, $\Phi_{e_i}^{\sigma_i \cup \rho}$ is compatible with $h_s$. By assumption, $\mathscr{P}_s \neq \emptyset$ for every $s < 3$. Since RWKL admits PA avoidance (Theorem 5.3.3), there is a decreasing sequence of sets $X \supseteq Y_0 \supseteq Y_1 \supseteq Y_2$ such that $Y_s$ is homogeneous for $\mathscr{P}_s$ for some color $i_s < 2$, and $Y_2 \oplus Y_1 \oplus Y_0 \oplus X$ is not of PA degree. By the pigeonhole principle, there exist some $s < t < 3$ and some color $i < 2$ such that $i = i_s = i_t$. The condition $(\sigma_0, \sigma_1, Y_2)$ is an extension of $p$ forcing $\Phi_{e_i}^{G_i}$ to be compatible with $h_s$ and $h_t$, hence forcing $\Phi_{e_i}^{G_i}$ to be partial.
- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.2.3. ■

As in the proof of strong cone avoidance, we are getting a $\Pi^0_1$ class of instances of $\mathsf{RT}^1_2$. In the proof of strong cone avoidance, we simply picked a member of this class using the cone avoidance basis theorem. Here, since we need to avoid PA degrees, we cannot pick a member, so we use RWKL instead of WKL. The true complexity of this construction is hidden in the proof that RWKL admits PA avoidance.

We are now ready to prove Theorem 5.4.3. Let $\mathscr{F}$ be a sufficiently generic filter for this notion of forcing, and for each $i < 2$, let $G_i = G_{\mathscr{F},i}$. By Lemma 5.4.4, both sets are infinite. Moreover, by Lemma 5.4.5, either $G_0$ or $G_1$ is not of PA degree. Letting $H$ be this set, it satisfies the statement of Theorem 5.4.3. ■

We can now prove Liu's theorem by combining PA avoidance of COH and strong PA avoidance of $\mathsf{RT}^1_2$.

**Theorem 5.4.6 (Liu [10])**
*Every computable coloring $f : [\mathbb{N}]^2 \to 2$ has an infinite $f$-homogeneous set of non-PA degree.*

Proof. The proof follows the one of Theorem 3.4.1. Fix $f$. Let $\vec{R} = R_0, R_1, \dots$ be the computable sequence of sets defined for every $x \in \mathbb{N}$ by $R_x = \{y \in \mathbb{N} : f(x, y) = 1\}$. By Exercise 5.4.1, there is an infinite $\vec{R}$-cohesive set $X \subseteq \mathbb{N}$ of non-PA degree. In particular, for every $x \in X$, $\lim_{y \in X} f(x, y)$ exists. Let $\hat{f} : X \to 2$ be the limit coloring of $f$, that is, $\hat{f}(x) = \lim_{y \in X} f(x, y)$. By Theorem 5.4.3, there is an infinite $\hat{f}$-homogeneous set $Y \subseteq X$ for some color $i < 2$ such that $Y \oplus X$ is of non-PA degree. Since for every $x \in Y$, $\lim_{y \in Y} f(x, y) = i$, one can thin out the set $Y$ to obtain an infinite $f$-homogeneous subset $H \subseteq Y$. ■

## 5.5 Randomness

Algorithmic randomness is a sub-field of computability theory studying the amount of randomness contained in binary sequences taken individually. Contrary to the notion of effective computability which admits a robust mathematical definition, randomness does not translate mathematically to a single notion, but to a hierarchy of concepts. Nonetheless, randomness admits its own form of robustness, by having many different characterizations based on multiple paradigms. See Downey and Hirschfeldt [31] or Nies [32] for an introduction on algorithmic randomness.

Among the notions of randomness, *Martin-Löf randomness* is widely considered as capturing the intuitive idea of a random sequence.[16] It can be equivalently defined using multiple paradigms:

- *Incompressibility*: There should be no recognizable pattern in the sequence, which would yield a possibility to compress the sequence. This approach due to Chaitin is based on Kolmogorov complexity.
- *Unpredicability*: One should not be able to predict the the bits of the sequence. This approach is formalized using martingales.
- *Measure*: Random sequences should not satisfy any "rare" properties which can be effectively described.

16: This is known as the Martin-Löf-Chaitin thesis, and plays the same role as the Church-Turing thesis for computability.

Kolmogorov complexity is probably the shortest way to define Martin-Löf randomness. A *prefix-free machine* is a partial computable function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ whose domain is prefix-free, that is, if $\sigma, \tau \in \operatorname{dom} M$ with $\sigma \neq \tau$, then they are incomparable. A prefix-free machine $M$ is *universal*[17] if for every prefix-free machine $N$, there is some $\rho \in 2^{<\mathbb{N}}$ such that $(\forall \sigma \in 2^{<\mathbb{N}})M(\rho\sigma) = N(\sigma)$.

17: The proof of the existence of a universal prefix-free machine goes as follows: Prove the existence of a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$, $\Phi_{f(e)}$ is prefix-free and if $\Phi_e$ is prefix-free, then $\Phi_{f(e)} = \Phi_e$. Then, let

$$M(1^e 0\sigma) = \Phi_{f(e)}(\sigma)$$

**Definition 5.5.1.** Fix a universal prefix-free machine $M$. The *Kolmogorov complexity* $K_M(\sigma)$ of a string $\sigma \in 2^{<\mathbb{N}}$ is the length of the shortest string $\tau \in 2^{<\mathbb{N}}$ such that $M(\tau) = \sigma$. ◊

The Kolmogorov complexity of a string depends on the choice of a universal prefix-free machine. Given another universal prefix-free machine $N$, $(\forall \sigma \in 2^{<\mathbb{N}})K_N(\sigma) = K_M(\sigma) + \mathcal{O}(1)$. Kolmogorov complexity is therefore an asymptotic notion of complexity. From now on, we omit the subscript $M$ and work with inequalities to additive constant, noted $\leq^+$.

**Exercise 5.5.2.** Show that for every $\sigma \in 2^{<\mathbb{N}}$, $K(\sigma) \leq^+ |\sigma| + 2\log_2(|\sigma|)$. ★

**Definition 5.5.3 (Chaitin [33] and Levin [34]).** A set $X \in 2^{\mathbb{N}}$ is *Martin-Löf random*[18] if for every $n \in \mathbb{N}$, $K(X \upharpoonright_n) \geq^+ n$. ◊

18: This definition is independently due to Chaitin and Levin, but coincides with the notion of Martin-Löf randomness based of measure.

The *Lebesgue measure* on Cantor space $2^{\mathbb{N}}$ is the measure $\mu$ induced by letting $\mu([\sigma]) = 2^{-|\sigma|}$ for every $\sigma \in 2^{<\mathbb{N}}$. In particular, every open class $\mathcal{U} \subseteq 2^{\mathbb{N}}$ being of the form $\bigcup_{\sigma \in W}[\sigma]$ for some prefix-free set $W \subseteq 2^{<\mathbb{N}}$, $\mu(\mathcal{U}) = \sum_{\sigma \in W}[\sigma]$. It follows that the Lebesgue measure of a closed class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is $1 - \mu(2^{\mathbb{N}} \setminus \mathcal{P})$. In the case of closed classes, one can give a more direct definition in terms of trees:

**Exercise 5.5.4.** The *measure* of a tree $T \subseteq 2^{<\mathbb{N}}$ is defined as

$$\mu(T) = \lim_n \frac{\operatorname{card}\{\sigma \in T : |\sigma| = n\}}{2^n}$$

Show that $\mu(T) = \mu([T])$. ★

The following exercise shows the existence of a $\Pi^0_1$ class of positive measure containing only (but not all) Martin-Löf random sets.

**Exercise 5.5.5.** Fix a universal prefix-free machine $M$. For every $c \geq 1$, let $\mathcal{U}_c$ be the $\Sigma^0_1$ class $\{X : \exists n K_M(X \upharpoonright_n) < n - c\}$ and let $V_c \subseteq 2^{<\mathbb{N}}$ be a prefix-free set of strings such that $[\![V_c]\!] = \mathcal{U}_c$ and such that for every $\sigma \in V_c$, $K_M(\sigma) < |\sigma| - c$.

1. Show that $\sum_{\sigma \in V_c} 2^{-|\sigma|+c} \leq \sum_{\sigma \in V_c} 2^{-K_M(\sigma)} \leq 1$. [19]
2. Deduce that $\mu(\mathcal{U}_c) \leq 2^{-c}$, hence that the $\Pi^0_1$ class $2^{\mathbb{N}} \setminus \mathcal{U}_c$ has positive measure.[20] ★

19: For every prefix-free machine $M$ and every set of strings $S \subseteq 2^{<\mathbb{N}}$,

$$\sum_{\sigma \in S} 2^{-K_M(\sigma)} \leq 1$$

20: If $V \subseteq 2^{<\mathbb{N}}$ is prefix-free, then

$$\mu([\![ V ]\!]) = \sum_{\sigma \in V} 2^{-|\sigma|}$$

Given a measurable class $\mathscr{C}$ and a cylinder $[\sigma]$, we write $\mu(\mathscr{C} | [\sigma]) = \frac{\mu(\mathscr{C} \cap [\sigma])}{\mu([\sigma])}$ for the measure of $\mathscr{C}$ relative to $[\sigma]$. The Lebesgue measure satisfies the following theorem which happens to be a very powerful tool for the computability-theoretic study of measure:

**Theorem 5.5.6 (Lebesgue density)**
*Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a measurable class of positive measure. For almost every $X \in \mathscr{C}$, $\lim_n \mu(\mathscr{C} | [X \upharpoonright_n]) = 1$.*

It follows from Lebesgue density theorem that for every $\epsilon > 0$, there is a cylinder $[\sigma]$ such that $\mu(\mathscr{C} | [\sigma]) > 1 - \epsilon$.

Weak weak König's lemma is the restriction of weak König's lemma to trees of positive measure, that is, the statement "Every infinite binary tree of positive measure admits an infinite path." $\mathsf{WWKL}_0$ is $\mathsf{RCA}_0$ augmented with weak weak König's lemma. By Exercise 5.5.5, there exists a $\Pi^0_1$ class of positive measure containing only Martin-Löf random sequences. Conversely, for every $\Pi^0_1$ class $\mathscr{P} \subseteq 2^{\mathbb{N}}$ of positive measure and every Martin-Löf random sequence $Z$, there exists a string $\sigma \in 2^{<\mathbb{N}}$ such that $\sigma \cdot Z \in \mathscr{P}$. Thus, $\mathsf{WWKL}_0$ is equivalent to the statement "For every set $X$, there exists a Martin-Löf random sequence relative to $X$". For these reasons, $\mathsf{WWKL}_0$ is considered as capturing probabilistic arguments.

Seeing $\mathsf{WWKL}_0$ as a restriction of $\mathsf{WKL}_0$ which itself captures compactness arguments, $\mathsf{WWKL}_0$ can be seen as a weaker notion of compactness. We now prove that weak weak König's lemma admits PA avoidance using a forcing with closed classes of positive measure.[21]

21: Note that we prove a much stronger statement since the closed class is not assumed to be effectively closed. This actually corresponds to a proof that weak weak König's lemma admits strong PA avoidance.

**Theorem 5.5.7**
*Every closed class $\mathscr{P} \subseteq 2^{\mathbb{N}}$ of positive measure admits a member of non-PA degree.*

Proof. Consider the notion of forcing $\mathbb{P}$ whose conditions are closed classes $\mathbb{Q} \subseteq 2^{\mathbb{N}}$ of positive measure, partially ordered by inclusion. A condition is its self interpretation.

**Lemma 5.5.8.** For every condition $\mathbb{Q} \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\mathscr{R} \leq \mathbb{Q}$ forcing $\Phi^G_e$ not to be a $\mathsf{DNC}_2$ function. ★

Proof. By Lebesgue density theorem (Theorem 5.5.6), there is some $\sigma \in 2^{<\mathbb{N}}$ such that $\mu(\mathbb{Q} | [\sigma]) > 0.9$. For every $x \in \mathbb{N}$ and $v < 2$, let $\mathcal{U}_{x,v} = \{X : \Phi^{\sigma \cdot X}_e(x)\downarrow = v\}$. Consider the following set

$$U = \{(x, v) \in \mathbb{N} \times 2 : \mu(\mathcal{U}_{x,v}) > 0.2\}$$

Note that the classes $\mathcal{U}_{x,v}$ are uniformly $\Sigma^0_1$, so the set $U$ is $\Sigma^0_1$. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. By assumption, $\mu(\mathcal{U}_{x,\Phi_x(x)}) > 0.2$. Let $\mathscr{C} \subseteq \mathcal{U}_{x,\Phi_x(x)}$ be a clopen[22] subclass such that $\mu(\mathscr{C}) > 0.2$. Let $\mathbb{Q}_\sigma = \{X \in 2^{\mathbb{N}} : \sigma \cdot X \in \mathbb{Q}\}$. By choice of $\sigma$,

22: A class is *clopen* if it is both closed and open. Here, we use the fact that if $\bigcup_{\sigma \in W}[\sigma]$ is an open class, for every $\epsilon > 0$, there is a finite subset $F \subseteq W$ such that

$$\mu(\bigcup_{\sigma \in F}[\sigma]) > \mu(\bigcup_{\sigma \in W}[\sigma]) - \epsilon$$

$\mu(\mathbb{Q}_\sigma) > 0.9$, so $\mu(\mathbb{Q}_\sigma \cap \mathscr{C}) > 0.1$. Finally, let $\mathscr{R} = \{\sigma \cdot X : X \in \mathbb{Q}_\sigma \cap \mathscr{C}\}$. The class $\mathscr{R}$ is a closed subclass of $\mathbb{Q}$ such that $\mu(\mathscr{R} | [\sigma]) > 0.1$, thus $\mathscr{R}$ is a valid extension. Furthermore, $\mathscr{R}$ forces $\Phi_e^G(x)\downarrow = \Phi_x(x)$.

- Case 2: $(x, 0), (x, 1) \notin U$ for some $x \in \mathbb{N}$. By assumption, $\mu(\mathbb{Q}_{x,0}) \leq 0.2$ and $\mu(\mathbb{Q}_{x,1}) \leq 0.2$, so $\mu(\mathbb{Q}_{x,0} \cup \mathbb{Q}_{x,1}) \leq 0.4$. Let $\mathscr{R} = \{\sigma \cdot X \in \mathbb{Q} : X \notin \mathbb{Q}_{x,0} \cup \mathbb{Q}_{x,1}\}$. Since $\mu(\mathbb{Q} | [\sigma]) > 0.9$, then $\mu(\mathscr{R} | [\sigma]) > 0.5$). So $\mathscr{R}$ is a valid extension of $\mathbb{Q}$ forcing $\neg(\Phi_e^G(x)\downarrow = 0) \wedge \neg(\Phi_e^G(x)\downarrow = 1)$, hence forcing $\Phi_e^G$ not to be a DNC$_2$ function.
- Case 3: None of Case 1 and Case 2 holds. Then $U$ is a $\Sigma_1^0$ graph of a $\{0, 1\}$-valued DNC function. This contradicts the fact that $\mathbf{0}$ is not PA. ■

We are now ready to prove Theorem 5.5.7. Given $e \in \mathbb{N}$, let $\mathscr{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi_e^G$ not to be a DNC$_2$ function. It follows from Lemma 5.5.8 that every $\mathscr{D}_e$ is dense, hence every sufficiently generic filter $\mathscr{F}$ is $\{\mathscr{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathscr{F}}$ is not of PA degree. This completes the proof of Theorem 5.5.7. ■

**Exercise 5.5.9.** Consider the notion of forcing of Theorem 5.5.7. Given a condition $\mathscr{P} \subseteq 2^{\mathbb{N}}$, a string $\sigma \in 2^{<\mathbb{N}}$ such that $\mu(\mathbb{Q} | [\sigma]) > 0.9$, and a $\Sigma_1^0$ formula $\varphi(G)$, let $\mathscr{P} \mathrel{?\vdash} \varphi(G)$ iff $\mu\{X : \varphi(\sigma \cdot X)\} > 0.2$.

1. Show that $\mathscr{C} \mathrel{?\vdash} \varphi(G)$ is a $\Sigma_1^0$-preserving, $\Pi_1^0$-merging forcing question.
2. Deduce that if $C$ is a non-computable set and $\mathscr{P} \subseteq 2^{\mathbb{N}}$ is a closed class of positive measure, there is a member $G \in \mathscr{P}$ such that $C \not\leq_T G$. ⋆

## 5.6 Avoiding closed classes

The notion of PA avoidance is an avoidance of a particular closed class: the $\Pi_1^0$ class $\mathscr{P} \subseteq 2^{\mathbb{N}}$ of DNC$_2$ functions. This class has two particularities: First, it is effectively closed, hence can be represented by a computable tree. Second, it is *homogeneous*, that is, if one considers the pruned[23] tree $T \subseteq 2^{<\mathbb{N}}$ corresponding to $\mathscr{P}$, for every $\sigma, \tau \in T$ at the same level, the sub-trees below $\sigma$ and $\tau$ coincide.

23: A tree is *pruned* it it has no leaves, in other words if every node is extendible.

In this section, we generalize PA avoidance to avoid a larger collection of closed classes, with no effectiveness or homogeneity constraint. Many natural closed classes in $2^{\mathbb{N}}$ with no computable member cannot even be computably approximated by giving arbitrarily large initial segments of members.

Given a closed class $\mathscr{C} \subseteq 2^{\mathbb{N}}$, a *trace* is a collection of finite coded sets of strings $F_0, F_1, \dots$ such that for each $n \in \mathbb{N}$, $F_n$ contains only strings of length exactly $n$, and $\mathscr{C} \cap \bigcup_{\sigma \in F_n} [\sigma] \neq \emptyset$.[24] In other words, for every $n \in \mathbb{N}$, there is a string $\sigma \in F_n$ and some $P \in \mathscr{C}$ such that $\sigma \prec P$. A $k$-*trace* is a trace such that $\operatorname{card} F_n = k$ for every $n \in \mathbb{N}$. A *constant-bound trace* (c.b-trace) of $\mathscr{C}$ is a $k$-trace for some $k \in \mathbb{N}$.

24: One usually writes $[\![F_n]\!]$ for the clopen class generated by $F_n$. Indeed, using $[F_n]$ would be confusing with the collection of paths through a tree.

**Definition 5.6.1.** A problem P admits *constant-bound trace avoidance*[25] if for every set $Z$ and every closed class $\mathscr{C} \subseteq 2^{\mathbb{N}}$ with no $Z$-computable c.b-trace, every $Z$-computable instance $X$ of P admits a solution $Y$ such that $\mathscr{C}$ has no $Z \oplus Y$-computable c.b-trace. ◇

25: We defined the notion of closed classes in Cantor space $2^{\mathbb{N}}$, but all the theorems work equally for effectively compact classes in Baire space $\mathbb{N}^{\mathbb{N}}$. More precisely, it works for every closed class $\mathscr{C} \subseteq h^{\mathbb{N}}$ for some total computable function $h : \mathbb{N} \to \mathbb{N}$.

Before proving that some problems admit constant-bound trace avoidance, we shall start with a few exercises to get familiar with this seemingly artificial notion. The two following exercises show that for a homogeneous $\Pi^0_1$ class, every constant-bound trace computes a member. Hence, c.b-trace avoidance generalizes PA avoidance.

**Exercise 5.6.2.** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a $\Pi^0_1$ class. Show that every $k$-trace of $\mathscr{C}$ computes a 1-trace of $\mathscr{C}$. ⋆

**Exercise 5.6.3.** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a homogeneous closed class. Show that every 1-trace of $\mathscr{C}$ computes a member of $\mathscr{C}$. ⋆

The following exercise shows that c.b-trace avoidance generalizes cone avoidance.

**Exercise 5.6.4.** Let $C$ be a non-computable set. Show that $\{C\}$ does not admit any computable c.b-trace. ⋆

As usual, the core lemma involved in proofs of constant-bound trace avoidance is based on a 3-case analysis. As in PA avoidance for weakly merging forcing questions, the case analysis for preservation of c.b-traces is non-trivial and based on a combinatorial lemma. Let us introduce some piece of terminology which will be helpful in working with constant-bound traces.

A *block* is a finite set of strings all of which have the same length. We write $\mathscr{B}_n$ for the set of all blocks $F \subseteq 2^n$ and $\mathscr{B} = \bigcup_n \mathscr{B}_n$. Given a closed class $\mathscr{C} \subseteq 2^{\mathbb{N}}$, a block $F \in \mathscr{B}_n$ is *$\mathscr{C}$-correct* if $F = \{\mu \in 2^n : \mathscr{C} \cap [\mu] \neq \emptyset\}$. In other words, $F$ is $\mathscr{C}$-correct if it is some level in the pruned tree representing $\mathscr{C}$. Given $n, k \in \mathbb{N}$, a finite collection of blocks $V \subseteq \mathscr{B}_n$ is *$k$-disperse* if for every $k$-partition $(P_s : s < k)$ of $V$, there is some $s < k$ such that $\bigcap_{F \in P_s} F = \emptyset$. The following exercise emphasises a core property of $k$-disperse sequences:

**Exercise 5.6.5.** Fix $n, k \in \mathbb{N}$, and let $V \subseteq \mathscr{B}_n$ be a $k$-disperse sequence. If $E \in \mathscr{B}_n$ is a block which intersects[26] every element of $V$, then $\operatorname{card} E > k$.⋆

26: By *intersects*, we mean that $F \cap E \neq \emptyset$ for every $F \in V$.

We now prove the core combinatorial lemma which frames the 3-case analysis.

**Lemma 5.6.6 (Liu [30]).** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable c.b-trace. Let $U \subseteq \mathscr{B}$ be a c.e. set of blocks. Either $U$ contains a $\mathscr{C}$-correct block, or for every $k \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that the set $\mathscr{B}_n \setminus U$ is $k$-disperse. ⋆

Proof. Suppose that $U$ does not contain any $\mathscr{C}$-correct block.[27] For every $n \in \mathbb{N}$, let $V_n = \mathscr{B}_n \setminus U$. Fix some $k \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$, $V_n$ is not $k$-disperse, otherwise we are done. Since $V_n$ is co-c.e. uniformly in $n$, there exists a co-c.e. enumeration $(V_{n,t})_{t \in \mathbb{N}}$ of $V_n$. Since $V_n$ is not $k$-disperse, there exists some $t \in \mathbb{N}$ and a $k$-partition $(P_{n,s} : s < k)$ of $V_{n,t}$ such that for each $s < k$, $\bigcap_{F \in P_{n,s}} F \neq \emptyset$. Such $k$-partition can be computed uniformly in $n$. Moreover, since $V_n$ contains a $\mathscr{C}$-correct block, then there is some $s < k$ such that $P_{n,s}$ contains a $\mathscr{C}$-correct block, hence for every $\sigma \in \bigcap_{F \in P_{n,s}} F$, $\mathscr{C} \cap [\sigma] \neq \emptyset$. For each $n$, let $E_n$ be obtain by picking a string in each set $\bigcap_{F \in P_{n,s}} F$ for each $s < k$. The sequence $(E_n)_{n \in \mathbb{N}}$ is a computable $k$-trace of $\mathscr{C}$, contradicting the hypothesis. ■

27: The proof actually shows that if $\mathscr{U}$ is a c.e. set of blocks with no $\mathscr{C}$-correct block and if there is no $k$-disperse sequence of blocks outside of $U$, then there is a computable $k$-trace of $\mathscr{C}$.

Let us illustrate preservation of constant-bound traces using the simplest notion of forcing, namely, Cohen forcing.

**Theorem 5.6.7**
*Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable c.b-trace. For every sufficiently Cohen generic set $G$, $\mathscr{C}$ admits no $G$-computable c.b-trace.*

Proof. It suffices to prove the following lemma.

**Lemma 5.6.8.** For every condition $\sigma \in 2^{<\mathbb{N}}$, every Turing index $e \in \mathbb{N}$ and every $k \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^G$ not to be a $k$-trace of $\mathscr{C}$.★

Proof. We can assume without loss of generality that $\Phi_e$ is a $k$-trace functional, that is, whenever $\Phi_e^X(n)\downarrow$, then the output is a block of size $k$, whose strings have length $n$. Fix a condition $\sigma$. Consider the following set:

$$U = \{F \in \mathscr{B}_n : n \in \mathbb{N},\ \exists \tau \geq \sigma\ \Phi_e^\tau(n)\downarrow \cap F = \emptyset\}$$

Note that the set $U$ is $\Sigma^0_1$. There are three cases:

- Case 1: there is some $n \in \mathbb{N}$ such that $U \cap \mathscr{B}_n$ contains some $\mathscr{C}$-correct block $F$. Let $\tau \geq \sigma$ witness $F \in U$, that is, let $\tau \geq \sigma$ be such that $\Phi_e^\tau(n)\downarrow \cap F = \emptyset$. Then $\tau$ forces $\Phi_e^G$ not to be a $k$-trace of $\mathscr{C}$.
- Case 2: there is some $n \in \mathbb{N}$ such that $\mathscr{B}_n \setminus U$ is $k$-disperse. We claim that for every $F \in \mathscr{B}_n \setminus U$, $\sigma$ forces $\Phi_e^G(n)\uparrow \vee \Phi_e^G(n)\downarrow \cap F \neq \emptyset$. Indeed, if for some $Z \in [\sigma]$, $\Phi_e^Z(n)\downarrow \cap F = \emptyset$, then by the use property, there is some $\tau \leq Z$ such that $\Phi_e^\tau(x)\downarrow \cap F = \emptyset$, contradicting the fact that $F \in \mathscr{B}_n \setminus U$. Thus $\sigma$ forces

  $$\Phi_e^G(n)\uparrow \vee (\forall F \in \mathscr{B}_n \setminus U)\ \Phi_e^G(n)\downarrow \cap F \neq \emptyset$$

  Since $\Phi_e$ is a $k$-trace functional, and $\mathscr{B}_n \setminus U$ is $k$-disperse, then by Exercise 5.6.5, $\sigma$ forces $\Phi_e^G(n)\uparrow$.
- Case 3: None of Case 1 and Case 2 holds. This cannot happen by Lemma 5.6.6. ■

We are now ready to prove Theorem 5.6.7. Given $e, k \in \mathbb{N}$, let $\mathscr{D}_{e,k}$ be the set of all conditions $\tau$ forcing $\Phi_e^G$ not to be a $k$-trace of $\mathscr{C}$. It follows from Lemma 5.6.8 that every $\mathscr{D}_{e,k}$ is dense, hence for every $\{\mathscr{D}_{e,k} : e, k \in \mathbb{N}\}$-generic set $G$, $\mathscr{C}$ admits no $G$-computable c.b-trace. ■

Looking more closely at the previous proof, the key feature of the forcing we exploited was the existence of a $\Sigma^0_1$-preserving forcing question such that, if it does not hold for a finite number of $\Sigma^0_1$ formulas, then there exists an extension forcing all negations simultaneously. This motivates the following definition, which is a strong form of $\Gamma$-merging.

**Definition 5.6.9.** Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas $\Gamma$, a forcing question is *finitely* $\Gamma$*-merging* if for every $p \in \mathbb{P}$ and every finite sequence of $\Gamma$-formulas $\varphi_0(G), \dots, \varphi_{\ell-1}(G)$, if $p \,?\!\vdash \varphi_s(G)$ holds for every $s < \ell$, then there is an extension $q \leq p$ forcing $\bigwedge_{s<\ell} \varphi_s(G)$. ◇

As for $\Gamma$-merging forcing questions, we say that a forcing question for $\Sigma^0_n$ formulas is finitely $\Pi^0_n$-merging if negation of the forcing question is finitely $\Pi^0_n$-merging. At this point, it should be clear how to prove the abstract theorem for constant-bound trace avoidance. We leave it as an exercise:

**Exercise 5.6.10.** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a $\Sigma^0_1$-preserving, finitely $\Pi^0_1$-merging forcing question. Prove that for every sufficiently generic filter $\mathcal{F}$, $\mathscr{C}$ admits no $G_{\mathcal{F}}$-computable constant-bound trace. ★

**Exercise 5.6.11.** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Adapt the proof of Theorem 3.2.4 to show that for any set $A$, there exists a set $G$ such that $G' \geq_T A$ and $\mathscr{C}$ admits no $G$-computable constant-bound trace. ★

**Exercise 5.6.12.** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Use computable Mathias forcing to prove that for every uniformly computable sequence of sets $\vec{R} = R_0, R_1, \dots$, there is an infinite $\vec{R}$-cohesive set $G$ such that $\mathscr{C}$ admits no $G$-computable constant-bound trace. ★

Recall that some disjunctive or tree-like forcing questions are not even $\Pi^0_1$-merging. One can generalize Exercise 5.6.10 to such notions as we did in Section 5.2.

**Definition 5.6.13.** Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas $\Gamma$, a forcing question is *weakly finitely* $\Gamma$*-merging* if for every $p \in \mathbb{P}$, there is a $d \in \mathbb{N}$ such that for every finite sequence of $\Gamma$-formulas $\varphi_0(G), \dots, \varphi_{\ell-1}(G)$, if $p \mathrel{?\vdash} \varphi_s(G)$ holds for every $s < \ell$, there is a $d$-partition $(P_t : t < d)$ of $\{0, \dots, \ell-1\}$ such that for every $t < d$, there is an extension $q \leq p$ forcing $\bigwedge_{s \in P_t} \varphi_s(G)$. ◊

The previous definition is quite technical, but contains exactly the hypothesis necessary to prove the following abstract theorem.

**Theorem 5.6.14**
*Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable c.b-trace. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a $\Sigma^0_1$-preserving weakly finitely $\Pi^0_1$-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $\mathscr{C}$ admits no $G_{\mathcal{F}}$-computable c.b-trace.*

Proof. It suffices to prove the following diagonalization lemma.

**Lemma 5.6.15.** For every condition $p \in \mathbb{P}$, every Turing index $e \in \mathbb{N}$ and every $k \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi^G_e$ not to be a $k$-trace of $\mathscr{C}$.★

Proof. Let $d \in \mathbb{N}$ witness that the forcing question is weakly finitely $\Pi^0_1$-merging for $p$. Consider the following set

$$U = \{F \in \mathscr{B}_n : n \in \mathbb{N}, p \mathrel{?\vdash} \Phi^G_e(n)\downarrow \cap F = \emptyset\}$$

Since the forcing question is $\Sigma^0_1$-preserving, the set $U$ is $\Sigma^0_1$. There are three cases:

- Case 1: there is some $n \in \mathbb{N}$ such that $\mathcal{U} \cap \mathscr{B}_n$ contains some $\mathscr{C}$-correct block $F$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi^G_e(n) \cap F = \emptyset$. In particular, $q$ forces $\Phi^G_e$ not to be a $k$-trace of $\mathscr{C}$.
- Case 2: there is some $n \in \mathbb{N}$ such that $\mathscr{B}_n \setminus \mathcal{U}$ is $k \cdot d$-disperse. Since the forcing question is weakly finitely $\Pi^0_1$-merging with witness $d$, there

is a $d$-partition $(P_t : t < d)$ of $\mathcal{B}_n \setminus \mathcal{U}$ such that for every $t < d$, there is an extension $q_t \leq p$ forcing

$$\bigwedge_{F \in P_t} \left(\Phi_e^G(n)\uparrow \vee \Phi_e^G(n) \cap F \neq \emptyset\right)$$

Let $t < d$ be such that $P_t$ is $k$-disperse.[28] Since $\Phi_e$ is a $k$-trace functional, by Exercise 5.6.5, the extension $q_t \leq p$ forces $\Phi_e^G(n)\uparrow$.

- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.6.6. ■

28: For any $d$-partition of a $k \cdot d$-disperse family, one of the parts is $k$-disperse. Indeed, otherwise, for each part $t < d$, there is a $k$-partition witnessing the failure. Putting all these $k$-partitions together, we obtain a failure of $k \cdot d$-dispersity of the family.

We are now ready to prove Theorem 5.6.14. Given $e, k \in \mathbb{N}$, let $\mathcal{D}_{e,k}$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi_e^G$ not to be a $k$-trace of $\mathscr{C}$. It follows from Lemma 5.2.5 that every $\mathcal{D}_{e,k}$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_{e,k} : e, k \in \mathbb{N}\}$-generic, so $\mathscr{C}$ admits no $G_{\mathcal{F}}$-computable c.b-trace. This completes the proof of Theorem 5.6.14. ■

Liu [30] proved that Ramsey's theorem for pairs admits constant-bound trace avoidance, following the same structure as his proof of PA avoidance, *mutatis mutandis*. We leave the steps as exercises.

**Exercise 5.6.16 (Liu [30]).** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Adapt the proof of Theorem 5.3.3 to show that for any non-empty $\Pi_1^0$ class $\mathscr{P} \subseteq 2^{\mathbb{N}}$, there exists an infinite set $H$ homogeneous for $\mathscr{P}$ such that $\mathscr{C}$ admits no $H$-computable constant-bound trace. ⋆

**Exercise 5.6.17 (Liu [30]).** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Adapt the proof of Theorem 5.4.3 using Exercise 5.6.16 to show that for any set $A$, there exists an infinite subset $H$ of $A$ or $\overline{A}$ such that $\mathscr{C}$ admits no $H$-computable constant-bound trace. ⋆

**Exercise 5.6.18 (Liu [30]).** Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Combine Exercise 5.6.12 and Exercise 5.6.17 to show that for any computable coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite $f$-homogeneous set $H \subseteq \mathbb{N}$ such that $\mathscr{C}$ admits no $H$-computable constant-bound trace. ⋆

The notion of constant-bound trace avoidance is the right invariant property strongly preserved by the pigeonhole principle to prevent it from computing a 1-trace of a closed class $\mathscr{C} \subseteq 2^{\mathbb{N}}$. Indeed, if $\mathscr{C}$ admits a computable $k$-trace $F_0, F_1, \dots$ for some $k \in \mathbb{N}$, one application of the pigeonhole principle for $k$ colors yields an infinite 1-trace of $\mathscr{C}$. This however leaves open the case of closed classes with no computable member, but admitting a computable 1-trace.

**Question 5.6.19.** Is there a natural characterization of the closed classes strongly avoided by the pigeonhole principle? ⋆

## 5.7 DNC and compactness

Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally non-computable* (DNC) if $\forall e\ f(e) \neq \Phi_e(e)$. PA degrees are those computing a $\{0,1\}$-valued DNC function. In this section, we consider the computational power of $\mathbb{N}$-valued DNC functions. We shall see that the existence of DNC functions is equivalent to a Ramsey-type form of compactness, called the Ramsey-type weak weak König's lemma. A Turing degree is *DNC* if it computes a DNC function. It is often useful to think of DNC degrees as those computing a function which can escape finite c.e. sets when a bound to their size is known.

**Proposition 5.7.1 (Bienvenu, Patey and Shafer [35]).** Let $X$ be a set. The following are equivalent:

1. $X$ computes a DNC function ;
2. $X$ computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, b \in \mathbb{N}$, if $\operatorname{card} W_e \leq b$, then $g(e, b) \notin W_e$. ⋆

Proof. $(1) \to (2)$[29]: Let $f : \mathbb{N} \to \mathbb{N}$ be a DNC function. For every $e, b \in \mathbb{N}$ and $i < b$, let $h(e, b, i)$ be the index of the partial computable function $\Phi_{h(e,b,i)}$ which on any input $x$, waits for the $i$th element $y_i$ of $W_e$ to appear, in order of apparition. It $\operatorname{card} W_e \leq i$, then the program will never terminate, and $\Phi_{h(e,b,i)}$ will be the nowhere-defined function. If $\operatorname{card} W_e > i$, then $y_i$ is eventually found. Then, interpret $y_i$ as a $b$-tuple $\langle y_i^0, \dots, y_i^{b-1} \rangle$ and output $y_i^i$. In this case, $\Phi_{h(e,b,i)}(h(e, b, i)) \downarrow= y_i^i$, and $f(h(e, b, i)) \neq y_i^i$. Let $g(e, b) = \langle f(h(e, b, 0)), \dots f(h(e, b, b-1)) \rangle$. Suppose for the contradiction that $\operatorname{card} W_e \leq b$ and $g(e, b) \in W_e$. Say $g(e, b) = y_i \in W_e$. Then $f(h(e, b, i)) = y_i^i = \Phi_{h(e,b,i)}(h(e, b, i))$, contradicting the fact that $f$ is a DNC function.

29: The idea is the following: Given a list $y_0, \dots, y_{b-1}$ of $b$ integers, interpret each integer as a $b$-tuple of integers, based on a computable bijection.

$$\begin{array}{c|cccc} y_0 & \boxed{y_0^0} & y_0^1 & \cdots & y_0^{b-1} \\ y_1 & y_1^0 & \boxed{y_1^1} & \cdots & y_1^{b-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{b-1} & y_{b-1}^0 & y_{b-1}^1 & \cdots & \boxed{y_{b-1}^{b-1}} \end{array}$$

Then, given $b$-many $b$-tuples of elements, by a diagonal argument, one can create a $b$-tuple of integers which is different from each element of this list, and re-interpret it as an integer.
The difficulty comes from the fact that the list $y_0, \dots, y_{b-1}$ is c.e., so one uses a DNC function to create this diagonal $b$-tuple.

$(2) \to (1)$: Let $g : \mathbb{N}^2 \to \mathbb{N}$ be such that for every $e, b \in \mathbb{N}$, if $\operatorname{card} W_e < b$, then $g(e, b) \notin W_e$. For every $e \in \mathbb{N}$, let $h(e)$ be an index of the partial computable function $\Phi_{h(e)}$ which, on input $x$, waits until $\Phi_e(e)\downarrow$. If $x = \Phi_e(e)\downarrow$, then the program halts, otherwise it loops forever. In other words, $W_{h(e)} = \{\Phi_e(e)\}$ if $\Phi_e(e)\downarrow$, and $W_{h(e)} = \emptyset$ otherwise. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(e) = g(h(e), 1)$ is diagonally non-computable. ■

DNC degrees can be expressed as a form of compactness as follows: The *Ramsey-type weak weak König lemma* (RWWKL) is the problem whose instances are binary trees of positive measure, and whose solutions are infinite homogeneous sets for the tree. It is a problem at the intersection between weak weak König's lemma – corresponding to the existence of random sequences – and the Ramsey-type König's lemma, – the compactness part of Ramsey's theorem for pairs.

**Proposition 5.7.2.** Let $X$ be a set. The following are equivalent:

1. $X$ computes a DNC function ;
2. Every $\Pi_1^0$ class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ of positive measure admits an infinite $X$-computable homogeneous set. ⋆

Proof. $(1) \to (2)$: Fix a $\Pi_1^0$ class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ with $\mu(\mathcal{P}) \geq 2^{-c}$ for some $c \geq 3$. Given a set $H \subseteq \mathbb{N}$, let $\mathbb{Q}_H = \{X \in 2^{\mathbb{N}} : H \subseteq X\}$, and let $\mathbb{Q}_n = \mathbb{Q}_{\{n\}}$. A finite set $F \subseteq \mathbb{N}$ is *valid* if $\mu(\mathcal{P} \cap \mathbb{Q}_F) \geq 2^{-c \cdot 2^{\operatorname{card} F}}$. Note that $\emptyset$ is valid, and that if $F$ is valid, then it is homogeneous for $\mathcal{P}$. For every finite set $F \subseteq \mathbb{N}$, let $W_{h(F)}$ be the c.e. set of all $n \in \mathbb{N}$ such that $F \cup \{n\}$ is not valid. Let $g : \mathbb{N}^2 \to \mathbb{N}$ be the function given by Proposition 5.7.1. By a measure-theoretic argument[30], for any valid set $F$, $\operatorname{card} W_{h(F)} < 2 \cdot c \cdot 2^{\operatorname{card} F}$, so $g(h(F), 2 \cdot c \cdot 2^{\operatorname{card} F}) \notin W_{h(F)}$. We can define an infinite set $H \subseteq \mathbb{N}$ such that every initial segment is valid. In particular, $H$ is homogeneous for $\mathcal{P}$.

30: If $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is a closed class with $\mu(\mathcal{C}) \geq 2^{-c}$ for some $c \geq 3$, then

$$\operatorname{card}\{n \in \mathbb{N} : \mu(\mathcal{C} \cap \mathbb{Q}_n) < 2^{-2c}\} < 2c.$$

Indeed, let $F$ be a subset of it of size $2c$ and let $\mathcal{R}_F = \{X \in 2^{\mathbb{N}} : F \cap X = \emptyset\}$. Note that

$$2^{\mathbb{N}} = \mathcal{R}_F \cup \bigcup_{n \in F} \mathbb{Q}_n$$

We have $\mu(\mathcal{C} \cap \mathcal{R}_F) \leq 2^{-2c}$, and $\mu(\mathcal{C} \cap \bigcup_{n \in F} \mathbb{Q}_n) < 2c \cdot 2^{-2c}$, so

$$2^{-c} \leq \mu(\mathcal{C}) \leq 2^{-2c} + 2c \cdot 2^{-2c}$$

which yields a contradiction when $c \geq 3$.

(2) $\to$ (1): For every $e \in \mathbb{N}$, let $\mathcal{P}_e$ be the $\Pi^0_1$ class of all elements $X$ such that if $\Phi_e(e)\downarrow$, then interpreting the output as a $(e+3)$-tuple $\langle x^0_e, \dots, x^{e+2}_e\rangle$, there is some $s < t < e+3$ such that $X(x^s_e) \neq X(x^t_e)$. Let $\mathcal{P} = \bigcap_e \mathcal{P}_e$. First, notice that for every infinite homogeneous set $H = \{y_0 < y_1 < \dots\}$ for $\mathcal{P}$, the $H$-computable function defined by $f(e) = \langle y_0, \dots, y_{e+1}\rangle$ is diagonally non-computable. Second, for every $e$, $\mu(2^{\mathbb{N}} \setminus \mathcal{P}_e) \leq 2 \cdot 2^{-e-3} = 2^{-e-2}$, so $\mu(\mathcal{P}) \geq 1 - \sum_e 2^{-e-2} = 1/2$. Thus, $\mathcal{P}$ has positive measure. ■

The Ramsey-type weak weak König lemma is a particular case of RWKL, hence follows from Ramsey's theorem for pairs. Thus, the existence of DNC functions does not imply the existence of random sequences, and a fortiori of PA degrees.

## 5.8 DNC avoidance

We now develop the techniques to prove that a problem does not imply the existence of this weak notion of compactness. The framework of closed classes avoidance of Section 5.6 admits a straightforward generalization to effectively compacts in the Baire space $\mathbb{N}^{\mathbb{N}}$. The class of $\mathbb{N}$-valued DNC functions is $\Pi^0_1$ in the Baire space, but not compact, thus it does not fall within the scope of this framework.

**Definition 5.8.1.** A problem P admits *DNC avoidance*[31] if for every pair of sets $Z$ and $D \leq_T Z$ such that $Z$ is not of DNC degree over $D$, every $Z$-computable instance $X$ of P admits a solution $Y$ such that $Y \oplus Z$ is not of DNC degree over $D$. ◊

31: Note the similarity between PA and DNC avoidance.

Due to the similar nature of $\{0,1\}$-valued and $\mathbb{N}$-valued DNC functions, proofs of DNC avoidance are very similar to those of PA avoidance.

**Exercise 5.8.2.** Adapt the proof of Theorem 5.1.3 to show that for every sufficiently Cohen generic set $G$, $G$ is not of DNC degree. ★

In the proof of PA avoidance, the $\Pi^0_1$-merging property of the forcing question is used in the second case, for forcing partiality. Since the functionals are $\{0,1\}$-valued, it suffices to merge two $\Pi^0_1$ properties simultaneously to force partiality. In the case of $\mathbb{N}$-valued functionals, infinitely many $\Pi^0_1$ properties need to be forced simultaneously.

**Definition 5.8.3.** Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas $\Gamma$, a forcing question is *countably $\Gamma$-merging* if for every $p \in \mathbb{P}$ and every countable sequence of $\Gamma$-formulas $(\varphi_s(G))_{s\in\mathbb{N}}$, if $p \mathrel{?\vdash} \varphi_s(G)$ for each $s \in \mathbb{N}$, then there is an extension $q \leq p$ forcing $\forall s \varphi_s(G)$. ◊

Being countably $\Pi^0_1$-merging is a very strong properties, satisfied by very few notions of forcing in practice. Indeed, DNC degrees being computationally very weak, many natural problems imply their existence.

**Theorem 5.8.4**
*Let $(\mathbb{P}, \leq)$ be a notion of forcing with a $\Sigma^0_1$-preserving, countably $\Pi^0_1$-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is not of DNC*

*degree.*

Proof. It suffices to prove the following lemma:

**Lemma 5.8.5.** For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi_e^G$ not to be a DNC function. ⋆

Proof. Consider the following set[32]

$$U = \{(x, v) \in \mathbb{N}^2 : p \,?\!\vdash \Phi_e^G(x)\downarrow = v\}$$

32: Note that contrary to PA avoidance, this set ranges over $\mathbb{N}\times\mathbb{N}$ instead of $\mathbb{N}\times 2$. This difference is important in Case 2, where one needs to force countably many $\Pi_1^0$ formulas simultaneously.

Since the forcing question is $\Sigma_1^0$-preserving, the set $U$ is $\Sigma_1^0$. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^G(x)\downarrow = \Phi_x(x)$.
- Case 2: there is some $x \in \mathbb{N}$ such that for every $y \in \mathbb{N}$, $(x, y) \notin U$. Since the forcing question is countably $\Pi_1^0$-merging, there is an extension $q \leq p$ forcing $\forall y \neg(\Phi_e^G(x)\downarrow = y)$, hence forcing $\Phi_e^G$ not to be a DNC function.
- Case 3: None of Case 1 and Case 2 holds. Then $U$ is a $\Sigma_1^0$ graph of a DNC function. This contradicts the fact that $\mathbf{0}$ is not DNC. ■

We are now ready to prove Theorem 5.8.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi_e^G$ not to be a DNC function. It follows from Lemma 5.8.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathcal{F}}$ is not of DNC degree. This completes the proof of Theorem 5.8.4. ■

**Exercise 5.8.6.** Adapt the proof of Theorem 3.2.4 to show that for any set $A$, there exists a set $G$ such that $G' \geq_T A$ and $G$ is not of DNC degree. ⋆

## 5.9 Comparing avoidances

We have seen in Sections 3.5 and 3.6 that cone avoidance coincides with other preservation notions, such as preservation of 1 non-$\Sigma_1^0$ definition and of 1 hyperimmunity. Cone avoidance does not imply PA avoidance, as WKL satisfies the former, but not the latter. On the other hand, one can prove that PA avoidance implies cone avoidance. For this, we need the following theorem, which informally says that the computational distance between a set and its Turing jump can be any non-zero Turing degree.

**Theorem 5.9.1 (Posner and Robinson [36])**
*Let $A$ be a non-computable set. There exists a set $G$ such that $A \oplus G \geq_T G'$.*

33: One can modify the construction to encode any set $Z$ instead of $\emptyset'$. The construction is then $A \oplus Z \oplus \emptyset'$-computable. This generalization is due to Jockusch and Shore [37].

Proof. The idea is to build a 1-generic set $G$, which will encode $\emptyset'$[33], so that $G$ and $A$ allow to find the construction sequence. The construction itself will be computable in $A \oplus \emptyset'$. We can assume without loss of generality that $A$ is not a c.e. set (otherwise, one replaces $A$ by its complement). Let $(W_e)_{e\in\mathbb{N}}$ be an enumeration of the $\Sigma_1^0$ subsets of $2^{<\mathbb{N}}$.

Let $\sigma_0 = \epsilon$, the empty word. Suppose $\sigma_e$ defined. Consider the set

$$D_e = \{m : \exists\tau \text{ such that } \sigma_e\emptyset'(e)0^m1\tau \in W_e\}.$$

Note that $D_e$ is a c.e. set. In particular as $A$ is not c.e. there is some $m \in D_e$ with $m \notin A$ or some $m \notin D_e$ with $m \in A$. Consider the smallest $m$ such that we are in one case or the other. Note that $\emptyset' \oplus A$ allows to find uniformly this integer $m$.

In the first case, let $\sigma_{e+1} = \sigma_e\emptyset'(e)0^m1\tau$ for the first string $\tau$ such that $\sigma_e\emptyset'(e)0^m1\tau$ is listed in $W_e$. In the second case, let $\sigma_{e+1} = \sigma_e\emptyset'(e)0^m1$. Note that in this case no string of $W_e$ can extend $\sigma_{e+1}$. We define $G$ as being $\sigma_0 \preceq \sigma_1 \preceq \sigma_2 \preceq \dots$. This completes the construction.

It is clear that $G$ is 1-generic and computable in $A \oplus \emptyset'$. How do you now compute $\emptyset'$ from $G \oplus A$? Suppose we know the string $\sigma_e$. We then necessarily know the $e$-th bit of $\emptyset'$: it is the bit $i$ such that $\sigma_e i \prec G$. We can then find $\sigma_{e+1}$ as follows: we look at the number $m$ of 0 which follows $\sigma_e i$ in $G$. If $m \in A$, this means that $\sigma_{e+1} = \sigma_e i0^m1$. If $m \notin A$, this means that $\sigma_{e+1} = \sigma_e i0^m1\tau$ for the first string $\tau$ found in $W_e$. Finding this string $\tau$ is then a computable process. We can therefore in all cases find $\sigma_{e+1}$, and by repeating the process, compute $\emptyset'$ from $A \oplus G$. Thus, $G \oplus \emptyset' \leq_T G \oplus A$. Since every 1-generic set is generalized low, then $G' \leq_T G \oplus A$. ■

**Corollary 5.9.2**
*If a problem* P *admits PA avoidance, then it admits cone avoidance.*

Proof. Fix a set $Z$, a non-$Z$-computable set $C$ and a P-instance $X \leq Z$. By Theorem 5.9.1 relativized to $Z$, there is a set $G$ such that $C \oplus Z \oplus G \geq_T (Z \oplus G)'$. Since P admits PA avoidance, there is a solution $Y$ to $X$ such that $Y \oplus Z \oplus G$ is not of PA degree over $Z \oplus G$. In particular, $Y \oplus Z \not\geq_T C$, otherwise $Y \oplus Z \oplus G \geq_T C \oplus Z \oplus G \geq_T (Z \oplus G)'$, but $(Z \oplus G)'$ is of PA degree over $Z \oplus G$. ■

Constant-bound trace avoidance generalizes PA avoidance, since the $\Pi^0_1$ class of $\{0,1\}$-valued DNC functions does not admit any computable constant-bound trace. On the other hand, some problems such as WWKL admit PA avoidance, but not constant-bound trace avoidance. Indeed, there is a $\Pi^0_1$ class of positive measure with no computable constant-bound trance.

An infinite set $X \subseteq \mathbb{N}$ is *immune* iff it has no computable infinite subset, or equivalently no c.e. infinite subset. We have already seen a strong form of immunity, namely, hyperimmunity, for which one cannot even approximate an infinite subset by pairwise disjoint blocks of finite sets.

**Definition 5.9.3.** A problem P admits *preservation of 1 immunity* if for every set $Z$ and every $Z$-immune set $I$, every $Z$-computable instance $X$ of P admits a solution $Y$ such that $I$ is $Z \oplus Y$-immune. ◊

As for DNC avoidance, the existence of a $\Sigma^0_1$-preserving, countably $\Pi^0_1$-merging forcing question is sufficient to prove preservation of 1 immunity.

**Theorem 5.9.4**
*Fix an infinite immune set $I$. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a $\Sigma^0_1$-*

*preserving, countably $\Pi^0_1$-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $I$ is $G_{\mathcal{F}}$-immune.*

Proof. It suffices to prove the following lemma:

**Lemma 5.9.5.** For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $W_e^G$ not to be an infinite subset of $I$. ⋆

Proof. Consider the following set

$$U = \{x \in \mathbb{N} : p \mathrel{?\vdash} x \in W_e^G\}$$

Since the forcing question is $\Sigma^0_1$-preserving, the set $U$ is $\Sigma^0_1$. There are three cases:

- Case 1: $x \in U \setminus I$ for some $x \in \mathbb{N}$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $x \in W_e^G$, hence forcing $W_e^G \nsubseteq I$.
- Case 2: $U$ is finite. Since the forcing question is countably $\Pi^0_1$-merging, there is an extension $q \leq p$ forcing $\forall x \notin U\ x \notin W_e^G$, hence forcing $W_e^G$ to be finite.
- Case 3: $U$ is an infinite c.e. subset of $I$. This contradicts the immunity of $I$. ■

We are now ready to prove Theorem 5.9.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $W_e^G$ not to be an infinite subset of $I$. It follows from Lemma 5.9.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $I$ is $G_{\mathcal{F}}$-immune. This completes the proof of Theorem 5.9.4. ■

There exists some problems, such as the Ascending Descending sequence principle (ADS) which admits DNC avoidance, but not preservation of 1 immunity. This naturally raises the following question:

**Question 5.9.6.** Does preservation of 1 immunity imply DNC avoidance? ⋆