Jump compactness avoidance 10

Prerequisites: Chapters 2 to 5 and 9

Jump compactness avoidance combines the complexity of two orthogonal problematics, namely, second-jump control and compactness avoidance. As one shall expect, from a purely abstract viewpoint, it can be reduced to the design of a forcing question for Σ^0_2 formulas with the appropriate merging properties. However, in real world applications, such as variants of Mathias forcing in reverse mathematics, both techniques do not necessarily combine well, adding an extra layer of complexity.

10.1 Context and motivation

Jump PA avoidance plays a particularly important role in reverse mathematics, due to its connections with the cohesiveness principle. Recall from Section 3.4 that an infinite set $C \subseteq \mathbb{N}$ is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$, where $\subseteq^*$ means "included up to finite changes". The *cohesiveness principle* is the problem COH whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets.

As mentioned in Chapter 9, COH should be considered as a statement about jump computation, as it is computably equivalent[1] to the statement "For every Δ^0_2 infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, there is a Δ^0_2-approximation of an infinite path." There exists a uniformly computable sequence of sets[2] such that the degrees of its cohesive sets are exactly those whose jump is PA over $\emptyset'$. Such an instance is *maximal*, in the sense that every solution to this instance compute a solution to every other computable instance. Moreover, for every set P of PA degree over $\emptyset'$, there exists an ω-model $\mathcal{M}$ of COH such that for every $X \in \mathcal{M}$, $X' \leq_T P$. Therefore, separating a problem from COH over ω-models can be reduced without loss of generality to jump PA avoidance.

1: This equivalence also holds over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, but not RCA_0 alone. Indeed, RCA_0 + COH is Π^1_1-conservative over RCA_0 (Exercise 7.3.14), while by Fiori-Carones et al. [62, Proposition 4.4], the other statement implies $\mathsf{B}\Sigma^0_2$ over RCA_0.

2: Actually, it suffices to consider the sequence of all primitive recursive sets.

Definition 10.1.1. A problem P admits *jump PA avoidance*[3] if for every pair of sets Z and $D \leq_T Z$ such that Z' is not of PA degree over D', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)'$ is not of PA degree over D'.[4] ◊

3: As usual, the unrelativized formulation with $Z = D = \emptyset$ is far more natural, but does not behave well with artificial problems.

4: One can also define the notion of strong jump PA avoidance, by considering arbitrary instances of P instead of Z-computable ones.

The cohesiveness principle can be considered as a sequential version of the pigeonhole principle. An instance is a countable sequences of instances of RT^1_2, that is, a countable sequence of sets $R_0, R_1, \dots$, and a solution is a single set which is, up to finite changes, a solution to every R_n. One can define a similar statement capturing the degrees whose jump are DNC over $\emptyset'$, in terms of the *thin set theorem*. The thin set theorem for n-tuples (TS^n) is a statement introduced by Friedman, whose instances are colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite sets $H \subseteq \mathbb{N}$ such that $f[H]^n \neq \mathbb{N}$. Such sets are called *f-thin*.

Exercise 10.1.2 (Patey [88]). Given a uniformly computable sequence $\vec{g} = g_0, g_1, \dots$ of functions of type $\mathbb{N} \to \mathbb{N}$, an infinite set $C \subseteq \mathbb{N}$ is *thin $\vec{g}$-cohesive* if for every $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $C \setminus [0, k]$ is g_n-thin.

1. Let $\vec{f} = f_0, f_1, \ldots$ be the sequence of all primitive recursive functions of type $\mathbb{N} \to \mathbb{N}$. Show that for every infinite thin $\vec{f}$-cohesive set C, C' is of DNC degree over $\emptyset'$.
2. Let $\vec{g} = g_0, g_1, \ldots$ be a uniformly computable sequence of functions of type $\mathbb{N} \to \mathbb{N}$ and D be a set whose jump is of DNC degree over $\emptyset'$. Show that D computes an infinite thin $\vec{g}$-cohesive set. ★

The degrees whose jump are DNC over $\emptyset'$ received less attention than their PA counterpart, but can be used to prove separations over another well-known statement: the rainbow Ramsey theorem for pairs. A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is k-*bounded* if for every $c \in \mathbb{N}$, $f^{-1}(c)$ has size at most k. A set $H \subseteq \mathbb{N}$ is an f-*rainbow* if f is injective on $[H]^n$, that is, each color is used at most once. The *rainbow Ramsey theorem* for n-tuples and k-bounded colorings (RRT^n_k) is the problem whose instances are k-bounded colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite f-rainbows.

Exercise 10.1.3 (Miller). Construct a computable 2-bounded coloring $f : [\mathbb{N}]^2 \to \mathbb{N}$ such that for every $\emptyset'$-c.e. set $W_e^{\emptyset'}$, if $\operatorname{card} W_e^{\emptyset'} \geq 2e+2$, then $W_e^{\emptyset'}$ is not extendible into an infinite f-rainbow. Deduce that every infinite f-rainbow is of DNC degree over $\emptyset'$.[5] ★

5: This uses the characterization of DNC degrees in terms of effectively immune functions. See Section 6.2 for more details. Miller actually proved a reversal: for every computable k-bounded coloring $f : [\mathbb{N}]^2 \to \mathbb{N}$, every DNC function over $\emptyset'$ computes an infinite f-rainbow.

It follows that if a problem P admits jump DNC avoidance in the following sense, then there is an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ which is not a model of RRT^2_2.

Definition 10.1.4. A problem P admits *jump DNC avoidance* if for every pair of sets Z and $D \leq_T Z$ such that Z' is not of DNC degree over D', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)'$ is not of DNC degree over D'. ◊

10.2 Jump PA avoidance

As explained, the pure theory of jump compactness avoidance is a simple adaptation of the techniques of compactness avoidance to Σ^0_2 formulas. In this section, we give two examples with Cohen genericity and tree forcing for the sake of concreteness, and then state the general abstract theorem, leaving its proof as an exercise.

Theorem 10.2.1
For every sufficiently Cohen generic set G, G' is not of PA degree over $\emptyset'$.

Proof. Consider Cohen forcing, that is, the set $2^{<\mathbb{N}}$ of binary strings, partially ordered by the prefix relation. We defined in Section 9.3 a forcing question for Σ^0_2 formulas.

Definition 10.2.2. Let σ be a Cohen condition, and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Define $\sigma \operatorname{?\vdash} \varphi(G)$ to hold if there exists some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that τ strongly forces $\psi(G, x)$, that is, for every $\rho \geq \tau$, $\psi(\rho, x)$ holds. ◊

This forcing question satisfies a strong version of its specifications, that is, if $\sigma \operatorname{?\vdash} \varphi(G)$ does not hold, then σ itself already forces $\neg\varphi(G)$. It follows that, given two Σ^0_2-formulas $\varphi_0(G)$ and $\varphi_1(G)$, if none of $\sigma \operatorname{?\vdash} \varphi_i(G)$ holds, then σ forces $\neg\varphi_0(G) \wedge \neg\varphi_1(G)$. This property is exploited in the following lemma:

Lemma 10.2.3. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$.[6] ⋆

6: Recall that a degree is PA iff it computes a $\{0,1\}$-valued DNC function. This equivalence also holds relative to any oracle.

Proof. Consider the following set:

$$U = \{(x, v) \in \mathbb{N} \times 2 : \sigma \mathrel{?\vdash} \Phi_e^{G'}(x)\downarrow = v\}$$

Since the forcing question is Σ^0_2-preserving, the set U is Σ^0_2. There are three cases:

- Case 1: $(x, \Phi_x^{\emptyset'}(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x^{\emptyset'}(x)\downarrow$. By Property (1) of the forcing question, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G'}(x)\downarrow = \Phi_x^{\emptyset'}(x)$.
- Case 2: there is some $x \in \mathbb{N}$ such that $(x, 0), (x, 1) \notin U$. Then σ already forces $\neg(\Phi_e^{G'}(x)\downarrow = 0)$, $\neg(\Phi_e^{G'}(x)\downarrow = 1)$, so σ forces $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_2 graph of a $\{0,1\}$-valued DNC function over $\emptyset'$. This contradicts the fact that $\mathbf{0}'$ is not PA over $\emptyset'$. ■

We are now ready to prove Theorem 10.2.1. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $\sigma \in 2^{<\mathbb{N}}$ forcing $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$. It follows from Lemma 10.2.3 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$. This completes the proof of Theorem 10.2.1. ■

If a problem P admits a low basis, then it admits jump PA avoidance. Thus, by the low basis theorem for Π^0_1 classes (Theorem 4.4.6), there exists a PA degree which is low, hence whose jump is not PA over $\emptyset'$. More generally, as explained in Section 9.2, it is preferable to use an effective first-jump construction rather than a second-jump one when available, as the former usually involves a simpler machinery.

Although WKL admits a low basis, it is sometimes necessary to use a forcing construction with a second-jump control, when trying for example to preserve a first-jump and second-jump property simultaneously, as it was the case for Theorem 9.4.1. We now prove that WKL can simultaneously avoid a cone, and have a jump of non-PA degree over $\emptyset'$.

Theorem 10.2.4
Let C be a non-computable set. For every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists a member $G \in \mathcal{P}$ such that $C \not\leq_T G$ and G' is not of PA degree over $\emptyset'$.

Proof. The proof is an adaptation of Theorem 9.4.1, using the same notion of forcing and the same forcing question. More precisely, we use a restriction of the Jockusch-Soare forcing to infinite *primitive recursive* binary trees, partially ordered by the inclusion relation. By Lemma 9.4.2, every Π^0_1 class in $2^{\mathbb{N}}$ can be represented as the class of paths of a primitive recursive binary tree.

The forcing question for Σ^0_1-formulas is the same as in Exercise 3.3.7 and Theorem 9.4.1. We recall it for the sake of completeness.

Definition 10.2.5. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_1 formula $\varphi(G)$, define $T \operatorname{?}\!\vdash \varphi(G)$ to hold if there is some level $\ell \in \mathbb{N}$ such that $\varphi(\sigma)$ holds for every node σ at level ℓ in T. ◊

This forcing question is Σ^0_1-preserving and admits strong properties: if $T \operatorname{?}\!\vdash \varphi(G)$, then σ already forces $\varphi(G)$. On the other hand, if $T \operatorname{?}\!\nvdash \varphi(G)$, then one must restrict T to an infinite primitive recursive sub-tree S in order to force $\neg\varphi(G)$ (see Lemma 9.4.4). By Theorem 3.3.4 for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$.

Definition 10.2.6. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $T \operatorname{?}\!\vdash \varphi(G)$ to hold if there is some $x \in \mathbb{N}$ and an extension $S \leq T$ such that $S \operatorname{?}\!\vdash \psi(G, x)$. ◊

The forcing question for Σ^0_2-formulas is Σ^0_2-preserving, and also satisfies strong properties, but on Π^0_2-formulas rather than Σ^0_2-formulas. By Lemma 9.4.6, if $T \operatorname{?}\!\nvdash \varphi(G)$, then T already forces $\neg\varphi(G)$. This property, similar to the case of Cohen forcing, is exploited to prove the following lemma:

Lemma 10.2.7. For every condition T and every Turing index $e \in \mathbb{N}$, there is an extension $S \subseteq T$ forcing $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$. ★

Proof. Consider the following set:

$$U = \{(x, v) \in \mathbb{N} \times 2 : T \operatorname{?}\!\vdash \Phi_e^{G'}(x)\downarrow = v\}$$

Since the forcing question is Σ^0_2-preserving, the set U is Σ^0_2. There are three cases:

- Case 1: $(x, \Phi_x^{\emptyset'}(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x^{\emptyset'}(x)\downarrow$. By Property (1) of the forcing question, there is an extension $S \subseteq T$ forcing $\Phi_e^{G'}(x)\downarrow = \Phi_x^{\emptyset'}(x)$.
- Case 2: there is some $x \in \mathbb{N}$ such that $(x, 0), (x, 1) \notin U$. Then T already forces $\neg(\Phi_e^{G'}(x)\downarrow = 0) \wedge \neg(\Phi_e^{G'}(x)\downarrow = 1)$, so T forces $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_2 graph of a $\{0,1\}$-valued DNC function over $\emptyset'$. This contradicts the fact that $\mathbf{0}'$ is not PA over $\emptyset'$. ■

Putting all the pieces together, for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$ by Theorem 3.3.4, and $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$ by Lemma 10.2.7. This completes the proof of Theorem 10.2.4. ■

Recall from Section 5.1 that given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*merging* if for every $p \in \mathbb{P}$ and every pair of Γ-formulas $\varphi_0(G), \varphi_1(G)$, if $p \operatorname{?}\!\vdash \varphi_0(G)$ and $p \operatorname{?}\!\vdash \varphi_1(G)$ both hold, then there is an extension $q \leq p$ forcing $\varphi_0(G) \wedge \varphi_1(G)$.

Exercise 10.2.8. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_2-preserving Π^0_2-merging forcing question. Adapt the proof of Theorem 5.1.9 to show that for every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$. ★

10.3 Mathias forcing and COH

Solutions to Ramsey-type theorems are usually built using variants of Mathias forcing. As seen in Proposition 9.5.1, Mathias-like notions of forcing tend to produce sets of high degree when the reservoirs are only under computability-theoretic restrictions. Indeed, by considering sufficiently sparse reservoirs, one can ensure that the principal function[7] generic set G eventually dominates every total computable function. By Martin's domination theorem, these sets are of high degree.

7: Recall that the *principal function* of an infinite set $X = \{x_0 < x_1 < \dots\}$ is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto x_n$.

We therefore developed in Section 9.6 a framework of partition regularity, yielding variants of Mathias forcing enjoying many of the combinatorial features of Mathias forcing, but with a good second-jump control.[8] Recall that a class $\mathscr{P} \subseteq 2^{\mathbb{N}}$ is *partition regular* if it is non-empty, it is closed under superset, and for every $X \in \mathscr{P}$ and every 2-cover $Y_0 \cup Y_1 \supseteq X$, there is some $i < 2$ such that $Y_i \in \mathscr{P}$. The idea is to work with Mathias conditions (σ, X) such that $X \in \mathscr{P}$, where $\mathscr{P}$ is a partition regular class containing only "non-sparse" infinite sets.

8: The reader must be familiar with Section 9.6 to understand the remainder of this section.

Restricting the reservoirs to a well-chosen partition regular class enabled to prevent the reservoirs from being too sparse, while still allowing the basic operations on reservoirs, such as finite truncation, or finite partitioning. Unfortunately, although this restriction is sufficient to obtain strong jump cone avoidance, there is no hope of obtaining jump PA avoidance using a notion of forcing which allows finite partitioning of the reservoir.

Proposition 10.3.1. Fix a partition regular class $\mathscr{P} \subseteq 2^{\mathbb{N}}$. Let $\mathbb{P}$ be the restriction of computable Mathias forcing where reservoirs belong to $\mathscr{P}$. For every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is of PA degree over $\emptyset'$. ⋆

Proof. Fix a uniformly computable sequence of sets $R_0, R_1, \dots$ such that for every infinite $\vec{R}$-cohesive set C, C' is of PA degree over $\emptyset'$. We claim that for every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is $\vec{R}$-cohesive. Indeed, given a condition (σ, X) and some n, either $X \cap R_n$, or $X \cap \overline{R}_n$ belongs to $\mathscr{P}$, so either $(\sigma, X \cap R_n)$ or $(\sigma, X \cap \overline{R}_n)$ is a valid extension. Any sufficiently generic filter $\mathcal{F}$ containing the former (latter) extension satisfies $G_{\mathcal{F}} \subseteq^* R_n$ ($G_{\mathcal{F}} \subseteq^* \overline{R}_n$). ■

The previous proposition can be considered as a sanity check, but does not help designing an appropriate notion of forcing. In order to better understand the problem, let us consider the forcing question for Σ^0_2-formulas for the most basic variant of Mathias forcing with a good second-jump control. For this, we need to reintroduce some pieces of notation from Section 9.6.

Letting $W^Z_0, W^Z_1, \dots$ be the list of all Z-c.e. sets of strings, this induce a list $\mathcal{U}^Z_0, \mathcal{U}^Z_1, \dots$ of all $\Sigma^0_1(Z)$ classes of sets, upward-closed by inclusion, as follows: $\mathcal{U}^Z_e = \{X : (\exists \rho \in W^Z_e)\rho \subseteq X\}$. Fix a countable Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$, coded by a set $M = \bigoplus_n Z_n$. Any set $X \in \mathcal{M}$ is represented by an integer $a \in \mathbb{N}$ such that $X = Z_a$. We then say that a is an M*-code* of X. One will consider exclusively partition regular classes of the form $\mathcal{U}^{\mathcal{M}}_C = \bigcap_{(e,i) \in C} \mathcal{U}^{Z_i}_e$, for some set of indices $C \subseteq \mathbb{N}^2$.

Thinking of a partition regular class as a "reservoir of reservoirs", the smaller the partition regular class is, the more positive information it imposes on the reservoirs. The idea is therefore to fix a maximal set of indices $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}^{\mathcal{M}}_C$ is partition regular. Such a class is then called $\mathcal{M}$*-minimal*. Consider the notion of forcing whose conditions are pairs (σ, X), where $X \in \mathcal{U}_C^{\mathcal{M}}$ and $X \in \mathcal{M}$, and whose extension is usual Mathias extension. The forcing question for Σ^0_2-formulas is defined as follows:

Definition 10.3.2. Given a condition (σ, X) and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $(\sigma, X) \,?\!\vdash \varphi(G)$ to hold if there is some finite $\rho \subseteq X$ and some $x \in \mathbb{N}$ such that the following class is not large[9]

$$\mathcal{U}_C^{\mathcal{M}} \cap \{Z : \exists \eta \subseteq Z \neg\psi(\sigma \cup \rho \cup \eta, x)\}$$

9: Recall that a class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *large* if it is upward-closed, and for every $k \in \mathbb{N}$ and every k-cover $Y_0 \cup \cdots \cup Y_{k-1} = \mathbb{N}$, there is some $i < k$ such that $Y_i \in \mathcal{A}$. By Proposition 9.6.10, an upward-closed class $\mathcal{A}$ is large iff it contains a partition regular subclass. An arbitrary union of partition regular classes being partition regular, $\mathcal{A}$ contains a maximal partition regular subclass, written $\mathcal{L}(\mathcal{A})$.

This forcing question is $\Sigma^0_1(M' \oplus C)$ and Π^0_2-merging, which is almost sufficient to apply Exercise 10.2.8. However, even in the case where the Scott set $\mathcal{M}$ is coded by a set of low degree, the natural algorithm to build an $\mathcal{M}$-minimal class $\mathcal{U}_C^{\mathcal{M}}$ produces a Δ^0_3 set of indices C (see Proposition 9.6.19), yielding a Σ^0_3 forcing question for Σ^0_2-formulas. In the case of jump cone avoidance, we circumvented this problem by considering a weaker notion of minimality, called *$\mathcal{M}$-cohesiveness*. By Proposition 9.6.25, PA degrees over M' are sufficient (and necessary[10]) to compute a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive, which is sufficient to obtain a diagonalization lemma by the cone avoidance basis theorem.

10: Le Houérou, Levy Patey and Mimouni [83, Lemma 4.15] gave a direct proof of the necessity of PA degrees over M', but there is a less direct argument: if there were an $\mathcal{M}$-cohesive class $\mathcal{U}_C^{\mathcal{M}}$ with $C \oplus M'$ of non-PA degree over $\emptyset'$, then one would be able to construct an infinite cohesive set whose jump is not of PA degree over $\emptyset'$, yielding a contradiction.

In the case of jump PA avoidance, however, having a Π^0_2-merging forcing question for Σ^0_2-formulas which is Σ^0_1 relative to a PA degree over $\emptyset'$ is not sufficient to apply Exercise 10.2.8. One must therefore give up the notions of $\mathcal{M}$-minimality and $\mathcal{M}$-cohesiveness, and work with evolving partition regular classes. Consider therefore a new notion of forcing, whose conditions are of the form (σ, X, C), where

1. (σ, X) is a Mathias condition;
2. $\mathcal{U}_C^{\mathcal{M}}$ is a partition regular subclass of $\mathcal{L}_X$;[11]
3. $X \in \mathcal{M}$ and $M' \oplus C$ is not of PA degree over $\emptyset'$.

11: Recall that

$$\mathcal{L}_X = \{Z : Z \cap X \text{ is infinite}\}$$

If one only asked X to belong to $\mathcal{U}_C^{\mathcal{M}}$, then by considering a partition regular subclass $\mathcal{U}_D^{\mathcal{M}} \subseteq \mathcal{U}_C^{\mathcal{M}}$, X might no belong to $\mathcal{U}_D^{\mathcal{M}}$, so (σ, X, D) would not be a valid extension. Requiring that $\mathcal{U}_C^{\mathcal{M}}$ is a partition regular subclass of $\mathcal{L}_X$ is a way to strongly ensure that X will belong to all partition regular subclasses of $\mathcal{U}_C^{\mathcal{M}}$.

A condition (τ, Y, D) *extends* (σ, X, C) if (τ, Y) Mathias extends (σ, X) and $D \supseteq C$. The latter constraint ensures that $\mathcal{U}_D^{\mathcal{M}} \subseteq \mathcal{U}_C^{\mathcal{M}}$, so the partition regular class becomes more and more restrictive during the construction. The new forcing question for Σ^0_2-formulas can be defined as follows:

Definition 10.3.3. Given a condition (σ, X, C) and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $(\sigma, X, C) \,?\!\vdash \varphi(G)$ to hold if the following class is not large[12]

$$\mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq X} \{Z : \exists \eta \subseteq Z \neg\psi(\sigma \cup \rho \cup \eta, x)\}$$

12: This forcing question coincides with Definition 10.3.2 in the case $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive by Lemma 9.6.23. However, in the more general case of an arbitrary partition regular class, one must use the latter formulation.

This new forcing question is again $\Sigma^0_1(M' \oplus C)$, but letting M be of low degree, one can ensure that $M' \oplus C \equiv_T \emptyset'$, hence that the forcing question is Σ^0_2-preserving. This improved complexity is at one cost: the new forcing question is not Π^0_2-merging. Indeed, suppose $(\sigma, X, C) \,?\!\nvdash \varphi(G)$, then letting $D \supseteq C$ be a set of indices such that

$$\mathcal{U}_D^{\mathcal{M}} = \mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq X} \{Z : \exists \eta \subseteq Z \neg\psi(\sigma \cup \rho \cup \eta, x)\}$$

the condition (σ, X, D) is an extension of (σ, X, C) forcing $\neg\varphi(G)$. However, suppose that $\varphi_0(G) \equiv \exists x \psi_0(G, x)$ and $\varphi_1(G) \equiv \exists x \psi_1(G, x)$ be two Σ^0_2-formulas, if $(\sigma, X, C) \,?\!\nvdash \varphi_i(G)$ for both $i < 2$, then letting $D_i \supseteq C$ be the corresponding set of indices for each $i < 2$, it might be that $\mathcal{U}_{D_0}^{\mathcal{M}}$ and $\mathcal{U}_{D_1}^{\mathcal{M}}$ are both partition regular, but $\mathcal{U}_{D_0 \cup D_1}^{\mathcal{M}} = \mathcal{U}_{D_0}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}}$ is not, and therefore one cannot choose $(\sigma, X, D_0 \cup D_1)$ as the desired extension. Again, by Proposition 10.3.1, this notion of forcing cannot admit a forcing question with the right properties, as it produces cohesive sets. One must therefore modify the notion of forcing.

The solution consists of keeping both partition regular classes $\mathcal{U}_{D_0}^{\mathcal{M}}$ and $\mathcal{U}_{D_1}^{\mathcal{M}}$ even if they are incompatible, and commit to preserve the positive information from both classes. Concretely, $\mathcal{U}_{D}^{\mathcal{M}} = \mathcal{U}_{D_0}^{\mathcal{M}} \times \mathcal{U}_{D_1}^{\mathcal{M}}$ is a class over $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is partition regular in the following sense: for every $(X_0, X_1) \in \mathcal{U}_{D}^{\mathcal{M}}$, for every $Z_0 \cup Z_1 \supseteq X_0$ and $R_0 \cup R_1 \supseteq X_1$, there is some $i, j < 2$ such that $(Z_i, R_j) \in \mathcal{P}$. We shall therefore obtain a generalized condition[13] of the form (σ, X_0, X_1, D), where X_0, X_1 are two reservoirs and $\mathcal{U}_{D}^{\mathcal{M}}$ is a partition regular class over $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is a sub-class of

$$\mathcal{L}_{X_0, X_1} = \{(Z_0, Z_1) : X_0 \cap Z_0 \text{ and } X_1 \cap Z_1 \text{ are both infinite}\}$$

Because the forcing question will be used multiple times, the dimension of the product space will increase over conditions extensions. Moreover, we shall manipulate partition regular classes over product spaces which cannot be expressed as the cartesian product of partition regular classes over $2^{\mathbb{N}}$. We therefore need to develop the framework of product partition regularity.

13: Generalizing Mathias conditions to multiple reservoirs is a way to get rid of the issue of Proposition 10.3.1. Indeed, if (σ, X_0, X_1, D) is a condition, and R is a set, then maybe neither $(\sigma, X_0 \cap R, X_1 \cap R, D)$ nor $(\sigma, X_0 \cap \overline{R}, X_1 \cap \overline{R}, D)$ will be a valid extension, so this notion of forcing does not produce in general cohesive sets.

10.4 Product largeness

The theory of product partition regularity is a fairly straightforward generalization of standard partition regularity and will therefore not receive as a detailed development as in Section 9.6. In particular, many proofs will be left as exercise. In what follows, fix a finite set I, which will serve as the index set[14] of the product space. We shall therefore work with sub-classes of $I \to 2^{\mathbb{N}}$.[15] Elements of the set I will be denoted ν or μ, which for now can be thought of as integers, but later will be better represented as strings.

14: From now on, we shall use *index set* to denote the set of indices in the product space. This should not be confused with the set $C \subseteq \mathbb{N}^2$ of indices representing the class $\mathcal{U}_{C}^{\mathcal{M}}$.

15: The reason we do not use $I = \{0, \dots, n-1\}$ and work with products of the form $2^{\mathbb{N}} \times \dots \times 2^{\mathbb{N}}$ will become apparent in the next section, where we will use a hierarchy of index sets forming a tree structure.

One could define partition regularity for product classes, yielding a well-behaving generalization of partition regularity over $2^{\mathbb{N}}$. However, in the next sections, all the necessary combinatorics can be formulated in terms of largeness rather than partition regularity. We shall therefore solely introduce largeness for product classes, to reduce the number of concepts.

Definition 10.4.1. A class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ is *large*[16] if

1. For all $\langle X_\nu : \nu \in I \rangle \in \mathcal{A}$ and $Y_\nu \supseteq X_\nu$, then $\langle Y_\nu : \nu \in I \rangle \in \mathcal{A}$.[17]
2. For every $k \in \mathbb{N}$ and every k-cover $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$, there is some $j : I \to k$ such that $\langle Y_{j(\nu)} : \nu \in I \rangle \in \mathcal{A}$. ◇

16: When I is a singleton, this corresponds to standard largeness over $2^{\mathbb{N}}$.

17: We use the notation $\langle X_\nu : \nu \in I \rangle$ to represent an element of $I \to 2^{\mathbb{N}}$. Any such element can be coded by an element of $2^{\mathbb{N}}$.

The following fundamental lemma generalizes Exercise 9.6.13 and plays an important role in the effective theory of large classes:

Lemma 10.4.2 (Monin and Patey [78]). Suppose $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence of large classes. Then $\bigcap_s \mathcal{A}_s$ is large. ★

Proof. If $\langle X_\nu : \nu \in I \rangle \in \bigcap_s \mathcal{A}_s$ and $Y_\nu \supseteq X_\nu$ for every $\nu \in I$, then for every s, since $\mathcal{A}_s$ is large, $\langle Y_\nu : \nu \in Y \rangle \in \mathcal{A}_s$, so $\langle Y_\nu : \nu \in Y \rangle \in \bigcap_s \mathcal{A}_s$. Let

$Y_0 \cup \cdots \cup Y_k = \mathbb{N}$ for some $k \in \mathbb{N}$. For every $s \in \mathbb{N}$, by largeness of $\mathcal{A}_s$, there is some $j : I \to k$ such that $\langle Y_{j(\nu)} : \nu \in I \rangle \in \mathcal{A}_s$. By the infinite pigeonhole principle, there is some $j : I \to k$ such that $\langle Y_{j(\nu)} : \nu \in I \rangle \in \mathcal{A}_s$ for infinitely many s. Since $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence, $\langle Y_{j(\nu)} : \nu \in I \rangle \in \bigcap_s \mathcal{A}_s$. ■

Recall that for every infinite set $X \in 2^{\mathbb{N}}$, the class $\mathcal{L}_X = \{Y : X \cap Y \text{ is infinite }\}$ is partition regular. We generalize the definition to product classes.

Definition 10.4.3. Given $\langle X_\nu : \nu \in I \rangle$, let

$$\mathcal{L}_{\langle X_\nu : \nu \in I \rangle} = \{\langle Y_\nu : \nu \in I \rangle : \forall \nu \in I, Y_\nu \cap X_\nu \text{ is infinite}\}$$

The following easy exercise simply states that the definition is invariant under finite modifications of the sets.

Exercise 10.4.4 (Monin and Patey [78]). Let $\langle X_\nu : \nu \in I \rangle$ and $\langle Y_\nu : \nu \in I \rangle$ be such that $X_\nu =^* Y_\nu$[18] for every $\nu \in I$. Then $\mathcal{L}_{\langle X_\nu : \nu \in I \rangle} = \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$. ★

18: The notation $X =^* Y$ means that X and Y are equal *up to finite changes*.

In general, $\mathcal{L}_X \cap \mathcal{L}_Y \supsetneq \mathcal{L}_{X \cap Y}$ for infinite sets X, Y. For instance, if X and Y are the sets of all odd and even numbers, respectively, then $\mathbb{N} \in \mathcal{L}_X \cap \mathcal{L}_Y$ but $\mathcal{L}_{X \cap Y} = \emptyset$. On the other hand, if $\mathcal{L}_X \cap \mathcal{L}_Y$ is large, then so is $\mathcal{L}_{X \cap Y}$. The following lemma generalizes this property.

Lemma 10.4.5 (Monin and Patey [78]). Let $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ be a large class and $\langle X_\nu : \nu \in I \rangle$, $\langle Y_\nu : \nu \in I \rangle$ be two tuples. If $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \cap \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$ is large, then so is $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu \cap Y_\nu : \nu \in I \rangle}$. ★

Proof. First, note that $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu \cap Y_\nu : \nu \in I \rangle}$ is upward-closed for inclusion. Let $Z_0 \cup \cdots \cup Z_{k-1} = \mathbb{N}$. By refining the covering, we can assume that for every $t < k$ and $\nu \in I$, Z_t is both X_ν and Y_ν-homogeneous. Since $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \cap \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$ is large, there is some $j : I \to k$ such that $\langle Z_{j(\nu)} : \nu \in I \rangle \in \mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \cap \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$. We claim that $Z_{j(\nu)} \subseteq X_\nu \cap Y_\nu$ for every $\nu \in I$. Indeed, since $\langle Z_{j(\nu)} : \nu \in I \rangle \in \mathcal{L}_{\langle X_\nu : \nu \in I \rangle}$, then $Z_{j(\nu)} \cap X_\nu$ is infinite, so by X_ν-homogeneity of $Z_{j(\nu)}$, $Z_{j(\nu)} \subseteq X_\nu$. Similarly, $Z_{j(\nu)} \subseteq Y_\nu$. Thus $\langle Z_{j(\nu)} : \nu \in I \rangle \in \mathcal{A} \cap \mathcal{L}_{\langle X_\nu \cap Y_\nu : \nu \in I \rangle}$. ■

Recall from Section 9.6 that every large class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ admits a maximal partition regular sub-class $\mathcal{L}(\mathcal{A})$, which admits a formulation purely in terms of largeness thanks to Exercise 9.6.12. We give a similar definition for product classes.

Proposition 10.4.6 (Monin and Patey [78]). Let $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ be a non-trivial large class. The class

$$\mathcal{L}(\mathcal{A}) = \{\langle X_\nu : \nu \in I \rangle \in \mathcal{A} : \mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \text{ is large }\}$$

is a large sub-class of $\mathcal{A}$. ★

Proof. First, $\mathcal{L}(\mathcal{A})$ is by definition a sub-class of $\mathcal{A}$. Moreover, it is upward-closed for inclusion. Suppose for the contradiction that $\mathcal{L}(\mathcal{A})$ is not large. Then there is some $k \in \mathbb{N}$ and some k-cover $X_0 \cup \cdots \cup X_{k-1} = \mathbb{N}$ such that for every $j : I \to k$, $\langle X_{j(\nu)} : \nu \in I \rangle \notin \mathcal{L}(\mathcal{A})$. Unfolding the definition, for every $j : I \to k$, $\mathcal{A} \cap \mathcal{L}_{\langle X_{j(\nu)}:\nu \in I\rangle}$ is not large. Thus for every $j : I \to k$, there is some $k_j \in \mathbb{N}$ and some k_j-cover $Y_0 \cup \dots \cup Y_{k_j-1} = \mathbb{N}$ such that for every $i : I \to k_j$, $\langle Y_{i(\nu)} : \nu \in I\rangle \notin \mathcal{A}$. Let $Z_0 \cup \dots Z_{\ell-1} = \mathbb{N}$ be the common refinement of all these covers. Then, for every $r : I \to \ell$, $\langle Z_{r(\nu)} : \nu \in I\rangle \notin \mathcal{A} \cap \mathcal{L}_{\langle Z_{r(\nu)}:\nu \in I\rangle}$. However, since $\mathcal{A}$ is large, there is some $r : I \to \ell$ such that $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{A}$, and since $\mathcal{A}$ is non-trivial, $Z_{r(\nu)}$ is infinite for every $\nu \in I$, so $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{L}_{\langle Z_{r(\nu)}:\nu \in I\rangle}$. It follows that $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{A} \cap \mathcal{L}_{\langle Z_{r(\nu)}:\nu \in I\rangle}$. Contradiction. ■

Exercise 10.4.7.

1. Define the notion of partition regularity of sub-classes of $I \to 2^{\mathbb{N}}$.
2. Show that if $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ is large, then $\mathcal{L}(\mathcal{A})$ is the maximal partition regular subclass of $\mathcal{A}$. ⋆

10.4.1 Effective classes

Let $W_0^{Z,I}, W_1^{Z,I}, \dots$ be a list of all Z-c.e. subsets of $I \to 2^{<\mathbb{N}}$. As above, this induces a list $\mathcal{U}_0^{Z,I}, \mathcal{U}_1^{Z,I}, \dots$ of all $\Sigma_1^0(Z)$ sub-classes of $I \to 2^{\mathbb{N}}$, upward-closed by inclusion. Fix a countable Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$ coded by a set $M = \bigoplus_n Z_n$. Given a set $C \subseteq \mathbb{N}^2$, we write $\mathcal{U}_C^{\mathcal{M},I}$ for $\bigcap_{(e,i)\in C} \mathcal{U}_e^{Z_i,I}$.

Lemma 10.4.8. Let $C \subseteq \mathbb{N}^2$ be a set. The statement "$\mathcal{U}_C^{\mathcal{M},I}$ is large" is $\Pi_1^0(C \oplus M')$ uniformly in C, M and I. ⋆

Proof. Let us first show that the statement "$\mathcal{U}_e^{Z,I}$ is large" is $\Pi_2^0(Z)$ uniformly in e, Z and I. Indeed, by compactness, $\mathcal{U}_e^{Z,I}$ is large iff for every $k \in \mathbb{N}$, there is some $\ell \in \mathbb{N}$ such that for every k-cover $Y_0 \cup \dots \cup Y_{k-1} = \{0, \dots, \ell\}$, there is some $j : I \to k$ and some $\rho \in W_e^I$ such that for each $\nu \in I$, $\rho(\nu) \subseteq Y_{j(\nu)}$. This statement is $\Pi_2^0(Z)$ uniformly in e and Z. Then, by Lemma 10.4.2, $\mathcal{U}_C^{\mathcal{M},I}$ is large iff for every finite set $F \subseteq C$, $\mathcal{U}_F^{\mathcal{M},I}$ is large. The resulting statement is therefore $\Pi_1^0(C \oplus M')$. ■

We shall work exclusively with non-trivial classes of the form $\mathcal{U}_C^{\mathcal{M},I}$ where $\mathcal{M}$ is a Scott ideal coded by a set of low degree, and $C \subseteq \mathbb{N}^2$ is Δ_2^0. The following exercise shows that such classes are Π_2^0.

Exercise 10.4.9. Let $\mathcal{M}$ be a Scott ideal, coded by a set M of low degree. Let $C \subseteq \mathbb{N}^2$ be Σ_2^0. Show that $\mathcal{U}_C^{\mathcal{M},I}$ is Π_2^0. ⋆

10.4.2 Projections

We developed so far a theory of product largeness for a fixed set of indices I. The main theorem of this chapter will invoke the pigeonhole principle over I to obtain a sub-set $J \subseteq I$ over which the large class admits better properties. We must therefore define a proper notion of projection of a class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ over a sub-set $J \subseteq I$.

19: There exist multiple candidate notions of projection. For instance, one could have asked the class to be non-empty instead of large. However, this definition enjoys better combinatorial properties.

Definition 10.4.10. Given a class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ and a subset $J \subseteq I$, let $\pi_J(\mathcal{A})$ be the class of all $\langle X_\nu : \nu \in J\rangle$ such that the following class is large:[19]

$$\{\langle X_\nu : \nu \in I \setminus J\rangle : \langle X_\nu : \nu \in I\rangle \in \mathcal{A}\}$$

It is not clear at first sight that this definition of projection is not too strong, that is, asking the residual class to be large instead of non-empty might yield a small projection. Thankfully, the following lemma states that a large number of elements satisfies this property.

Lemma 10.4.11 (Monin and Patey [78]). Let $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ be a large class, and $J \subseteq I$ be a subset. Then $\pi_J(\mathcal{A})$ is large. ⋆

Proof. The class $\pi_J(\mathcal{A})$ is upward-closed by upward-closure of $\mathcal{A}$. Let $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$ for some $k \in \mathbb{N}$. Suppose for the contradiction that for every $j : J \to k$, $\langle Y_{j(\nu)} : \nu \in J\rangle \notin \pi_J(\mathcal{A})$. Unfolding the definition, for every $j : J \to k$, the following class is not large:

$$\{\langle X_\nu : \nu \in I \setminus J\rangle : \langle X_\nu : \nu \in I \setminus J\rangle \cdot \langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{A}\}$$

Let $Z_0 \cup \dots \cup Z_{\ell-1} = \mathbb{N}$ be the common refinement of all the covers witnessing that these classes are not large, and of $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$. Since $\mathcal{A}$ is large, there is some $r : I \to \ell$ such that $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{A}$. Since the cover refines $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$, there is a function $j : J \to k$ such that for every $\nu \in J$, $Y_{j(\nu)} \supseteq Z_{r(\nu)}$. Let $i : I \setminus J \to \ell$ be the restriction of r to $I \setminus J$. Then by upward-closure of $\mathcal{A}$, $\langle Z_{i(\nu)} : \nu \in I \setminus J\rangle \cup \langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{A}$, which contradicts the fact that $Z_0 \cup \dots \cup Z_{\ell-1} = \mathbb{N}$ refines the witness of non-largeness for j. ■

The following lemma states the existence of a commutative diagram between large classes and their projections. It will be very useful to consider each projection independently, and obtain a decreasing sequence of large sub-classes of $I \to 2^{\mathbb{N}}$.

Lemma 10.4.12 (Monin and Patey [78]). Let $\mathcal{U}_C^{\mathcal{M},I} \subseteq I \to 2^{\mathbb{N}}$ be a large class for some Δ^0_2 set $C \subseteq \mathbb{N}^2$, $J \subseteq I$ be a subset of indices and $\mathcal{A} \subseteq \pi_J(\mathcal{U}_C^{\mathcal{M},I})$ be a Π^0_2 large class. Then there is a Δ^0_2 set $D \supseteq C$ such that $\mathcal{U}_D^{\mathcal{M},I} \subseteq \mathcal{U}_C^{\mathcal{M},I}$ is large, and $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) = \mathcal{A}$. ⋆

Proof. Say $\mathcal{A} = \mathcal{U}_E^{\mathcal{M},J}$ for some Δ^0_2 set $E \subseteq \mathbb{N}^2$. There exists an increasing computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$ and every oracle Z, $\mathcal{U}_{f(e)}^{Z,I} = \{\langle X_\nu : \nu \in I\rangle : \langle X_\nu : \nu \in J\rangle \in \mathcal{U}_e^{Z,J}\}$. Let $D = C \cup \{(f(e), i) : (e, i) \in E\}$. Then D is Δ^0_2 and $\mathcal{U}_D^{\mathcal{M},I}$ is the class of all $\langle X_\nu : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$ such that $\langle X_\nu : \nu \in J\rangle \in \mathcal{A}$. Since $D \supseteq C$, $\mathcal{U}_D^{\mathcal{M},I} \subseteq \mathcal{U}_C^{\mathcal{M},I}$.

20: This claim is precisely the reason we defined projection in terms of largeness rather than non-emptiness.

We claim that $\mathcal{U}_D^{\mathcal{M},I}$ is large.[20] Note that it is upward-closed, as both $\mathcal{U}_C^{\mathcal{M},I}$ and $\mathcal{A}$ are. Let $k \in \mathbb{N}$ and $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$. Since $\mathcal{A} \subseteq J \to 2^{\mathbb{N}}$ is large, there is some $j : J \to k$ such that $\langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{A}$. Moreover, since $\mathcal{A} \subseteq \pi_J(\mathcal{U}_C^{\mathcal{M},I})$, the class

$$\{\langle X_\nu : \nu \in I \setminus J\rangle : \langle X_\nu : \nu \in J \setminus I\rangle \cup \langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{U}_C^{\mathcal{M},I}\rangle$$

is large. Therefore, there is some $i : I \setminus J \to k$ such that $\langle Y_{i(\nu)} : \nu \in I \setminus J\rangle$ belongs to this class. Letting $r : I \to k$ be the common extension of i and j, $\langle Y_{r(\nu)} : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$. Thus, $\langle Y_{r(\nu)} : \nu \in I\rangle \in \mathcal{U}_D^{\mathcal{M},I}$. This proves our claim.

We claim that $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) = \mathcal{A}$. By definition, given $\langle Y_\nu : \nu \in J\rangle \in \mathcal{A}$, the class $\mathcal{B} = \{\langle Y_\nu : \nu \in I \setminus J\rangle : \langle Y_\nu : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}\}$ is large since $\mathcal{A} \subseteq \pi_J(\mathcal{U}_C^{\mathcal{M},I})$. By construction of $\mathcal{U}_D^{\mathcal{M},I}$, $\mathcal{B} = \{\langle Y_\nu : \nu \in I \setminus J\rangle : \langle Y_\nu : \nu \in I\rangle \in \mathcal{U}_D^{\mathcal{M},I}\}$, so $\langle Y_\nu : \nu \in J\rangle \in \pi_J(\mathcal{U}_D^{\mathcal{M},I})$. It follows that $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) \supseteq \mathcal{A}$. Suppose now that $\langle Y_\nu : \nu \in J\rangle \in \pi_J(\mathcal{U}_D^{\mathcal{M},I})$. Then the class $\mathcal{D} = \{\langle Y_\nu : \nu \in I \setminus J\rangle : \langle Y_\nu : \nu \in I\rangle \in \mathcal{U}_D^{\mathcal{M},I}\}$ is large, and in particular non-empty. By definition of $\mathcal{U}_D^{\mathcal{M},I}$, $\langle Y_\nu : \nu \in J\rangle \in \mathcal{A}$. Thus $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) \subseteq \mathcal{A}$. ■

Exercise 10.4.13. Let $I = \{0, 1\}$, $J = \{0\}$, let Odd and Even be the sets of odd and even numbers, respectively. Let $\mathcal{B} = (\mathcal{L}_{\mathsf{Odd}} \times 2^{\mathbb{N}}) \cup (\mathcal{L}_{\mathsf{Even}} \times \{\mathbb{N}\})$. Let $\hat{\pi}_J(\mathcal{B})$ be the set of all $X \in 2^{\mathbb{N}}$ such that $(X, Y) \in \mathcal{B}$ for some set Y.[21]

1. Show that $\mathcal{B}$ is large.
2. What is $\pi_J(\mathcal{B})$? What is $\hat{\pi}_J(\mathcal{B})$?
3. Show that $\mathcal{L}_{\mathsf{Even}}$ is a Π^0_2 sub-class of $\hat{\pi}_J(\mathcal{B})$, but there is no large sub-class $\mathcal{D} \subseteq \mathcal{B}$ such that $\hat{\pi}_J(\mathcal{D}) = \mathcal{L}_{\mathsf{Even}}$. ★

21: In other words, $\hat{\pi}_J(\mathcal{B})$ is the alternative notion of projection. The goal of this exercise is to show that such version does not satisfy Lemma 10.4.12.

10.4.3 Index sets

So far, we only manipulated large classes over product spaces for a fixed index set I, and reduced the dimension of a space using projection. One of the main interest of product spaces is to force multiple positive information on the reservoirs by considering the cartesian product of two large classes. Given two index sets I and K, there exists a natural one-to-one correspondence between the following two classes:[22]

$$K \to (I \to 2^{\mathbb{N}}) \quad \text{and} \quad K \times I \to 2^{\mathbb{N}}$$

22: The translation from the second class to the first class is known in computer science as *curryfication*.

We therefore identify the two classes, and given a class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$, we consider $K \to \mathcal{A}$ as a sub-class of $K \times I \to 2^{\mathbb{N}}$.

Definition 10.4.14. Given two index sets I and J, we write $J \leq I$ if there is an index set K such that $J = K \times I$. Given two classes $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ and $\mathcal{B} \subseteq J \to 2^{\mathbb{N}}$, we write $\mathcal{B} \leq \mathcal{A}$ if $J = K \times I$ for some K and $\mathcal{B} \subseteq K \to \mathcal{A}$.◇

If $J \leq I$ as witnessed by an index set K, we call *canonical surjection* the function $f : J \to I$ defined for every $(\mu, \nu) \in J \times I$ by $f(\mu, \nu) = \nu$.

Exercise 10.4.15. Let $I_0 \geq I_1 \geq I_2$ be three index sets and $\mathcal{A}_i \subseteq I_i \to 2^{\mathbb{N}}$ be classes for each $i < 3$. Show that if $\mathcal{A}_3 \leq \mathcal{A}_2$ and $\mathcal{A}_2 \leq \mathcal{A}_1$, then $\mathcal{A}_3 \leq \mathcal{A}_1$.★

10.5 Product Mathias forcing

Let us now exemplify the concepts introduced in this chapter by designing a variant of Mathias forcing whose generic sets have a jump of non-PA degree over $\emptyset'$. The main theorem of this chapter will be an elaboration of this notion of forcing, with many subtleties due to the disjunctive nature of the pigeonhole principle.

Fix a countable Scott ideal $\mathcal{M}$, coded by a set M of low degree. Consider the notion of forcing[23] whose conditions[24] are tuples $(\sigma, \langle X_\nu : \nu \in I\rangle, C)$, where

23: This notion of forcing may seem quite complex at first sight, but it is arguably the natural refinement of Mathias forcing with a good second-jump control which produces non-cohesive solutions.

24: One could have merged the sets $\langle X : \nu \in I\rangle$ into a single set $X = \bigcup_{\nu \in I} X_\nu$, and worked with tuples (σ, X, I, C), such that $\mathcal{U}_C^{\mathcal{M},I}$ is a large sub-class of $\mathcal{L}_{\langle X : \nu \in I\rangle}$. The use of multiple reservoirs will however be needed for our later refinement of Mathias forcing.

1. I is a finite index set;
2. $(\sigma, \bigcup_{\nu\in I} X_\nu)$ is a Mathias condition;
3. $\mathcal{U}_C^{\mathcal{M},I}$ is a large sub-class of $\mathcal{L}_{\langle X_\nu:\nu\in I\rangle}$;
4. $\langle X_\nu : \nu \in I\rangle \in \mathcal{M}$ and C is Δ^0_2.

A condition $(\tau, \langle Y_\mu : \mu \in J\rangle, D)$ *extends* $(\sigma, \langle X_\nu : \nu \in I\rangle, C)$ if $(\tau, \bigcup_{\mu\in J} Y_\mu)$ Mathias extends $(\sigma, \bigcup_{\nu\in I} X_\nu)$, $J \leq I$ with canonical surjection $f : J \to I$, $\mathcal{U}_D^{\mathcal{M},J} \leq \mathcal{U}_C^{\mathcal{M},I}$, and for every $\mu \in J$, $Y_\mu \subseteq X_{f(\mu)}$.

Every filter $\mathcal{F}$ for this notion of forcing induces a set $G_\mathcal{F} = \bigcup\{\sigma : (\sigma, \langle X_\nu : \nu \in I\rangle, C) \in \mathcal{F}\}$. The following extension lemma states that not only for every sufficiently generic filter $\mathcal{F}$, the set $G_\mathcal{F}$ is infinite, but if $\mathcal{F}$ contains a condition $(\sigma, \langle X_\nu : \nu \in I\rangle, C)$, then $G_\mathcal{F} \cap X_\nu$ is infinite for every $\nu \in I$.

Lemma 10.5.1. Let $(\sigma, \langle X_\nu : \nu \in I\rangle, C)$ be a condition and $x \in X_\nu$ for some $\nu \in I$. Then $(\sigma \cup \{x\}, \langle X_\nu \setminus [0, x] : \nu \in I\rangle, C)$ is a valid extension. ⋆

Proof. Immediate by Exercise 10.4.4. ■

As one expects, the use of multiple reservoirs prevents $G_\mathcal{F}$ to be cohesive as a set. The following lemma states that for every computable instance $\vec{R}$ of COH with no computable solution, and every sufficiently generic filter $\mathcal{F}$, the set $G_\mathcal{F}$ is not $\vec{R}$-cohesive.

Lemma 10.5.2. Let $\vec{R} = R_0, R_1, \dots$ be a uniformly computable sequence of sets with no computable infinite $\vec{R}$-cohesive set. For every condition $(\sigma, \langle X_\nu : \nu \in I\rangle, C)$, and every $\mu \in I$, there is an extension $(\sigma, \langle Y_{(i,\nu)} : (i, \nu) \in 2 \times I\rangle, D)$ and some $n \in \mathbb{N}$ such that $Y_{(0,\mu)} \subseteq R_n$ and $Y_{(1,\mu)} \subseteq \overline{R}_n$. ⋆

Proof. Pick any $\mu \in I$ and let $\mathcal{A} = \pi_{\{\mu\}}(\mathcal{U}_C^{\mathcal{M},I})$. Note that $\mathcal{A}$ is a Π^0_2 subclass of $\mathcal{L}_{X_\mu}$. By Exercise 9.6.27, there is some $n \in \mathbb{N}$ such that $\mathcal{A} \cap \mathcal{L}_{R_n}$ and $\mathcal{A} \cap \mathcal{L}_{\overline{R}_n}$ are both large. By Lemma 10.4.5, $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{L}_{R_n\cap X_\mu}$ and $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{L}_{\overline{R}_n\cap X_\mu}$ are both large. By Lemma 10.4.12, there are two Δ^0_2 sets $D_0, D_1 \supseteq C$ such that $\mathcal{U}_{D_i}^{\mathcal{M},I} \subseteq \mathcal{U}_C^{\mathcal{M},I}$ is large and $\pi_{\{\mu\}}(\mathcal{U}_{D_i}^{M,I}) = \mathcal{A}_i$ for each $i < 2$. Let $J = 2 \times I$, $D \subseteq \mathbb{N}^2$ be such that $\mathcal{U}_D^{\mathcal{M},J} = \mathcal{U}_{D_0}^{\mathcal{M},I} \times \mathcal{U}_{D_1}^{\mathcal{M},I}$. Then $\mathcal{U}_D^{\mathcal{M},J} \leq \mathcal{U}_C^{\mathcal{M},I}$. Let $Y_{(0,\mu)} = X_\mu \cap R_n$, $Y_{(1,\mu)} = X_\mu \cap \overline{R}_n$, and $Y_{(i,\nu)} = X_\nu$ otherwise. Then the condition $(\sigma, \langle Y_\nu : \nu \in J\rangle, D)$ is the desired extension. ■

Having a notion of forcing producing non-cohesive generic sets is a sanity check, but it might be the case that the generic set *computes* a cohesive set for a computable instance of COH. We shall prove later that this does not happen, by designing a Π^0_2-merging and Σ^0_2-preserving forcing question for Σ^0_2-formulas.

Forcing question for Σ^0_1-formulas. We now design a forcing question for Σ^0_1-formulas. It essentially corresponds to the forcing question for computable Mathias forcing.[25]

25: Contrary to the proof of Theorem 9.7.1, the reservoirs belong to $\mathcal{M}$, so the forcing question can directly involve the reservoirs rather than using an over-approximation in terms of largeness. The forcing question therefore has a good definitional complexity and is Π^0_1-extremal.

Definition 10.5.3. Given a Mathias condition (σ, X) and a Σ^0_1 formula $\varphi(G)$, define $(\sigma, X) \,?\!\vdash \varphi(G)$ to hold there exists some $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◇

Note that this relation is $\Sigma^0_1(X)$. The proof of validity of the forcing question for Σ^0_1-formulas is straightforward and is left as an exercise.

Exercise 10.5.4. Let $p = (\sigma, \langle X_\nu : \nu \in I \rangle, C)$ be a condition and $\varphi(G)$ be a Σ^0_1 formula. Prove that

1. if $(\sigma, \bigcup_\nu X_\nu) ?\!\vdash \varphi(G)$, then there is an extension of p forcing $\varphi(G)$;
2. if $(\sigma, \bigcup_\nu X_\nu) ?\!\nvdash \varphi(G)$, then there is an extension of p forcing $\neg\varphi(G)$. ★

Syntactic forcing relation. As in the proof of Theorem 9.7.1, it will be convenient to define a syntactic forcing relation for Π^0_2-formulas.

Definition 10.5.5. Let $p = (\sigma, \langle X_\nu : \nu \in I \rangle, C)$ be a condition and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. Let $p \Vdash \varphi(G)$ hold if for every $\rho \subseteq \bigcup_{\nu \in I} X_\nu$ and every $x \in \mathbb{N}$,[26] [27]

$$\mathcal{U}^{\mathcal{M},I}_C \subseteq \{\langle Y_\nu : \nu \in I \rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) ?\!\vdash \psi(G, x)\}$$

26: One would be tempted to only require that the intersection of the left and right-hand side of the inclusion is large. However, since $\mathcal{U}^{\mathcal{M},I}_C$ may decrease over condition extension, this forcing relation would not be closed under extension. Asking for inclusion is a way to strongly enforce the largeness of the intersection, for every further restriction of $\mathcal{U}^{\mathcal{M},I}_C$.

27: Technically, we should have used

$$(\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu \setminus [0, \max \rho])$$

to ensure that the minimum of the reservoirs is larger than the stems, but we drop this restriction for simplicity of the notation.

Since the size of the index set may increase over condition extension, it is not completely clear that this syntactic forcing relation is closed under extension. The following lemma shows that it is the case.

Lemma 10.5.6. Let p be a condition and $\varphi(G)$ be a Π^0_2-formula such that $p \Vdash \varphi(G)$. For every extension $q \leq p$, $q \Vdash \varphi(G)$. ★

Proof. Say $p = (\sigma, \langle X_\nu : \nu \in I \rangle, C)$, $q = (\tau, \langle Y_\mu : \mu \in J \rangle, D)$, and $\varphi(G) \equiv \forall x \psi(G, x)$. Let K be such that $J = K \times I$, and let $f : J \to I$ be the canonical surjection. Fix some $x \in \mathbb{N}$ and some $\rho \subseteq \bigcup_{\mu \in J} Y_\mu$. Since $(\tau, \bigcup_{\mu \in J} Y_\mu)$ Mathias extends $(\sigma, \bigcup_{\nu \in I} X_\nu)$, there is some $\eta \subseteq \bigcup_{\nu \in I} X_\nu$ such that $\tau \cup \rho = \sigma \cup \eta$. Since $p \Vdash \varphi(G)$, then

$$\mathcal{U}^{\mathcal{M},I}_C \subseteq \{\langle R_\nu : \nu \in I \rangle : (\sigma \cup \eta, \bigcup_{\nu \in I} R_\nu) ?\!\vdash \psi(G, x)\}$$

We claim that

$$\mathcal{U}^{\mathcal{M},J}_D \subseteq \{\langle Z_\mu : \mu \in J \rangle : (\tau \cup \rho, \bigcup_{\mu \in J} Z_\mu) ?\!\vdash \psi(G, x)\}$$

Fix some $\langle Z_\mu : \mu \in J \rangle \in \mathcal{U}^{\mathcal{M},J}_D$. Since $\mathcal{U}^{\mathcal{M},J}_D \leq \mathcal{U}^{\mathcal{M},I}_C$, $\mathcal{U}^{\mathcal{M},J}_D \subseteq K \to \mathcal{U}^{\mathcal{M},I}_C$. It follows that there is some $\langle R_\nu : \nu \in I \rangle \in \mathcal{U}^{\mathcal{M},I}_C$ such that $\bigcup_{\mu \in J} Z_\mu \supseteq \bigcup_{\nu \in I} R_\nu$. Since $(\sigma \cup \eta, \bigcup_{\nu \in I} R_\nu) ?\!\vdash \psi(G, x)$, then $(\tau \cup \rho, \bigcup_{\mu \in J} Z_\mu) ?\!\vdash \psi(G, x)$. ∎

Together with Lemma 10.5.6, the following lemma states that, for every sufficiently generic filter $\mathcal{F}$, if $p \Vdash \varphi(G)$ for some $p \in \mathcal{F}$, then p forces $\varphi(G)$.

Lemma 10.5.7. Let $p = (\sigma, \langle X_\nu : \nu \in I \rangle, C)$ be a condition and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. If $p \Vdash \varphi(G)$, then for every $x \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\psi(G, x)$. ★

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G)$, then in particular, for $\rho = \emptyset$,

$$\mathcal{U}^{\mathcal{M},I}_C \subseteq \{\langle Y_\nu : \nu \in I \rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) ?\!\vdash \psi(G, x)\}$$

Since $\langle X_\nu : \nu \in I \rangle \in \mathcal{U}^{\mathcal{M},I}_C$, then $(\sigma, \bigcup_{\nu \in I} X_\nu) ?\!\vdash \psi(G, x)$. By Exercise 10.5.4, there is an extension of p forcing $\psi(G, x)$. ∎

Forcing question for Σ^0_2-formulas. We now have all the necessary tools to define a forcing question for Σ^0_2-formulas with good definitional and combinatorial properties.

Definition 10.5.8. Let $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ be a condition and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Let $p \operatorname{?\vdash} \varphi(G)$ hold if the following class is not large:

$$\mathcal{U}^{\mathcal{M},I}_C \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq \bigcup_{\nu \in I} X_\nu} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \operatorname{?\vdash} \psi(G, x)\}$$

By Lemma 10.4.8, the forcing question is $\Sigma^0_1(C \oplus M')$, hence Σ^0_2 since M is low and C Δ^0_2. It follows that the forcing question is Σ^0_2-preserving. We now prove that it meets its specifications.

Lemma 10.5.9. Let p be a condition and $\varphi(G)$ a Σ^0_2-formula.

1. If $p \operatorname{?\vdash} \varphi(G)$, then there is an extension of p forcing $\varphi(G)$.
2. If $p \operatorname{?\nvdash} \varphi(G)$, then there is an extension q of p with $q \Vdash \neg\varphi(G)$. ★

Proof. Say $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ and $\varphi(G) \equiv \exists x \psi(G, x)$.

Suppose first $p \operatorname{?\vdash} \varphi(G)$. Then there is some finite set $F \subseteq C$, some $\ell \in \mathbb{N}$ and some $x_0, \dots, x_{\ell-1} \in \mathbb{N}$ and $\rho_0, \dots, \rho_{\ell-1} \subseteq \bigcup_{\nu \in I} X_\nu$ such that

$$\mathcal{A} = \mathcal{U}^{\mathcal{M},I}_F \cap \bigcap_{s<\ell} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho_s, \bigcup_{\nu \in I} Y_\nu) \operatorname{?\vdash} \psi(G, x_s)\}$$

is not large. Given $k \in \mathbb{N}$, let $\mathcal{C}_k$ be the $\Pi^0_1(\mathcal{M})$ class of all $Y_0 \oplus \dots \oplus Y_{k-1} \in 2^{\mathbb{N}}$ such that $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$ and for every $j : I \to k$, $\langle Y_{j(\nu)} : \nu \in I\rangle \notin \mathcal{A}$. There is some $k \in \mathbb{N}$ such that $\mathcal{C}_k \neq \emptyset$. Since $\mathcal{M}$ is a Scott ideal, there is some $Y_0 \oplus \dots \oplus Y_{k-1} \in \mathcal{C}_k \cap \mathcal{M}$. By Proposition 10.4.6, there is some $j : I \to k$ such that $\mathcal{U}^{\mathcal{M},I}_C \cap \mathcal{L}_{\langle Y_{j(\nu)} : \nu \in I\rangle}$ is large. Since $\langle Y_{j(\nu)} : \nu \in I\rangle \notin \mathcal{A}$, there is some $s < \ell$ such that $(\sigma \cup \rho_s, \bigcup_{\nu \in I} Y_{j(\nu)}) \operatorname{?\vdash} \psi(G, x_s)$. By definition of a condition, $\mathcal{U}^{\mathcal{M},I}_C \subseteq \mathcal{L}_{\langle X_\nu : \nu \in I\rangle}$, so by Lemma 10.4.5, $\mathcal{U}^{\mathcal{M},I}_C \cap \mathcal{L}_{\langle X_\nu \cap Y_{j(\nu)} : \nu \in I\rangle}$ is large. For every $\nu \in I$, let $Z_\nu = X_\nu \cap Y_{j(\nu)}$. Let $D \supseteq C$ be a Δ^0_2 set such that $\mathcal{U}^{\mathcal{M},I}_D = \mathcal{U}^{\mathcal{M},I}_C \cap \mathcal{L}_{\langle Z_\nu : \nu \in I\rangle}$. Then $q = (\sigma \cup \rho_s, \langle Z_\nu : \nu \in I\rangle, D)$ is an extension of p such that $(\sigma \cup \rho_s, \bigcup_{\nu \in I} Y_{j(\nu)}) \operatorname{?\vdash} \psi(G, x_s)$. By Exercise 10.5.4, there is an extension of q forcing $\psi(G, x_s)$, hence forcing $\varphi(G)$.

Suppose first $p \operatorname{?\nvdash} \varphi(G)$. Let $D \supseteq C$ be a Δ^0_2 set such that

$$\mathcal{U}^{\mathcal{M},I}_D = \mathcal{U}^{\mathcal{M},I}_C \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq \bigcup_{\nu \in I} X_\nu} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \operatorname{?\vdash} \psi(G, x)\}$$

Then $q = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ is an extension of p such that $q \Vdash \neg\varphi(G)$. ■

Our last lemma states that the forcing question for Σ^0_2-formulas is Π^0_2-merging. It follows from Exercise 10.2.8 that for every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$.

Lemma 10.5.10. Let p be a condition and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_2-formulas. If $p \operatorname{?\nvdash} \varphi_0(G)$ and $p \operatorname{?\nvdash} \varphi_1(G)$, then there is an extension q of p with $q \Vdash \neg\varphi_0(G)$ and $q \Vdash \neg\varphi_1(G)$. ★

Proof. Say $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ and $\varphi_i(G) \equiv \exists x \psi_i(G, x)$ for each $i < 2$. For each $i < 2$, let $D_i \supseteq C$ be a Δ^0_2 set such that

$$\mathcal{U}^{\mathcal{M},I}_{D_i} = \mathcal{U}^{\mathcal{M},I}_{C} \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq \bigcup_{\nu \in I} X_\nu} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \mathrel{?\!\vdash} \psi_i(G, x)\}$$

Let $D \subseteq \mathbb{N}^2$ be a Δ^0_2 set such that $\mathcal{U}^{\mathcal{M},2\times I}_{D} = \mathcal{U}^{\mathcal{M},I}_{D_0} \times \mathcal{U}^{\mathcal{M},I}_{D_1}$. For each $(i, \nu) \in 2 \times I$, let $Y_{(i,\nu)} = X_\nu$. Then $q = (\sigma, \langle Y_{(i,\nu)} : (i, \nu) \in 2 \times I\rangle, D)$ is the desired extension of p. ■

Exercise 10.5.11. Fix a uniformly computable sequence $\vec{g} = g_0, g_1, \dots$ of functions of type $\mathbb{N} \to \mathbb{N}$. Use product Mathias forcing to show that there exists an infinite thin $\vec{g}$-cohesive[28] set $C \subseteq \mathbb{N}$ such that C' is not of PA degree over $\emptyset'$. ⋆

28: Recall that an infinite set $C \subseteq \mathbb{N}$ is *thin $\vec{g}$-cohesive* if for every $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $C \setminus [0, k]$ is g_n-thin.

10.6 Pigeonhole principle

As explained in Section 3.4, Ramsey's theorem for pairs can be decomposed into the cohesiveness principle (COH) and the pigeonhole principle for Δ^0_2 instances ($(\mathsf{RT}^1_2)'$). It is natural to wonder whether this decomposition is strict, that is, whether COH implies $(\mathsf{RT}^1_2)'$ or $(\mathsf{RT}^1_2)'$ implies COH over RCA_0. The former question can easily be answered negatively by a first-jump control argument (see Hirschfeldt et al. [47]), while the former was a long-standing open question. It was first answered negatively by Chong, Slaman and Yang [29] using non-standard models.[29] More recently, Monin and Patey [78] proved that $(\mathsf{RT}^1_2)'$ does not imply COH over ω-models, by proving that $(\mathsf{RT}^1_2)'$ admits jump PA avoidance using a variant of the product Mathias forcing.

29: Chong, Slaman and Yang [29] constructed a non-standard model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + (\mathsf{RT}^1_2)'$ in which every set is of low degree (from the viewpoint of the model). Such a model cannot be standard, as Downey et al. [28] constructed a Δ^0_2 set with no infinite subset of it or its complement of low degree.

Theorem 10.6.1 (Monin and Patey [78])
Let $A \subseteq \mathbb{N}$ be a Δ^0_2 set. There exists an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that H' is not of PA degree over $\emptyset'$.[30]

30: The statement relativizes as follows: For every set Z such that Z' is not of PA degree over $\emptyset'$, and every $\Delta^0_2(Z)$ set A, there exists an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that $(H \oplus Z)'$ is not of PA degree over $\emptyset'$.

The natural attempt would be to adapt product Mathias forcing to construct solutions to $(\mathsf{RT}^1_2)'$, the same way Mathias forcing was adapted in the proof of Theorem 3.4.6. Fix a Δ^0_2 set A and a countable Scott ideal $\mathcal{M}$, coded by a set M of low degree. Let $A_0 = A$ and $A_1 = \overline{A}$, and consider the notion of forcing $(\mathbb{Q}, \leq)$ whose conditions are tuples of the form $(\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$, where $(\sigma_i, \langle X_\nu : \nu \in I\rangle, C)$ is a product Mathias forcing condition for each $i < 2$, and $\sigma_i \subseteq A_i$. Condition extension is defined accordingly. One must really think of such notion of a condition as two product Mathias conditions sharing the reservoirs and notions of largeness. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$, defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C) \in \mathcal{F}\}$.

Syntactic forcing relation. The syntactic forcing relation for Π^0_2-formulas is a straightforward adaptation of Definition 10.5.5. The only difference comes from the structural constraint of homogeneity, which requires ρ to be included in A_i.

Definition 10.6.2. Let $p = (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ be a condition, $i < 2$ be a part and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. Let $p \Vdash \varphi(G_i)$ hold if for every $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$ and every $x \in \mathbb{N}$,

$$\mathcal{U}_C^{\mathcal{M},I} \subseteq \{\langle Y_\nu : \nu \in I\rangle : (\sigma_i \cup \rho, \bigcup_{\nu \in I} Y_\nu) \mathrel{?}\!\vdash \psi(G,x)\}$$

The proof of stability of the syntactic forcing relation under condition extension is left as an exercise.

Exercise 10.6.3. Adapt the proof of Lemma 10.5.6 to show that if p is a condition and $\varphi(G)$ is a Π^0_2-formula such that $p \Vdash \varphi(G_i)$ for some $i < 2$, then for every extension $q \leq p$, $q \Vdash \varphi(G_i)$. ★

Contrary to product Mathias forcing, this syntactic forcing relation does not entail the semantic one in general, because the stem must be a subset of A_i. One must therefore introduce a notion of validity as in Theorem 9.7.1.

Definition 10.6.4. We say that part i of $(\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ is *valid* if $\langle X_\nu \cap A_i : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$. Part i of a filter $\mathcal{F}$ is *valid* if part i is valid for every condition in $\mathcal{F}$.[31] ◊

31: One could have strengthened the definition of validity by requiring that $\mathcal{U}_C^{\mathcal{M},I} \cap \mathcal{L}_{\langle X_\nu \cap A_i : \nu \in I\rangle}$ is large. Indeed, Lemma 10.6.13 already proves the existence of a valid part in the stronger sense.

A new problem arises in the realm of product spaces: if $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is large, there is not necessarily some $i < 2$ such that $(A_i, A_i) \in \mathcal{A}$. It follows that every condition does not necessarily have a valid side. We shall leave this issue for now. The notion of validity is designed so that the following lemma holds.

Lemma 10.6.5 (Monin and Patey [78]). Let $p = (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ be a condition with valid part i and $\varphi(G) \equiv \forall x \psi(G,x)$ be a Π^0_2 formula. If $p \Vdash \varphi(G_i)$, then for every $x \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\psi(G_i, x)$. ★

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G_i)$, then in particular, for $\rho = \emptyset$,

$$\mathcal{U}_C^{\mathcal{M},I} \subseteq \{\langle Y_\nu : \nu \in I\rangle : (\sigma_i \cup \rho, \bigcup_{\nu \in I} Y_\nu) \mathrel{?}\!\vdash \psi(G,x)\}$$

By validity of part i of p, $\langle X_\nu \cap A_i : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$, so $(\sigma_i, A_i \cap \bigcup_{\nu \in I} X_\nu) \mathrel{?}\!\vdash \psi(G,x)$. Let $\mu \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$ be such that $\psi(\sigma_i \cup \mu, x)$ holds. Let $\tau_i = \sigma_i \cup \mu$, $\tau_{1-i} = \sigma_{1-i}$, and for each $\nu \in I$, let $Y_\nu = X_\nu \setminus \{0, \dots, \max \mu\}$. Then $(\tau_0, \tau_1, \langle Y_\nu : \nu \in I\rangle, C)$ is an extension forcing $\psi(G_i, x)$. ■

Together with Exercise 10.6.3, the previous lemma implies that, for every sufficiently generic filter $\mathcal{F}$ with valid part i, if $p \Vdash \varphi(G_i)$ for some $p \in \mathcal{F}$, then p forces $\varphi(G_i)$.[32]

32: This statement might be vacuous as the existence of a sufficiently generic filter with a valid part is not clear.

Exercise 10.6.6 (Monin and Patey [78]). Let $p, q \in \mathbb{Q}$ be two conditions such that $q \leq p$. Show that if part i of q is valid, then so is part i of p. ★

The following exercise implies that for every sufficiently generic filter $\mathcal{F}$ with valid part i, $G_{\mathcal{F},i}$ is infinite.

Exercise 10.6.7 (Monin and Patey [78]). Let $p = (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ be a condition. Show that if part i of p is valid, then there is an extension $q = (\tau_0, \tau_1, \langle Y_\nu : \nu \in I\rangle, D)$ such that $\operatorname{card} \tau_i > \operatorname{card} \sigma_i$.[33] ★

33: Note that the extension has the same index set as the condition. This will be useful in combination with Lemma 10.6.14.

Index sets. As mentioned, if $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is large, there is not necessarily some $i < 2$ such that $(A_i, A_i) \in \mathcal{A}$. On the other hand, if $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, by the pigeonhole principle, there is some $i < 2$ and some $a < b < 3$ such that $(A_i, A_i) \in \pi_{\{a,b\}}(\mathcal{A})$. We shall therefore work with a more complex notion of condition over a larger index set, representing multiple $\mathbb{Q}$-conditions by projections. To do this, we shall define an infinite sequence of big index sets $\mathcal{I}_0 \geq \mathcal{I}_1 \geq \dots$ where $\mathcal{I}_n$ contains only finite sequences of length n, satisfying some appropriate Ramsey property on its index subsets.

Example 10.6.8. Say $\mathcal{I}_1 = \{0, 1, 2\}$ and let $I \lhd \mathcal{I}_1$ if $I \subseteq \mathcal{I}_1$ and $\operatorname{card} I = 2$. By the pigeonhole principle, for every 2-partition of $\mathcal{I}_1$, there is some monochromatic $I \lhd \mathcal{I}_1$.

We now generalize the previous example for argument for every n. Let $u_0, u_1, \dots$ be inductively defined by $u_0 = 1$ and $u_{n+1} = \binom{2u_n+1}{2} u_n$.

Definition 10.6.9. Given $n \in \mathbb{N}$, the *meta n-index set* $\mathcal{I}_n$ is defined inductively defined as follows: $\mathcal{I}_0 = \{\epsilon\}$, and

$$\mathcal{I}_{n+1} = (2u_n + 1) \times \mathcal{I}_n = \{x \cdot \nu : x \leq 2u_n \wedge \nu \in I_n\}$$

Technically, meta index sets are nothing but index sets. However, they differ by their role, as they should be thought of families of index sets $\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\}$, for some relation $\lhd$ that we define now:

Definition 10.6.10. Let $\lhd$ be the smallest relation satisfying $\{\epsilon\} \lhd \mathcal{I}_0$, and if $I \lhd \mathcal{I}_n$ and $x < y \leq 2u_n$, then $(x \cdot I \cup y \cdot I) \lhd \mathcal{I}_{n+1}$.[34] ◇

34: The notation $x \cdot I$ means $\{x \cdot \nu : \nu \in I\}$.

Note that if $I \lhd \mathcal{I}_n$, then $I \subseteq \mathcal{I}_n$. Moreover, if $J \lhd \mathcal{I}_{n+1}$, then there is some $I \lhd \mathcal{I}_n$ such that $J \leq I$. An easy counting argument yields the following lemma.

Lemma 10.6.11 (Monin and Patey [78]). For every $n \in \mathbb{N}$, $\operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = u_n$. ★

Proof. By induction over n. For $n = 0$, there is exactly one $I \subseteq \mathcal{I}_0$ such that $I \lhd \mathcal{I}_0$, namely, $\{\epsilon\}$, and $u_0 = 1$. Suppose $\operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = u_n$. Then $\operatorname{card}\{J \subseteq \mathcal{I}_{n+1} : J \lhd \mathcal{I}_{n+1}\} = \binom{2u_n+1}{2} \operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = \binom{2u_n+1}{2} u_n = u_{n+1}$. ■

The following lemma states that the meta index sets satisfy some desired Ramsey property. It will play an essential role in proving that every meta-condition contains a branch with a valid side.

Lemma 10.6.12 (Monin and Patey [78]). For every $n \in \mathbb{N}$ and every 2-cover $B_0 \cup B_1 = \mathcal{I}_n$, there is some $I \lhd \mathcal{I}_n$ and some $i < 2$ such that $I \subseteq B_i$. ★

Proof. By induction on n. The case $n = 0$ is trivial. Assume it holds for n. Let $B_0 \cup B_1 = \mathcal{I}_{n+1}$. For every $x \leq 2u_n$ and $i < 2$, let $B_{x,i} = \{\nu : x \cdot \nu \in B_i\}$. Note that for each $x \leq 2u_n$, $B_{x,0} \cup B_{x,1} = \mathcal{I}_n$, so by induction hypothesis, there is some $I_x \lhd \mathcal{I}_n$ and $i_x < 2$ such that $I_x \subseteq B_{x,i_x}$. By Lemma 10.6.11, $\operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = u_n$, so by the pigeonhole principle, there is some $x < y \leq 2u_n$, some $I \lhd \mathcal{I}_n$ and $i < 2$ such that $I = I_x = I_y$ and $i = i_x = i_y$. Letting $J = x \cdot I \cup y \cdot I$, we have $J \lhd \mathcal{I}_{n+1}$ and $J \subseteq B_i$. ■

Meta-conditions. We now define a more complex notion of forcing $(\mathbb{P}, \leq)$, whose conditions are of the form $(\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ for some $n \in \mathbb{N}$, where

1. $\sigma_i^I \subseteq A_i$ for each $i < 2$ and $I \lhd \mathcal{I}_n$;
2. $(\sigma_i^I, \bigcup_{\nu \in I} X_\nu)$ is a Mathias condition for each $i < 2$ and $I \lhd \mathcal{I}_n$;
3. $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \subseteq \mathcal{I}_n \to 2^{\mathbb{N}}$ is a large sub-class of $\mathcal{L}_{\langle X_\nu : \nu \in \mathcal{I}_n\rangle}$;
4. $\langle X_\nu : \nu \in \mathcal{I}_n\rangle \in \mathcal{M}$ and C is Δ_2^0.

We write $\mathbb{P}_n$ for the set of meta-conditions indexed by $\mathcal{I}_n$, and $\mathbb{Q}_n$ for the set of conditions indexed by some $I \lhd \mathcal{I}_n$. One should really think of a meta-condition $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ as u_n-many parallel $\mathbb{Q}$-conditions $c^{[I]} = (\sigma_0^I, \sigma_1^I, \langle X_\nu : \nu \in I\rangle, C^I)$ for each $I \lhd \mathcal{I}_n$, where $C^I \subseteq \mathbb{N}^2$ is such that $\mathcal{U}_{C^I}^{\mathcal{M}, I} = \pi_I(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$. We shall refer to $c^{[I]}$ as *branches* of c. The notion of meta-condition has been design so that it satisfies the following validity lemma:

Lemma 10.6.13 (Monin and Patey [78]). For every meta-condition $c \in \mathbb{P}_n$, there is some $I \lhd \mathcal{I}_n$ such that $c^{[I]}$ admits a valid part. ⋆

Proof. Say $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$. Since $A_0 \cup A_1 = \mathbb{N}$ and by Proposition 10.4.6, $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$ is large, there is some $j : \mathcal{I}_n \to 2$ such that $\langle A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle \in \mathcal{L}(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$. Thus, $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu : \nu \in \mathcal{I}_n\rangle} \cap \mathcal{L}_{\langle A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle}$ is large, so by Lemma 10.4.5, $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle}$ is large.

Let $B_i = \{\nu \in \mathcal{I}_n : j(\nu) = i\}$ for each $i < 2$. Since $B_0 \cup B_1 = \mathcal{I}_n$, then by Lemma 10.6.12, there is some $I \lhd \mathcal{I}_n$ and some $i < 2$ such that $I \subseteq B_i$. Since $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle}$ is large, then $\langle X_\nu \cap A_{j(\nu)} : \nu \in I\rangle \in \pi_I(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$. As $I \subseteq B_i$, $\langle X_\nu \cap A_i : \nu \in I\rangle = \langle X_\nu \cap A_{j(\nu)} : \nu \in I\rangle \in \pi_I(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$, so part i of the $\mathbb{Q}$-condition $c^{[I]}$ is valid. ■

A meta-condition $d = (\langle \tau_0^J, \tau_1^J : J \lhd \mathcal{I}_m\rangle, \langle Y_\mu : \mu \in \mathcal{I}_m\rangle, D)$ *extends* $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ if $m \geq n$, and for every $J \lhd \mathcal{I}_m$, letting $I \lhd \mathcal{I}_n$ be the unique index set such that $J \leq I$, $d^{[J]} \leq c^{[I]}$ as $\mathbb{Q}$-conditions. The following commutative diagram will be very useful to propagate lemmas from $(\mathbb{Q}, \leq)$ forcing to $(\mathbb{P}, \leq)$ forcing.

Lemma 10.6.14 (Monin and Patey [78]). Fix a meta-condition $c \in \mathbb{P}_n$ and $I \lhd \mathcal{I}_n$. For every $\mathbb{Q}_n$-condition $q \leq c^{[I]}$, there is a meta-condition $d \leq c$ in $\mathbb{P}_n$ such that $d^{[I]} = q$.[35] ⋆

35: One must be a bit careful when using this lemma: it only states the existence of a commutative diagram for a fixed n.

Proof. Say $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ and $q = (\tau_0^I, \tau_1^I, \langle Y_\nu : \nu \in I\rangle, D^I)$. By Lemma 10.4.12, there is a Δ_2^0 set $D \supseteq C$ such that $\mathcal{U}_D^{\mathcal{M}, \mathcal{I}_n} \subseteq \mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n}$ is a large class and $\pi_I(\mathcal{U}_D^{\mathcal{M}, \mathcal{I}_n}) = \mathcal{U}_{D^I}^{\mathcal{M}, I}$. For every $J \lhd \mathcal{I}_n$ with $J \neq \mathcal{I}$ and $i < 2$, let $\tau_i^J = \sigma_i^J$. For every $\nu \in \mathcal{I}_n \setminus I$, let $Y_\nu = X_\nu$. The meta-condition $d = (\langle \tau_0^I, \tau_1^I : I \lhd \mathcal{I}_n\rangle, \langle Y_\nu : \nu \in \mathcal{I}_n\rangle, D)$ is an extension of c such that $d^{[I]} = q$. ■

Forcing question for Σ_2^0-formulas. A meta-condition representing multiple $\mathbb{Q}$-conditions, requirements must be forced on every branch of the meta-condition.

Definition 10.6.15. Given a requirement $\mathcal{R}(G)$, a part $i < 2$ and a meta-condition $c \in \mathbb{P}_n$, let $\mathcal{R}(c, i)$ be the set of all $I \lhd \mathcal{I}_n$ such that $c^{[I]}$ does not force $\mathcal{R}(G_i)$.[36] ◊

One could define a non-disjunctive Σ^0_2-preserving forcing question for Σ^0_2-formulas on $\mathbb{Q}$-conditions which would meet its specifications, and witness the answer by an extension with the same index set. For a single Σ^0_2-formula, one could then use Lemma 10.6.14 to define a finite decreasing sequence of meta-conditions $c = c_0 \geq c_1 \geq \dots \geq c_k$ such that $\mathcal{R}(c_{s+1}, i) \subsetneq \mathcal{R}(c_s, i)$, eventually yielding $\mathcal{R}(c_k, i) = \emptyset$ for each $i < 2$, thus forcing the requirement on every part of every branch.

However, in order to obtain jump PA avoidance, one must design a Π^0_2-merging forcing question. The forcing question for Σ^0_2-formulas on $\mathbb{Q}$-conditions is Π^0_2-merging, but the witnessed extension is obtained by considering the cartesian product of multiple large classes, hence increasing the index set. Trying to adapt Lemma 10.6.14 to increasing index sets would yield an extension d with more branches. Then $\mathcal{R}(d, i)$ might be larger than $\mathcal{R}(c, i)$, which would not yield a progress towards forcing the requirements on all the branches.

We shall therefore directly design a forcing question for Σ^0_2-formulas on meta-conditions c, parameterized by the set $\mathcal{R}(c, i)$, with the following property: either there exists an extension d with the same index set forcing $\mathcal{R}(G_i)$ on some branch $I \in \mathcal{R}(c, i)$, yielding $\mathcal{R}(d, i) \subseteq \mathcal{R}(c, i) \setminus \{I\}$, or there exists an extension $d \in \mathbb{P}_m$ with a larger index set, but forcing $\mathcal{R}(G_i)$ on every branch $J \lhd \mathcal{I}_m$ such that $J \leq I$ for some $I \in \mathcal{R}(c, i)$, so $\mathcal{R}(d, i) = \emptyset$.[37]

Definition 10.6.16. Let $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C)$ be a meta-condition, $H \subseteq \{I \lhd \mathcal{I}_n\}$, $i < 2$ and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Let $c \mathrel{?\vdash_H} \varphi(G_i)$ hold if the following class is not large:

$$\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \bigcap_{\substack{I \in H, x \in \mathbb{N}, \\ \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu}} \{\langle Z_\mu : \mu \in \mathcal{I}_n \rangle : (\sigma_i \cup \rho, \bigcup_{\nu \in I} Z_\nu) \mathrel{?\nvdash} \psi(G, x)\}$$

Note that the relation in Σ^0_2 uniformly in H, i and $\varphi(G)$. The following lemma states that the forcing question meets its specifications and the witnessed extension has the same index set.

Lemma 10.6.17 (Monin and Patey [78]). Let $c \in \mathbb{P}_n$ be a meta-condition, $H \subseteq \{I \lhd \mathcal{I}_n\}$, $i < 2$, and $\varphi(G)$ be a Σ^0_2 formula.

1. If $c \mathrel{?\vdash_H} \varphi(G_i)$, then there is an extension $d \leq c$ in $\mathbb{P}_n$ and some $I \in H$ such that $d^{[I]}$ strongly forces[38] $\varphi(G_i)$.
2. If $c \mathrel{?\nvdash_H} \varphi(G_i)$, then there is an extension $d \leq c$ in $\mathbb{P}_n$ such that for every $I \in H$, $d^{[I]} \Vdash \neg\varphi(G_i)$. ★

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$ and $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C)$. For every $I \in H$, $x \in \mathbb{N}$ and $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$, let

$$\mathcal{A}_{I,x,\rho} = \{\langle Z_\mu : \mu \in \mathcal{I}_n \rangle : (\sigma_i^I \cup \rho, \bigcup_{\nu \in I} Z_\nu) \mathrel{?\nvdash} \psi(G, x)\}$$

Suppose first $c \mathrel{?\vdash_H} \varphi(G_i)$. Then there is some finite set $F \subseteq C$ and some $t \in$

36: This definition and the following explanation is slightly approximative in the sense given to "forcing". In our setting, a positive answer to the forcing question yields an extension strongly forcing the Σ^0_2 formula, while the witness of a negative answer syntactically forces its negation. As seen, the syntactical forcing relation implies the semantical one only on valid parts. A requirement being often a disjunction between wrong computation and partiality, the formal sense given to "forcing" actually depends on the side of the disjunction. We will therefore give a more formal sense in the case of jump PA avoidance in Definition 10.6.20.

37: The idea was already present in the proof of Liu's theorem [12], who designed a forcing question for Σ^0_1-formulas with the same features. It is also present in Theorem 5.3.3.

38: Recall that given a notion of forcing $(\mathbb{P}, \leq)$, a condition p *strongly forces* a formula $\varphi(G)$ if the formula holds for *every* filter containing p.

$\mathbb{N}$ such that the following class is not large:

$$\mathcal{B} = \mathcal{U}_F^{\mathcal{M},\mathcal{I}_n} \bigcap_{I \in H, x < t, \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu \upharpoonright t} \mathcal{A}_{I,x,\rho}$$

Since $\mathcal{B}$ is $\Sigma^0_1(\mathcal{M})$ and $\mathcal{M}$ is a Scott ideal, there is some $k \in \mathbb{N}$ and a k-cover $Z_0 \cup \dots \cup Z_{k-1} = \mathbb{N}$ in $\mathcal{M}$ such that for every $j : \mathcal{I}_n \to k$, $\langle Z_{j(\nu)} : \nu \in I \rangle \notin \mathcal{B}$. By Proposition 10.4.6, $\mathcal{L}(\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n})$ is large, so there is some $j : \mathcal{I}_n \to k$ such that $\langle Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle \in \mathcal{L}(\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n})$. In particular, $\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu : \nu \in \mathcal{I}_n \rangle} \cap \mathcal{L}_{\langle Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle}$ is large, so by Lemma 10.4.5, so is $\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle}$. In particular, $\langle X_\nu \cap Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle \in \mathcal{U}_F^{\mathcal{M},\mathcal{I}_n}$, so there is some $I \in H$, some $x < t$ and some $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu \upharpoonright t$ such that $\langle X_\nu \cap Z_{j(\nu)} \rangle \notin \mathcal{A}_{I,x,\rho}$. Unfolding the definition of $\mathcal{A}_{I,x,\rho}$, $(\sigma^I_i \cup \rho, \bigcup_{\nu \in I} Z_{j(\nu)}) ?\nvdash \psi(G, x)$, so $(\sigma^I_i \cup \rho, \bigcup_{\nu \in I} Z_{j(\nu)})$ strongly forces $\psi(G, x)$, hence strongly forces $\varphi(G)$. Let $D \subseteq C$ be a Δ^0_2 set such that $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_n} = \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle}$. For every $\nu \in \mathcal{I}_n$, let $Y_\nu = (X_\nu \cap Z_{j(\nu)}) \setminus \{0, \dots, t\}$. Let $\tau^I_i = \sigma^i_I \cup \rho$ and $\tau^I_{1-i} = \sigma^I_{1-i}$. For every $J \lhd \mathcal{I}_n$ with $J \neq I$, let $\tau^J_0 = \sigma^J_0$ and $\tau^J_1 = \sigma^J_1$. The meta-condition $d = (\langle \tau^J_0, \tau^J_1 : J \lhd \mathcal{I}_n \rangle, \langle Y_\nu : \nu \in \mathcal{I}_n \rangle, D)$ is an extension of c such that $d^{[I]}$ strongly forces $\varphi(G_i)$.

Suppose now $c ?\nvdash_H \varphi(G_i)$. Let $D \supseteq C$ be a Δ^0_2 set such that

$$\mathcal{U}_D^{\mathcal{M},\mathcal{I}_n} = \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \bigcap_{I \in H, x \in \mathbb{N}, \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu} \mathcal{A}_{I,x,\rho}$$

The meta-condition $d = (\langle \sigma^I_0, \sigma^I_1 : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, D)$ is an extension of c such that $d^{[I]} \Vdash \neg\varphi(G_i)$ for every $I \in H$. ■

Recall from Section 5.2 that given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *weakly Γ-merging*[39] if for every $p \in \mathbb{P}$, there is some $k \in \mathbb{N}$ such that for every k-tuple of Γ-formulas $\varphi_0(G), \dots, \varphi_{k-1}(G)$, if $p ?\vdash \varphi_i(G)$ for each $i < k$, then there is an extension $q \leq p$ and two indices $i < j < k$ such that q forces $\varphi_i(G) \wedge \varphi_j(G)$. Thanks to Liu's notion of valuation (see Section 5.2), if a notion of forcing admits a Σ^0_2-preserving and weakly Π^0_2-merging forcing question for Σ^0_2-formulas, then every sufficiently generic filter yields a set whose jump is not of PA degree over $\emptyset'$.

39: Note that in the definition of a weakly Γ-merging forcing question, the parameter k might depend on the condition p.

This notion of weak Π^0_2-merging forcing question does not apply directly on meta-conditions due to the branching and disjunctive nature of meta-conditions, but the same combinatorial argument holds, with the necessary adaptation. In particular, the following lemma informally states that the forcing question on meta-conditions for Σ^0_2-formulas is weakly Π^0_2-merging.

Lemma 10.6.18 (Monin and Patey [78]). Let $c \in \mathbb{P}_n$ be a meta-condition, $H \subseteq \{I \lhd \mathcal{I}_n\}$, $i < 2$ and $\varphi_0(G), \dots, \varphi_{2u_n}(G)$ be $2u_n + 1$ many Σ^0_2 formulas. Suppose that for every $s \leq 2u_n$, $c ?\nvdash_H \varphi_s(G_i)$. Then there is some extension $d \in \mathbb{P}_{n+1}$ such that for every $I \in H$ and every $J \lhd \mathcal{I}_{n+1}$ such that $J \leq I$, there are some $a < b \leq 2u_n$ such that

$$d^{[J]} \Vdash \neg\varphi_a(G_i) \quad \text{and} \quad d^{[J]} \Vdash \neg\varphi_b(G_i)$$

Proof. Say $c = (\langle \sigma^I_0, \sigma^I_1 : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C)$ and $\varphi_s(G) \equiv \exists x \psi_s(G, x)$ for each $s \leq 2u_n$. For every $s \leq 2u_n$, the following class is large:

$$\mathcal{A}_s = \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \bigcap_{\substack{I\in H, x\in\mathbb{N},\\ \rho\subseteq A_i\cap\bigcup_{\nu\in I} X_\nu}} \{\langle Z_\mu : \mu \in \mathcal{I}_n\rangle : (\sigma_i^I \cup \rho, \bigcup_{\nu\in I} Z_\nu) \mathrel{?\nvdash} \psi_s(G, x)\}$$

Let $D \subseteq \mathbb{N}^2$ be a Δ_2^0 set such that $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}} = \prod_{j\leq 2u_n} \mathcal{A}_s$. In particular, $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}}$ is large. For every $(j,\nu) \in \mathcal{I}_{n+1}$, let $Y_{(j,\nu)} = X_\nu$. For every $J \lhd \mathcal{I}_{n+1}$, let $\tau_0^J = \sigma_0^I$ and $\tau_1^J = \sigma_1^I$, where $I \lhd \mathcal{I}_n$ is the unique index set such that $J \leq I$. Note that $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}} \subseteq \mathcal{L}_{\langle Y_\mu:\mu\in\mathcal{I}_{n+1}\rangle}$ and $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}} \leq \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n}$. The meta-condition $d = (\langle \tau_0^J, \tau_1^J : J \lhd \mathcal{I}_{n+1}\rangle, \langle Y_\mu : \mu \in \mathcal{I}_{n+1}\rangle, D)$ is an extension of c.

Fix $I \in H$ and $J \lhd \mathcal{I}_{n+1}$ such that $J \leq I$. Let $a < b \leq 2u_n$ be such that $J = \{a,b\} \times I$. We claim that $d^{[J]} \Vdash \neg\varphi_a(G_i)$ and $d^{[J]} \Vdash \neg\varphi_b(G_i)$. We prove the former, the latter being symmetric. Fix some $x \in \mathbb{N}$ and $\rho \subseteq A_i \cap \bigcup_{\mu\in J} Y_\mu$. In particular, $\rho \subseteq A_i \cap \bigcup_{\nu\in I} X_\nu$. Fix $\langle Z_\mu : \mu \in J\rangle \in \pi_J(\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}})$. In particular,

$$\langle Z_{(a,\nu)} : \nu \in I\rangle \in \mathcal{A}_a \subseteq \{\langle Z_\mu : \mu \in \mathcal{I}_n\rangle : (\sigma_i^I \cup \rho, \bigcup_{\nu\in I} Z_\nu) \mathrel{?\nvdash} \psi_a(G, x)\}$$

so $(\sigma_i^I \cup \rho, \bigcup_{\nu\in I} Z_{(a,\nu)}) \mathrel{?\nvdash} \psi_a(G,x)$. As $\sigma_i^I = \tau_i^J$ and $\bigcup_{\nu\in I} Z_{(a,\nu)} \subseteq \bigcup_{\mu\in J} Z_\mu$, then $(\tau_i^J \cup \rho, \bigcup_{\mu\in J} Z_\mu) \mathrel{?\nvdash} \psi_a(G,x)$. Thus, for every $x \in \mathbb{N}$ and $\rho \subseteq A_i \cap \bigcup_{\mu\in J} Y_\mu$, $\pi_J(\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}}) \subseteq \{\langle Z_\mu : \mu \in J\rangle : (\tau_i^J \cup \rho, \bigcup_{\mu\in J} Z_\mu) \mathrel{?\nvdash} \psi_a(G,x)\}$, so $d^{[J]} \Vdash \neg\varphi_a(G_i)$. ■

Diagonalization. We now use the forcing question for Σ_2^0-formulas to prove the appropriate diagonalization lemmas in the context of jump PA avoidance. Because of the weakly Π_2^0-merging nature of the forcing question for meta-conditions, one needs to use the valuation machinery introduced by Liu [12].

Recall from Section 5.2 that a *valuation* is a partial $\{0,1\}$-valued function $h \subseteq \mathbb{N} \to 2$. A valuation is finite if it has finite support, that is, $\operatorname{dom} h$ is finite. A valuation h is Z*-correct* if for every $n \in \operatorname{dom} h$, $\Phi_n^Z(n)\downarrow\neq h(n)$. Two valuations f and h are *compatible* if for every $n \in \operatorname{dom} f \cap \operatorname{dom} h$, $f(n) = h(n)$. The following lemma is a relativization of Lemma 5.2.3.

Lemma 10.6.19 (Liu [12]). Fix a set Z. Let U be a Z-c.e. set of finite valuations. Either U contains a Z-correct[40] valuation, or for every $k \in \mathbb{N}$, there are k pairwise incompatible finite valuations outside of U. ★

40: Note that the appropriate relativization of Lemma 5.2.3 requires to relativize the notion of correctness, as it is a computability-theoretic property.

For every $e \in \mathbb{N}$, let $\mathcal{R}_e(G)$ be the requirement "either $\Phi_e^{G'}$ is partial, or $\Phi_e^{G'}(x)\downarrow = \Phi_x^{\emptyset'}(x)$ for some $x \in \mathbb{N}$." As mentioned in a note next to Definition 10.6.15, we overload the forcing relation for the requirement $\mathcal{R}_e(G)$.

Definition 10.6.20. Given a $\mathbb{Q}$-condition p, some index $e \in \mathbb{N}$ and a part $i < 2$, we say that p *forces* $\mathcal{R}_e(G_i)$ if

1. either p strongly forces "$\Phi_e^{G_i'}$ is incompatible with h" for a $\emptyset'$-correct valuation h,
2. or $p \Vdash$ "$\Phi_e^{G_i'}$ is compatible with h_s" for two incompatible valuations h_0, h_1.[41] ◊

41: The statement "$\Phi_e^{G'}$ is incompatible with h" is $\Sigma_2^0(G)$, as it is equivalent to $\exists x \Phi_e^{G'}(x)\downarrow\neq h(x)$.

According to Definition 10.6.15, given a meta-condition $c \in \mathbb{P}_n$ we write $\mathcal{R}_e(c,i)$ for the set of index sets $I \lhd \mathcal{I}_n$ such that $c^{[I]}$ does not force $\mathcal{R}_e(G_i)$.

Lemma 10.6.21 (Monin and Patey [78]). For every meta-condition c, every part $i < 2$ and index $e \in \mathbb{N}$ such that $\mathcal{R}_e(c, i) \neq \emptyset$, there is an extension $d \le c$ such that $\operatorname{card} \mathcal{R}_e(d, i) < \operatorname{card} \mathcal{R}_e(c, i)$. ★

Proof. Let $H = \mathcal{R}_e(c, i)$, and let U be the set of all valuations h such that $c \,?\!\vdash_H$ "$\Phi_e^{G_i'}$ is incompatible with h". Note that the set U is $\emptyset'$-c.e., so by Lemma 10.6.19, we have two cases. Case 1: $h \in U$ for some $\emptyset'$-correct valuation h. Then, by Lemma 10.6.17, there is an extension $d \le c$ in $\mathbb{P}_n$ and some $I \in H$ such that $d^{[I]}$ strongly forces $\Phi_e^{G_i'}$ to be incompatible with h. In particular, $\mathcal{R}_e(d, i) \subsetneq \mathcal{R}_e(c, i)$, hence $\operatorname{card} \mathcal{R}_e(d, i) < \operatorname{card} \mathcal{R}_e(c, i)$. Case 2: $h_0, \dots, h_{2u_n} \notin U$ for $2u_n + 1$ pairwise incompatible valuations. By Lemma 10.6.18, there is an extension $d \le c$ in $\mathbb{P}_{n+1}$ such that for every $I \in H$ and every $J \lhd \mathcal{I}_{n+1}$ such that $J \le I$, there are some $a < b \le 2u_n$ such that $d^{[J]} \Vdash$ "$\Phi_e^{G_i'}$ is compatible with h_a" and $d^{[J]} \Vdash$ "$\Phi_e^{G_i'}$ is compatible with h_b", hence $d^{[J]}$ forces $\mathcal{R}_e(G_i)$. It follows that $\mathcal{R}_e(d, i) = \emptyset$, so $\operatorname{card} \mathcal{R}_e(d, i) < \operatorname{card} \mathcal{R}_e(c, i)$. ■

We say that a meta-condition $c \in \mathbb{P}_n$ *forces* $\mathcal{R}_e(G)$ if $c^{[I]}$ forces $\mathcal{R}_e(G_i)$ for every $I \lhd \mathcal{I}_n$ and $i < 2$.

Lemma 10.6.22 (Monin and Patey [78]). For every meta-condition c and $e \in \mathbb{N}$, there is an extension $d \le c$ forcing $\mathcal{R}_e(G)$. ★

Proof. Apply iteratively Lemma 10.6.21 to obtain a meta-condition $d_0 \le c$ such that $\mathcal{R}_e(d_0, 0) = \emptyset$. Then, apply again iteratively Lemma 10.6.21 to obtain a meta-condition $d_1 \le d_0$ such that $\mathcal{R}_e(d_1, 1) = \emptyset$. ■

Tree structure. The partial order of meta-conditions being countable, every $\mathbb{P}$-filter can be identified with an infinite decreasing sequence of meta-conditions $c_0 \ge c_1 \ge \dots$ Each meta-conditions represents multiple $\mathbb{Q}$-conditions, each of which admits two parts. By Lemma 10.6.13, every meta-condition admits a branch with a valid part, and by Exercise 10.6.6, the valid parts a upward-closed under the extension relation. The valid parts of $\mathbb{Q}$-conditions along a decreasing sequence of meta-conditions therefore naturally form a tree structure, motivating the following definition.

Definition 10.6.23. A *path* through a $\mathbb{P}$-filter $\mathcal{F}$ is a pair $\langle P, i \rangle$ where $i < 2$, such that

1. for every $n \in \mathbb{N}$, $P(n) \lhd \mathcal{I}_n$ such that $P(n+1) \le P(n)$;
2. for every $c \in \mathcal{F} \cap \mathbb{P}_n$, part i of $c^{[P(n)]}$ is valid. ◇

By Lemma 10.6.13 and Exercise 10.6.6, every $\mathbb{P}$-filter admits a path. For every $\mathbb{P}$-filter $\mathcal{F}$ and every path $\langle P, i \rangle$, let

$$G_{\mathcal{F},P,i} = \bigcup \{\sigma_i^{P(n)} : (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C) \in \mathcal{F}\}$$

If $\mathcal{F}$ is a sufficiently generic $\mathbb{P}$-filter and $\langle P, i \rangle$ is a path through $\mathcal{F}$, then $\mathcal{F}_P = \{c^{[P(n)]} : c \in \mathcal{F} \cap \mathbb{P}_n, n \in \mathbb{N}\}$ might not be a sufficiently generic $\mathbb{Q}$-filter. Thankfully, if a $\mathbb{Q}$-condition p strongly forces a Σ^0_1, a Π^0_2 or a Σ^0_2-formula, then the property holds for *every* $\mathbb{Q}$-filter containing p, with no consideration of genericity. The following lemma states that the syntactic forcing relation for Π^0_2-formulas holds along paths of every sufficiently generic $\mathbb{P}$-filter.

Lemma 10.6.24 (Monin and Patey [78]). Let $\mathcal{F}$ be a sufficiently generic $\mathbb{P}$-filter, and let $\langle P, i\rangle$ be a path through $\mathcal{F}$. Let $\varphi(G)$ be a Π^0_2-formula and $c \in \mathcal{F}$. If $c^{[P(n)]} \Vdash \varphi(G_i)$, then $\varphi(G_{\mathcal{F},P,i})$ holds. ⋆

Proof. Fix some $x \in \mathbb{N}$ and say $\varphi(G) \equiv \forall x \psi(G, x)$. Let $\mathcal{D}_x$ be the set of meta-conditions $d \leq c$ such that $d^{[I]}$ forces $\psi(G_i, x)$ for every branch $I \leq P(n)$ such that part i of $d^{[I]}$ is valid. By Exercise 10.6.3, Lemma 10.6.5 and Lemma 10.6.14, the set $\mathcal{D}_x$ is dense below c, so by genericity of $\mathcal{F}$, there is some $d \in \mathcal{D}_x \cap \mathcal{F}$. Say $d \in \mathcal{P}_m$. Since $P(m) \leq P(n)$ and part i of $d^{[I]}$ is valid, $d^{[P(m)]}$ forces $\psi(G_i, x)$, so $\psi(G_{\mathcal{F},P,i}, x)$ holds. Thus $\varphi(G_{\mathcal{F},P,i})$ holds.■

We are now ready to prove Theorem 10.6.1.

Proof of Theorem 10.6.1. Let $\mathcal{F}$ be a sufficiently generic $\mathbb{P}$-filter, and let $\langle P, i\rangle$ be a path through $\mathcal{F}$. By definition of a meta-condition, $G_{\mathcal{F},P,i} \subseteq A_i$. By Exercise 10.6.7 and Lemma 10.6.14, $G_{\mathcal{F},P,i}$ is infinite. By Lemma 10.6.22, for every $e \in \mathbb{N}$, the set of meta-conditions forcing $\mathcal{R}_e(G)$ is dense, hence there is some $d_e \in \mathbb{P} \cap \mathcal{F}$ such that d_e forces $\mathcal{R}_e(G)$. By Lemma 10.6.24, it follows that $\mathcal{R}_e(G_{\mathcal{F},P,i})$ holds for every $e \in \mathbb{N}$, so $G'_{\mathcal{F},P,i}$ is not of PA degree over $\emptyset'$. This completes the proof of Theorem 10.6.1. ■

10.7 Jump DNC avoidance

As mentioned in the introduction, jump DNC avoidance did not receive as much attention as jump PA avoidance since the DNC counterpart to COH did not occur naturally in reverse mathematics.

Exercise 10.7.1. Adapt the proof of Theorem 10.2.1 to show that for every sufficiently Cohen generic set G, G' is not of DNC degree over $\emptyset'$. ⋆

Exercise 10.7.2. Adapt the proof of Theorem 10.2.4 to show that given a non-computable set C and a non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists a member $G \in \mathcal{P}$ such that $C \not\leq_T G$ and G' is not of DNC degree over $\emptyset'$. ⋆

Recall from Section 5.8 that given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *countably Γ-merging* if for every $p \in \mathbb{P}$ and every countable sequence of Γ-formulas $(\varphi_s(G))_{s\in\mathbb{N}}$, if $p \mathrel{?\vdash} \varphi_s(G)$ for each $s \in \mathbb{N}$, then there is an extension $q \leq p$ forcing $\forall s \varphi_s(G)$.

Exercise 10.7.3. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_2-preserving, countably Π^0_2-merging forcing question. Adapt the proof of Theorem 5.8.4 to show that for every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is not of DNC degree over $\emptyset'$.⋆

Both in the cases of Cohen forcing and WKL, we actually exploited a stronger feature of the forcing question for Σ^0_2-formulas. A forcing question for Σ^0_n-formulas is *Π^0_n-extremal* if for every Σ^0_n-formula φ and every condition $p \in \mathbb{P}$, if $p \mathrel{?\nvdash} \varphi(G)$, then p forces $\neg\varphi(G)$.

Exercise 10.7.4. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Π^0_n-extremal forcing question. Show that the forcing question is countably Π^0_n-merging. ⋆

The status of the pigeonhole principle with respect to DNC degrees is slightly different than PA degrees. First of all, contrary to PA degrees (see Theorem 5.4.3), for every set X, there exists an instance of RT^1_2 such that every solution is of DNC degree over X. Such instance is constructed thanks to the notion of effective immunity. Recall from Section 6.2 that given a function $h : \mathbb{N} \to \mathbb{N}$, an infinite set A is *h-immune* if for every c.e. set W_e such that $W_e \subseteq A$, then $\operatorname{card} W_e \leq h(e)$. An infinite set is *effectively immune* if it is h-immune for some computable function $h : \mathbb{N} \to \mathbb{N}$.

42: The relativization of effective immunity has two parameters: a set A is Y-effectively X-immune if there is an Y-computable function $h : \mathbb{N} \to \mathbb{N}$ such that for every X-c.e. set W^X_e with $W^X_e \subseteq A$, then $\operatorname{card} W^X_e \leq h(e)$.

Proposition 10.7.5 (Hirschfeldt et al. [47]). For every set X, there is an X'-computable effectively bi-X-immune[42] set A. ⋆

Proof. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by $h(e) = 3e + 2$. We build an h-X-immune set A by stages using an X'-computable construction. At stage e, assume $A\restriction_e$ is defined, and $A(n)$ is defined for at most $2e$ other n's. Decide X'-computably whether W^X_e has at least $3e + 2$ many elements. If so, then there are at least two elements $n_0, n_1 \in W^X_e$ for which A has not yet been decided. Let $A(n_0) = 0$ and $A(n_1) = 1$. In any case, if $A(e)$ is not defined yet, let $A(e)$ be any value among 0 and 1. This completes the construction. ■

In particular, letting $X = \emptyset'$, there exists a Δ^0_3 instance of RT^1_2 such that every solution computes a DNC function over $\emptyset'$. This implies that RT^1_2 does not admit strong DNC avoidance, and *a fortiori* does not admit strong jump DNC avoidance.

Exercise 10.7.6. Use Proposition 5.7.2 to prove the existence, for every set X, of an X'-computable set A such that every infinite subset of A or of $\overline{A}$ is of DNC degree over X. ⋆

Of course, the pigeonhole principle being computably true, every Δ^0_2 instance of RT^1_2 admits a Δ^0_2 solution, hence a solution which is not of DNC degree over $\emptyset'$. The following question remains open:

Question 10.7.7. Is there a Δ^0_2 instance of RT^1_2 such that for every solution H, H' is of DNC degree over $\emptyset'$? ⋆

One would naturally want to adapt the proof of Theorem 10.6.1 and work with ω-product largeness to obtain a countably Π^0_2-merging forcing question for Σ^0_2-formulas. However, ω-product spaces do not behave as nicely as finite product spaces, leaving the question open.