Lowness and avoidance

A gentle introduction to iterated jump control

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Introduction | 1

The mathematical practice is full of meta-mathematical considerations, even at the high school level. It is common to find in textbooks statements such as "the intermediate value theorem is equivalent to the least upper bound property" or "give an elementary proof of Euclid's theorem". Every mathematician will be convinced that the use of Fermat's last theorem to prove the irrationality of $2^{1/n}$ is overly sophisticated, and the very distinction between a theorem and a corollary – which are both mathematically true and logically equivalent statements – is purely meta-mathematical. What does it mean for one theorem to imply another? What are the optimal axioms necessary to prove ordinary theorems? These are all questions that reverse mathematics tries to answer. *Reverse mathematics* is originally a meta-mathematical program started in 1972 by Harvey Friedman, seeking for the optimal axioms to prove ordinary theorems, using subsystems of second-order arithmetic. The appellation took over time a broader meaning, encompassing all the sets of tools from proof theory and computability theory to study theorems from a computational perspective.

Intuitively, a theorem A implies a theorem B, or a statement B is a corollary of a theorem A if one can prove B with only elementary methods, using A as a blackbox. The whole difficulty is to find a robust, theory-agnostic notion of "elementary methods"[1] to formalize this intuition. This is where computability theory comes into play: Thanks to the Church-Turing thesis, there is a consensual and robust formalization of the ontological concept of "effective process". Furthermore, with the popularization of computers and their integration in everyday's life, the notion of algorithm started to be part of the common knowledge. Last, but not least, by a theorem of Gödel, there is a correspondence between the computably enumerable sets, and the sets definably by a Σ_1-formula in first-order arithmetic, paving the way to a translation of the computability-theoretic concepts to the proof-theoretic realm. All these considerations make the notion of "computable" a good candidate for the definition of "elementary".

1: Beware, we make here an important distinction between "elementary proof" and "simple proof". The former concept should be understood as "logically elementary", that is, involving only logically weak axioms, while the latter is a more human concept which seems harder to formalize. In particular, one can win a Fields medal by proving theorems requiring only weak axioms.

The formal setting of reverse mathematics is therefore subsystems of second-order arithmetic, that is, theories in a two-sorted language with a set of integers and collection of sets of integers.[2] The base theory, RCA_0, captures "computable mathematics". Thanks to the correspondence between computability and definability, proofs of implications are often witnessed by a computable procedure, and separation proofs mainly consist in constructing models of RCA_0 satisfying some specific computability-theoretic weakness properties.

2: Hilbert and Bernays used second-order arithmetic as a foundational language to reprove ordinary mathematics. They showed through their book *Grundlagen der Mathematik* that a large part of classical mathematics could be casted in this setting and proven using second-order Peano arithmetic (Z_2).

Since the start of reverse mathematics, many theorems have been studied from the core areas of mathematics, including analysis, algebra, topology, and highlighted two main empirical phenomena. First of all, mathematics seem very structured, that is, most theorems from ordinary mathematics are either computationally trivial, or computably equivalent to one of four subsystems of second-order arithmetic, linearly ordered by the implication. Second, a large part of ordinary mathematics requires very weak axiomatic and computability-theoretic power. As mentioned, these phenomena are empirical observations, and there exist two main areas of mathematics escaping these observations: logics and Ramsey theory. Logics, by essence, is meta-mathematical and contains constructions that are designed to outgrow the usual proof-theoretic strengths. Ramsey theory, on the other hand, has no *a priori* reason to be a counter-example to these phenomena, and its study represents one of the most active branches of modern reverse mathematics.

Beyond the comparison of theorems based on a formal notion of elementary proof, reverse mathematics play an important foundational and philosophical role in mathematics thanks to these empirical observations. Indeed, the second observation yields that mathematics is somewhat robust, in the sense that if some inconsistencies were to be discovered in ZFC, one could safely remove many strong axioms while keeping a large part of mathematics. Moreover, all the finitary consequences of RCA_0 are already provable over primitive recursive arithmetic (PRA), a very weak theory arguably capturing finitary mathematics. From this perspective, reverse mathematics can be seen as a partial realization of Hilbert's program as an answer to the foundational crisis of mathematics [1].

1.1 Mathematical problems

Many theorems from ordinary mathematics can be seen as *mathematical problems*, formulated in terms of *instances* and *solutions*. Consider for example the *intermediate value theorem* (IVT), which states, for every continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) < 0 < f(1)$ or $f(1) < 0 < f(0)$, the existence of a real number $x \in [0, 1]$ such that $f(x) = 0$. An instance of IVT is a continuous function $f : [0, 1] \to \mathbb{R}$ changing its sign over the interval, and a solution to f is a real number $x \in [0, 1]$ such that $f(x) = 0$. What is the axiomatic power needed to prove the intermediate value theorem?

First of all, one needs to cast this theorem in the setting of second-order arithmetic, with an appropriate coding. A real number can be represented as a fast-converging Cauchy sequence of rational numbers, hence as a set of integers. At first sight, a continuous function from $\mathbb{R}$ to $[0, 1]$ is a third-order object, but since it is fully specified by its values on the rationals, one can also represent a continuous function in second-order arithmetic. Having fixed the representation, both the frameworks of subsystems of second-order arithmetic and computability theory can be applied to the intermediate value theorem.

Thanks to the choice of the base theory, RCA_0, the proof-theoretic analysis of the intermediate value theorem translates to the following computability-theoretic question: *Given a computable instance of the intermediate value theorem, what is the computational content of a solution?* The classical proof of the intermediate value theorem provides an algorithm to find the solution: a dichotomic search. Following the proof, given a computable instance $f : [0, 1] \to \mathbb{R}$, one can define a computable fast-converging Cauchy sequence whose limit is a real number x such that $f(x) = 0$, with one subtlety: the natural order between Cauchy sequences is not decidable. Thankfully, one can circumvent this issue using a case analysis, and show the existence of a computable solution. On the other hand, there is provably no single algorithm which takes a code of such a continuous function as an input, and outputs a solution. From a proof-theoretic perspective, the dichotomic search can be formalized with weak induction assumptions, and the intermediate value theorem is provable over RCA_0.

More generally, the reverse mathematical analysis of a theorem, seen as a mathematical problem, answers two families of problematics:

- The *strength* of the theorem as an individual. What axioms are *necessary* and *sufficient* to prove a theorem? Based on the correspondence between definability and computability, these questions are reformulated in the computability-theoretic language as "What is the *computational strength* of a theorem?" One proves lower bounds by constructing instances such that every solution is computationally strong, and upper bounds by proving that every instance admits some computationally weak solution. Consider for example *König's lemma* (KL), which states that every infinite, finitely branching tree admits an infinite path. By a classical result in computability theory, every computable infinite, finitely branching tree admits an infinite $\emptyset''$-computable path, while there exists a computable infinite, finitely branching tree such that every infinite path computes $\emptyset'$. In the reverse mathematical formalism, this translates into an equivalence between KL and ACA_0 over RCA_0, where ACA_0 is a system capturing the arithmetic hierarchy.
- The *comparison* of two theorems. Does theorem A imply theorem B over RCA_0? Let us compare for example König's lemma, and *Ramsey's theorem for pairs and two colors* (RT^2_2). The latter theorem states the existence, for every graph with infinitely many vertices, of an infinite subset of vertices such that the induced sub-graph is either a clique, or an anti-clique. Given an infinite graph (V, E), one can easily compute an infinite, finitely branching tree such that every infinite path codes for a clique or an anti-clique. Intuitively, König's lemma, seen as a mathematical problem, is at least as hard to solve as Ramsey's theorem for pairs. In reverse mathematics, this construction yields a proof that KL implies RT^2_2 over RCA_0. On the other hand, the reverse implication does not hold: a famous theorem from Seetapun states that Ramsey's theorem for pairs and two colors has no coding power, in the sense that for every computable instance of RT^2_2, if every solution computes a fixed set of integers A, then A is computable. From this, one can build a model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ which does not contain the halting set, and therefore is not a model of KL, thus RT^2_2 does not imply KL over RCA_0. Note that, while the implication from KL to RT^2_2 is elementary, the proof of Seetapun's theorem involves some very clever techniques from effective forcing.

As it happens, when a problem P implies another problem Q from a proof-theoretic or computability-theoretic viewpoint, the reduction is most of the time rather short, if not straightforward, while the proofs of separations usually involve elaborate forcing arguments to preserve a computability-theoretic weakness property. Separating problems in reverse mathematics and proving upper bounds was at the origin of many developments in effective forcing, with the design of new notions of forcing and preservations properties, tailored to witness subtle combinatorial differences between problems. This resulted into a coherent whole of what could be now called a *separation theory*.

1.2 Separation theory

In classical reverse mathematics, proving that a problem P does not imply another problem Q over RCA_0 requires to construct a model of $\mathsf{RCA}_0 + \mathsf{P}$ which is not a model of Q. Furthermore, one usually wants to build counter-examples which are as close to the intended model a possible. In the case of second-order arithmetic, structures are of the form $\mathcal{M} = (M, S, <, +, \times, 0, 1)$ where M denotes the integers of the model (the first-order part) and $S \subseteq \mathcal{P}(M)$ represents the sets of integers (the second-order part). Almost all the proofs of separations in reverse mathematics involve models $\mathcal{M}$ where the set M is the true set of integers ω, equipped with the standard operations. These models are called *ω-models*, and are fully specified by their second-order part S. It is convenient to identify an ω-model $\mathcal{M}$ with the set S. To summarize, the goal is to obtain an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ which is not a model of Q.

Models of RCA_0 are well-understood and easy to construct, thank to the clear computability-theoretic interpretation of the axioms of RCA_0. An ω-model $\mathcal{M}$ with second-order part S satisfies RCA_0 if and only if S is a *Turing ideal*, that is, S is a collection of sets satisfying the following two closure properties: First, if $X \in S$ and X computes a set Y, then $Y \in S$. Second, if X and Y belong to S, then their effective union $X \oplus Y = \{2n : n \in X\} \cup \{2n + 1 : n \in Y\}$ also belongs to S. For instance, the collection of all the computable sets forms a Turing ideal, and more generally, given any fixed set X, the collection $\{Y : Y \leq_T X\}$ is a Turing ideal. Last, a union of an increasing sequence of Turing ideals is again a Turing ideal.

The idea to construct an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ which is not a model of Q goes as follows: First, construct a computable instance X_Q of Q with no computable solution. The solutions of this instance should be as hard to compute as possible, to simplify the construction of the model $\mathcal{M}$. Let $\mathcal{M}_0$ be the ω-model whose second-order part consists of the computable sets. In particular, $\mathcal{M}_0 \models \mathsf{RCA}_0$ but $\mathcal{M}_0$ does not satisfy Q, as the instance X_Q belongs to $\mathcal{M}_0$, but has no solution in $\mathcal{M}_0$. The problem is that $\mathcal{M}_0$ will usually not satisfy P either.

Given an instance $X_0 \in \mathcal{M}_0$ of P with no solution in $\mathcal{M}_0$, we shall construct a solution Y_0, and and extend $\mathcal{M}_0$ into another model $\mathcal{M}_1$ of RCA_0 containing Y_0. In order to obtain a model of RCA_0, the second-order part $\mathcal{M}_1$ must not only contain Y_0, but all the Y_0-computable sets. The initial model $\mathcal{M}_0$ might contain infinitely many P-instances with no solution in $\mathcal{M}_0$, and when extending $\mathcal{M}_0$ into $\mathcal{M}_1$, one might add even more P-instances. We shall therefore carefully list all these instances, and build an increasing sequence $\mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \mathcal{M}_2 \subsetneq \dots$ of ω-models of RCA_0, such that every P-instance $X \in \mathcal{M}_n$ has a solution in $\mathcal{M}_m$ for some $m \geq n$. Then, letting $\mathcal{M} = \bigcup_n \mathcal{M}_n$, the second-order part is again a Turing ideal, so $\mathcal{M} \models \mathsf{RCA}_0$, and by construction, $\mathcal{M} \models \mathsf{P}$.

There is an important issue in the previous construction: when extending a model $\mathcal{M}_n$ into a larger model $\mathcal{M}_{n+1}$ containing a solution Y_n to a P-instance X_n, one adds many sets, including the Y_n-computable ones, but also the $Y_n \oplus Z$-computable ones for any $Z \in \mathcal{M}_n$. During this extension process, one might inadvertently add a solution to the Q-instance X_Q, loosing our witness of failure of Q. If one is not careful, the final model $\mathcal{M}$ will also satisfy Q. Thankfully, there is some degree of freedom in the choice of a solution Y_n to a P-instance X_n. With an appropriate construction, if $\mathcal{M}_n$ does not contain any Q-solution to X_Q, one might build a P-solution Y_n to X_n such that $\mathcal{M}_{n+1}$ still does not contain any Q-solution to X_Q.

Not containing a solution to X_Q is usually not the good invariant, and part of the difficulty of a proof of separation consists in finding the appropriate computability-theoretic notion of weakness, such that

- There exists a computable instance X_Q of Q with no weak solution.
- For every weak instance X of P, there exists a weak solution.

Thus, a proof of separation of a problem P from a problem Q in reverse mathematics reduces to proving lower bounds to Q and upper bounds to P for an appropriate computability-theoretic notion specific for P and Q.

1.3 Jump control

There are two main families of constructions of solutions to an instance of a problem P: *effective* constructions and *forcing* constructions, the former being often an effectivization of the latter. Forcing therefore plays a central role in reverse mathematics, and in computability theory in general.

Forcing was originally introduced by Paul Cohen to answer open problems in set theory. The main idea is to start with a *ground model* $\mathcal{M}$, and construct a new mathematical object G by approximating it with a set $\mathbb{P}$ of *conditions*. These conditions are partially ordered by a relation $\leq$, intuitively meaning that $q \leq p$ if q is a more precise approximation of G than p. The resulting object G, combined with the model $\mathcal{M}$, defines an *extended model* $\mathcal{M}[G]$, which may not satisfy the same properties. Surprisingly, complex properties of the extended model can already be decided by conditions, in the sense that there exists a *forcing relation* $\Vdash$ between conditions and properties such that, if $p \Vdash \varphi(G)$, then the property $\varphi(G)$ will hold for every appropriate construction containing p. Moreover, the forcing relation is definable with only parameters in the ground model, and because of this, many properties of the extended model $\mathcal{M}[G]$ are *inherited* from the ground model $\mathcal{M}$. Indeed, thanks to the forcing relation, a formula with parameters in the extended model can be translated into another formula in the ground model.

The forcing technique in the computability-theoretic setting shares many features with the set-theoretic setting, with some notable differences: The comprehension scheme in set theory being over all definable formulas, it is sufficient for the forcing relation to be definable in the ground model, to propagate many properties from the ground model to the extended model. In computability theory, on the other hand, the computational content of definable sets is sensitive to the complexity of the defining formula, and one needs to have a forcing relation which is not only definable, but also preserves the complexity of the formulas it forces, in order to propagate computability-theoretic properties. Unfortunately, except for some simple cases such as Cohen forcing, the notions of forcing considered in computability theory do not admit a forcing relation with the desired definitional properties.

The novelty of this book is the emphasis of a related concept, called *forcing question*, which usually admits better definitional features that the associated forcing relation, and is sufficient to propagate computability-theoretic properties from the ground model to the extended model. This notion is not relevant in set theory, as the axioms are coarse enough to define a trivial forcing question from the forcing relation, but are of central interest in computability theory. We call "forcing question" any relation $?\vdash$ between a condition p and a formula $\varphi(G)$, such that if $p \mathrel{?\vdash} \varphi(G)$ holds, then there is an extension $q \leq p$ forcing $\varphi(G)$, and it not, then there is an extension $q \leq p$ forcing $\neg\varphi(G)$. A forcing question can be thought of as a completion of the forcing relation, dividing the set of conditions into two categories. Contrary to the forcing relation, there is no canonical forcing question, as any condition which forces neither a formula nor its negation can be put in either category. The whole difficulty is to design a forcing question with the appropriate definitional complexity. As we shall see throughout the book, beyond the definitional complexity of the forcing question, its combinatorial properties have a strong impact on the computability-theoretic features of the constructed object. The nth-fold Turing jump of G being $\Sigma^0_n(G)$-complete, the set of techniques for deciding Σ^0_n-formulas is known as *nth jump control*, and essentially consists in designing a forcing question for Σ^0_n-formulas with the appropriate definitional and combinatorial properties.

Although our main motivation is reverse mathematics, the techniques of iterated jump control have applications in many domains of computability theory and weak arithmetic.

1.4 Audience

This book aims at bridging the gap between the general introductory textbooks on computability theory and reverse mathematics on one hand, and the state-of-the-art research articles in reverse mathematics on the other hand. It is therefore not meant to be read as first intention, and assumes a prior knowledge of computability theory. Some familiarities with reverse mathematics would also be beneficial to the reader to give some motivation, although the basic concepts are re-introduced in Chapter 2.

The primary audience is graduate students in computability theory and researcher from other fields wanting to get familiar with the techniques used in reverse mathematics, but I believe it could also be of interest to some other well-established researchers in computability theory, given the recent identification of the forcing question as a central tool to study the computability-theoretic weakness of a forcing notion.

1.5 Book structure

This monograph is not meant to be read linearly, but each chapter forms almost a monolithic block focusing on one aspect of iterated jump control. Because of this, each chapter starts with a list of dependencies.

- *Chapter 2: Prerequisites* presents computability theory, reverse mathematics and forcing in a nutshell. It should not be considered as a proper introduction to these theories, and mostly fixes notation. This chapter can be safely skipped by any researcher familiar with them.
- *Chapter 3: Cone avoidance* introduces the core idea of forcing question through the simplest notion of avoidance, namely, cone avoidance. Although not technically difficult, this is a conceptually important chapter, as it contains many of the important concepts which will be used throughout the book. The highlight application is Seetapun's theorem, stating that Ramsey's theorem for pairs admits cone avoidance.
- *Chapter 4: Lowness* presents an effective version of first-jump control, enabling to construct sets belonging to the arithmetic hierarchy. Besides the intrinsic interest of classifying sets thanks to their definitional complexity, this chapter contains a proof of the low basis theorem for Π^0_1 classes and defines coded Turing ideals, both important notions for

higher jump control. It also contains a proof of a theorem by Cholak, Jockusch and Slaman, stating that every computable instance of Ramsey's theorem for pairs admits solutions of low_2 degree.

- *Chapter 5: Compactness avoidance* summarizes the interrelationship between the use of compactness argument in theorems and structural properties of the forcing question. It contains, among others, a proof of Liu's theorem, which says that Ramsey's theorem for pairs does not imply weak König's lemma.
- *Chapter 6: Custom properties* gives some examples of separations between combinatorial theorems with custom preservation properties, when the classical computability-theoretic notions fail to separate them. These separations involve the Erdős-Moser theorem, the ascending descending sequence and the chain anti-chain principles.
- *Chapter 7: Conservation theorems* applies a formalized version of the first-jump control techniques to prove conservation theorems over weak theories of second-order arithmetic. It contains a proof of the isomorphism theorem for weak König's lemma by Fiori-Carones, Kołodziejczyk, Wong and Yokoyama. This chapter can be skipped by anyone interested in purely computability-theoretic results.
- *Chapter 8: Forcing design* is the missing link in the thought process leading to a separation between two combinatorial theorems. It rationalizes the steps to design a notion of forcing with a good first-jump control, through the examples of the Erdős-Moser and the free set theorems. This is an independent chapter which, although quite short, I believe is of great importance for the researcher in reverse mathematics. It can be read after Chapter 3.
- *Chapter 9: Jump cone avoidance* studies the relationships between the forcing question and second-jump control through jump cone avoidance. The non-continuous nature of jump functionals raise many new challenges, and the core concepts introduced are of central importance for the remaining chapters. It contains a proof by Monin and Patey that every instance of the pigeonhole principle admits a solution of non-high degree.
- *Chapter 10: Jump compactness avoidance* is probably the most technical chapter of this book, as it combines the complexity of second-jump control with the techniques of compactness avoidance, which happens to raise many issues. The main theorem of this chapter is a theorem by Monin and Patey that every Δ^0_2 set admits an infinite subset in its or its complement whose jump is not of PA degree over $\emptyset'$.
- *Chapter 11: Higher jump cone avoidance* generalizes first and second jump control to higher levels of the arithmetic and the hyperarithmetic hierarchy. The conceptual difficulty mainly comes from the generalization of computability theory to the transfinite realm, known as higher recursion theory.

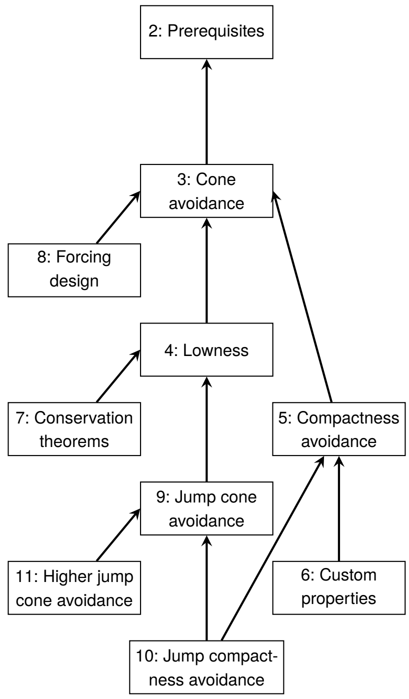

Figure 1.1: Dependencies between the chapters

Prerequisites 2

This textbook *is not* an introduction to computability theory or to reverse mathematics. The reader is assumed to have attended at least a first course in computability theory, and have a general background in mathematical logics, especially first-order logic and forcing. This chapter will recall basic facts of common knowledge, for the sake of self-containment and mostly to fix notation.

This book is a pedagogical resource to learn some specific techniques for computability-theoretic analysis for combinatorial theorems. It tries to bridge the gap between introductory textbooks in computability theory, and research articles on the field. The emphasis is put on the intellectual process of research rather than the actual theorems and end-results.

Where to learn computability theory? There are many books about computability theory. Cooper [2] is probably the most accessible resource for a first introduction to the subject. Soare [3] is a good alternative, although slightly more technical. Monin and Patey [4] provides a general overview of both computability theory and reverse mathematics.

Where to learn reverse mathematics? The field being younger, there are only a few options to learn reverse mathematics. The historical book is Simpson [5], is still a good reference, but its very formal style might be off-putting. A first reader might prefer Dzhafarov and Mummert [6] or Monin and Patey [4] as a gentle introduction. Hirschfeldt [7] monograph is also a good starting point for a reader familiar with computability theory.

2.1 Computability theory

Computability theory is essentially the study of mathematical objects or processes from a computational perspective. It has a primary focus on the structure of the degrees of computation, known as *Turing degrees*.

> **Definition 2.1.1.** Fix a reasonable programming language. A set $X \subseteq \mathbb{N}$ is *computable*[1] if there is an algorithm which, on input $n \in \mathbb{N}$, decides whether n belongs to X or not. $\diamond$

1: Computability theory used to be called Recursion theory. Some literature might use *recursive* for computable and *recursively enumerable* for computably enumerable.

All mainstream programming languages are mutually interpretable, thus the notion of computable set is robust. Moreover, by the Church-Turing thesis, this captures the informal notion of *effectively computable* set. One of the main features of models of computation is their relativization to *oracles*. A set X is Y*-computable* or *Turing reducible to* Y (written $X \leq_T Y$) if it is computable in a programming language enriched with the characteristic function of Y as a primitive.

We write $\Phi_0^Y, \Phi_1^Y, \Phi_2^Y, \dots$ for an effective listing of all programs[2] with oracle Y. The notation $\Phi_e^Y(x)\downarrow = v$ means that the eth program with oracle Y halts on input x and outputs v. If the program does not halt, we write $\Phi_e^Y(x)\uparrow$. Similarly, the notation $\Phi_e^Y(x)[s]\downarrow = v$ means that $\Phi_e^Y(x)\downarrow = v$ in at most s steps of computation. By convention, if $\Phi_e^Y(x)[s]\downarrow = v$, then $v, x < s$. Otherwise,

2: Depending on the context, we may furthermore assume that the programs are $\{0,1\}$-valued, or satisfy some additional decidable structural properties.

$\Phi_e^Y(x)[s]\uparrow$. We may further abstract oracle programs, and consider them as *Turing functionals* from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$, defined by $Y \mapsto \Phi_e^Y$. We then use $\Phi_0, \Phi_1, \Phi_2, \ldots$ as an effective listing of all Turing functionals.

Whenever a program halts, it halts on finite time, and thus with finitely many calls to its oracle. Thus, if $\Phi_e^Y(x)\downarrow$, not only there is some $s \in \mathbb{N}$ such that $\Phi_e^Y(x)[s]\downarrow$, but furthermore there is a shortest initial segment $\sigma \prec Y$ such $\Phi_e^Z(x)[s]\downarrow = \Phi_e^Y(x)$ for every $Z \succ \sigma$. This finite binary string[3] σ is called the *use* of the computation. From a topological viewpoint, this means that Turing functionals are partial continuous functions over the Cantor space[4] $2^{\mathbb{N}}$. We extend Turing functionals to partial oracles, and write $\Phi_e^\sigma(x)\downarrow = v$ to say that the eth program with oracle σ halts on input x and outputs v in less than $|\sigma|$ steps, whose only calls to the oracle are within its domain of definition.

3: We write $2^{<\mathbb{N}}$ for the set of all finite binary strings. Elements of $2^{<\mathbb{N}}$ are written with small greek letters $\sigma, \tau, \rho, \ldots$. We denote by $|\sigma|$ the *length* of the string σ and write $\sigma \preceq \tau$ if σ is a prefix of τ.

4: We write $2^{\mathbb{N}}$ for the class of all infinite binary sequences, also known as Cantor space. It is in one-to-one correspondence with the class of sets of integers, seeing an infinite binary sequence as the characteristic function of a set of integers. We shall therefore identify the two notions and write indistinctly $X \in 2^{\mathbb{N}}$ and $X \subseteq \mathbb{N}$.

2.1.1 Turing degree

Sets of integers are not the appropriate notion to capture the notion of *computational power*. For instance, if X equals Y up to finite changes, or if we let $Y = \{2n : n \in X\}$, then X and Y are mutually computable. The Turing reduction $\leq_T$ is a pre-order on $2^{\mathbb{N}}$. It induces an equivalence relation defined by $X \equiv_T Y$ iff $X \leq_T Y$ and $Y \leq_T X$.

Definition 2.1.2. A *Turing degree* is an equivalence class over $2^{\mathbb{N}}/\equiv_T$. ◇

We write $\deg_T(X) = \{Y \in 2^{\mathbb{N}} : X \equiv_T Y\}$ for the Turing degree of X. The Turing reduction naturally extends to the Turing degrees. The Turing degrees[5] $(\mathcal{D}, \leq)$ form an upper semilattice, with join $\deg_T(X) \cup \deg_T(Y) = \deg_T(X \oplus Y)$, where $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$. The Turing degree $\mathbf{0}$ of the computable sets is the smallest element of this semilattice.

5: We write $\leq_T$ for the Turing reduction over sets, and $\leq$ for the reduction over Turing degrees. We use small boldface letters $\mathbf{a}, \mathbf{b}, \ldots$ to denote Turing degrees.

The *Turing jump* of a set X is the set $X' = \{e : \Phi_e^X(e)\downarrow\}$. The operator $X \mapsto X'$ is Turing-invariant, and therefore induces an operation $\mathbf{a} \mapsto \mathbf{a}'$ over the Turing degrees. By the undecidability of the halting set, $\mathbf{a} < \mathbf{a}'$ for every Turing degree $\mathbf{a}$. The Turing jump can be iterated as follows: $\mathbf{a}^{(0)} = \mathbf{a}$, and $\mathbf{a}^{(n+1)} = (\mathbf{a}^{(n)})'$. Any Turing degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{0}'$ is *low*, and the degrees $\mathbf{b}$ such that $\mathbf{b}' \geq \mathbf{0}''$ are *high*.[6]

6: High degrees used to be defined as $\mathbf{b} \leq \mathbf{0}'$ and $\mathbf{b}' \geq \mathbf{0}''$. Indeed, $\mathbf{0}'$ and $\mathbf{0}''$ are respectively the lowest and the highest value that can take the jump of a degree $\mathbf{d} \leq \mathbf{0}'$, so low and high degrees where Turing degrees at these extremes.

2.1.2 Arithmetic hierarchy

Arithmetically definable sets of integers can be classified based on alternations of quantifiers.

Definition 2.1.3. For $n \geq 1$, a set X is Σ_n^0 if it can be written of the form

$$\{x \in \mathbb{N} : \exists y_1 \forall y_2 \ldots Q y_n\ P(x, y_1, \ldots, y_n)\}$$

where P is a computable predicate, and $Q = \forall$ if n even and $Q = \exists$ if n is odd. Π_n^0 sets are defined accordingly by starting with a universal quantifier. A set is Δ_n^0 if it is both Σ_n^0 and Π_n^0. ◇

By Post theorem, there is a correspondence between definability and computability. The Δ_1^0 sets are precisely the computable sets, and the Σ_1^0 sets are the *computably enumerable* (c.e.) ones, that is, sets of the form $W_e = \operatorname{dom} \Phi_e$

for some $e \in \mathbb{N}$. We write $W_0, W_1, \dots$ for an effective enumeration of the c.e. sets. More generally, the hierarchy can be relativized to any oracle Y by considering Y-computable predicates P. A set is $\Delta^0_n(Y)$ iff it is $Y^{(n-1)}$-computable, and $\Sigma^0_n(Y)$ if it is $Y^{(n-1)}$-c.e.[7]

A c.e. set X can be approximated by an uniformly computable sequence of increasing sets $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ with $X = \bigcup_s X_s$. Such a sequence is a called a *c.e. approximation* of X. Indeed, if $X = \operatorname{dom} \Phi_e$, one can let $X_s = \{x : \Phi_e(x)[s]\downarrow\}$. By Shoenfield's limit lemma, a Δ^0_2 set X can be approximated by a uniformly computable sequence of sets $X_0, X_1, X_2, \dots$ such that for every $n \in \mathbb{N}$, $\lim_s X_s(n)$ exists and equals $X(n)$. Such an approximation is called a *Δ^0_2 approximation of X.*[8]

7: There are three important families of sets:

Computable sets: Given n, it is possible to know whether it belongs to X or not, after a finite amount of time.

C.e. sets: If $n \in X$, then it will be enumerated in X after some point, but if $n \notin X$, we might never known whether it belongs to X or not.

Δ^0_2 sets: These are the $\emptyset'$-computable sets. Given some n, our belief of ownership to X might change finitely often over time, and then stabilize. However, we never know whether we have reached our limit or not.

8: Formally, a Δ^0_2 approximation of X is nothing but a computable function $f : \mathbb{N}^2 \to 2$ such that for every n, $\lim_s f(n,s)$ exists an equals $X(n)$.

2.1.3 Function growth

There is a duality between function growth and computational power. For example, any function dominating the halting time of programs computes the halting set. A function $f : \mathbb{N} \to \mathbb{N}$ *dominates* a function $g : \mathbb{N} \to \mathbb{N}$ if $f(x) \geq g(x)$ for every $x \in \mathbb{N}$. The *principal function* p_X of an infinite set $X = \{x_0 < x_1 < \dots\}$ is defined by $p_X(n) = x_n$.

Definition 2.1.4. A function f is *hyperimmune* if it is not dominated by any computable function. An infinite set X is *hyperimmune* it its principal function is hyperimmune.[9] ◊

9: Equivalently, an infinite set X is hyperimmune if for every c.e. array $\{F_n : n \in \mathbb{N}\}$, there is some $n \in \mathbb{N}$ such that $X \cap F_n = \emptyset$. A *c.e. array* is a c.e. sequence of finite coded non-empty sets which are pairwise disjoint.

A Turing degree $\mathbf{d}$ is *hyperimmune* if it computes (or equivalently contains) a hyperimmune function. Otherwise, $\mathbf{d}$ is *computably dominated* or *hyperimmune-free*. Every non-computable Δ^0_2 set is of hyperimmune degree, but there exists non-zero computably dominated degrees.

Definition 2.1.5. A function f is *dominating* if it eventually dominates every computable function. ◊

By Martin's domination theorem, a function is dominating iff it is of high degree. These degrees are precisely those able to uniformly list the computable sets, with repetitions.

2.1.4 DNC and PA degrees

By Kleene's recursion theorem, there is no total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\Phi_{f(e)} \neq \Phi_e$ for every $e \in \mathbb{N}$. The Turing degrees of fixpoint-free functions are those of diagonally non-computable functions.

Definition 2.1.6. A function f is *diagonally non-computable*[10] (DNC) if for every e, $f(e) \neq \Phi_e(e)$. ◊

10: A DNC function must always give a value, even if $\Phi_e(e)\uparrow$. An immediate diagonal argument shows that no such function is computable.

It might be useful to think of a DNC degree as the power, given a finite c.e. set W_e and a bound $b > \operatorname{card} W_e$, to find a value outside of W_e. A degree is DNC or high iff it contains a function which is almost-everywhere different from every total computable function.

A *binary tree* is a set $T \subseteq 2^{<\mathbb{N}}$ closed under prefix. A *path* through T is an infinite binary sequence $P \in cs$ such that every initial segment belongs to T.

We write $[T]$ for the class of all paths through T. A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is Π^0_1 if it is for the form $[T]$ for some computable (or equivalently for some co-c.e.) tree $T \subseteq 2^{<\mathbb{N}}$. The Π^0_1 classes are the effectively closed classes in Cantor space.

Definition 2.1.7. A degree $\mathbf{d}$ is *PA*[11] if for every infinite computable binary tree $T \subseteq 2^{<\mathbb{N}}$, $\mathbf{d}$ computes an infinite path. ◊

11: Historically, a degree is PA if it contains a completion of Peano Arithmetic. The new definition is more useful in practice.

The PA degrees are precisely those which compute (or equivalently contain) a $\{0, 1\}$-valued DNC function. The class of such functions is Π^0_1, hence there exists a universal computable tree. By the low basis theorem and the computably dominated basis theorem, there are low and computably dominated PA degrees, respectively. A degree is PA or high iff it codes a uniform list of sets which contain, among others, all the computable sets.

2.2 Reverse mathematics

Reverse mathematics is a foundational program at the intersection of computability theory and proof theory, whose goal is to find optimal axioms to prove ordinary theorems.[12] The general idea consists in fixing a very weak base theory capturing *computable mathematics*, and given a theorem T, finding a set of axioms provably equivalent to T over this base theory. More recently, the term "reverse mathematics" took the broader meaning of studying mathematical theorems from the viewpoint of computability theory and proof theory.

12: By "ordinary", we mean theorem which belong to the core of mathematics, outside logics. Indeed, constructions in logics are metamathematical, and thus are often designed to escape the axiomatic strength of the standard mathematical practice.

Traditional reverse mathematics[13] use the language of *second-order arithmetic*, that is, a two-sorted language with integers and sets of integers. In this language, every infinite mathematical object is represented by a set of integers. This enables to apply the framework of computability theory thanks to the correspondence between computability and definability. There are however two drawbacks: First, this restricts the scope to countable mathematics, or at least to mathematics which can be approximated through countable objects. Second, one must define an appropriate coding for every mathematical object. Thankfully, in many cases, the various natural representations of the same mathematical object are computably equivalent.

13: There exists variants of reverse mathematics using the higher-order setting, or intuitionistic logic.

2.2.1 Base theory

The base theory RCA_0, standing for Recursive Comprehension Axiom, consists of Robinson arithmetic Q, together with the Σ^0_1-induction scheme and the Δ^0_1-comprehension scheme. More precisely, Robinson arithmetic[14] is the universal closure of the following axioms:

14: Robinson arithmetic is Peano arithmetic without the induction scheme.

(1) $x + 1 \neq 0$	(5) $x + (y + 1) = (x + y) + 1$
(2) $x = 0 \vee \exists y\ (x = y + 1)$	(6) $x \times 0 = 0$
(3) $x + 1 = y + 1 \rightarrow x = y$	(7) $x \times (y + 1) = (x \times y) + x$
(4) $x + 0 = x$	(8) $x < y \leftrightarrow \exists z\ (z \neq 0 \wedge x + z = y)$

A formula is *arithmetic* if it does not contain any second-order quantifier, but may contain second-order parameters. One can define a syntactic hierarchy of arithmetic formulas similar to the arithmetic hierarchy, by replacing the computable predicate with a Δ^0_0 formula.[15] A Δ^0_0 *formula* contains only bounded first-order quantifiers, that is, quantifiers of the form $\forall x < y$ and $\exists x < y$.

15: Note that some computable sets (and even some primitive recursive sets) are not definable by Δ^0_0 formulas, but every c.e. set is definable by a Σ^0_1 formula, so the hierarchies coincide.

The Σ^0_1-*induction scheme* says, for every Σ^0_1 formula $\varphi(x)$,

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\ \varphi(x)$$

Restricting the induction scheme to capture computable mathematics might seem strange at first sight, as this scheme seems talk only about integers. An integer is a finite object, hence is computable. However, in non-standard models, a bounded set is considered as finite from inside the model, but if the bound is non-standard, it is actually infinite from an external viewpoint, and might be non-computable. Restricting induction restricts the complexity of the finite sets in the model.

The Δ^0_1-*comprehension scheme*[16] says, for every Σ^0_1 formula $\varphi(x)$ and Π^0_1 formula $\psi(x)$,

16: Being Δ^0_1 is not a syntactic notion. One therefore uses the trick of adding $\forall x(\varphi(x) \leftrightarrow \psi(x))$ as a premise, to ensure that the predicate is Δ^0_1.

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall y(\varphi(y) \leftrightarrow y \in X)$$

By relativization of Post's theorem, $X \leq_T Y$ iff X is $\Delta^0_1(Y)$. Therefore, the Δ^0_1-comprehension scheme ensures that the second-order part is downward-closed under the Turing reduction.

2.2.2 Models of RCA_0

A *model* in second-order arithmetic is of the form

$$\mathcal{M} = (M, S, +, \times, <, 0, 1)$$

where $S \subseteq \mathcal{P}(M)$. The *first-order part* M constitutes the integers, and the *second-order part* S are the sets of integers. An ω-*model* is a model whose first-order part is the set of standard integers ω, together with the usual operations $+, \times, <$. An ω-model is therefore fully specified by its second-order part, and is often identified with it. The ω-models of RCA_0 are precisely those whose second-order part is a Turing ideal.

Definition 2.2.1. A *Turing ideal*[17] is a class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ closed under the following two operations:

(1) Turing reduction: $\forall X \in \mathcal{I}\ \forall Y \leq_T X\ Y \in \mathcal{I}$;
(2) Effective join: $\forall X \in \mathcal{I}\ \forall Y \in \mathcal{I}\ X \oplus Y \in \mathcal{I}$. ◇

17: Natural classes of Turing ideals are rare in computability theory. Besides *topped* Turing ideals of the form $\{Z \in 2^{\mathbb{N}} : Z \leq_T X\}$ for a fixed set X, the most notable ideal is the *K-trivials*, used in algorithmic randomness. The low degrees do not form a Turing ideal: there exists two low degrees joining to $\mathbf{0}'$.

The class of all computable sets is the smallest Turing ideal for inclusion. Thus, RCA_0 admits a least ω-model, consisting of only computable sets. It follows that if a theorem implies the existence of a non-computable object, then it is not provable over RCA_0. In this sense, RCA_0 captures computable mathematics.

2.2.3 Big Five

The early study of reverse mathematics witnessed the emergence of four main systems of axioms, linearly ordered by logical strength, such that most of mathematics is either provable in RCA_0, or provably equivalent to one of the four systems over RCA_0. These systems, together with RCA_0, are known as the Big Five. We shall focus on the first two systems, namely, WKL_0 and ACA_0.

- WKL_0, standing for Weak König's lemma, is RCA_0 augmented with the statement "Every infinite binary tree admits an infinite path". This system informally captures compactness arguments. It is equivalent to the Borel-Lebesgue compactness theorem and Gödel's completeness theorem, among others. Contrary to RCA_0, WKL_0 does not admit a least ω-model. The second-order parts of its ω-models are closed under PA degrees, and are called *Scott ideals*.
- ACA_0, standing for Arithmetic Comprehension Axiom, is RCA_0 with the comprehension scheme for every arithmetic formula. Many important theorems, such as the Bolzano-Weierstrass theorem, are equivalent to ACA_0. Since the halting set is Σ^0_1-definable, the second-order parts of its ω-models are closed under the Turing jump, and called *jump ideals*. ACA_0 admits a least ω-model, whose second-order part corresponds to the arithmetic sets.

2.2.4 Computable reductions

More recently, the reverse mathematical framework was enriched with new reductions belonging to the computability-theoretic realm. A *problem*[18] is a relation $\mathsf{P} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$. An *instance* of P is an element of $\operatorname{dom} \mathsf{P} = \{X \in 2^{\mathbb{N}} : \exists Y\ (X, Y) \in \mathsf{P}\}$. Given an instance X of P, we denote by $\mathsf{P}(X) = \{Y : (X, Y) \in \mathsf{P}\}$ the class of *solutions* to X.

18: For instance, König's lemma is the problem whose instances are infinite, finitely branching trees, and a solution to a tree is an infinite path.

Definition 2.2.2. A problem P is *computably reducible* to Q (denoted $\mathsf{P} \leq_c \mathsf{Q}$) if for any instance X of P, there exists an instance $\tilde{X}$ of Q computable in X, such that for any Q-solution $\tilde{Y}$ to $\tilde{X}$, $X \oplus \tilde{Y}$ computes a P-solution to X.[19] ◊

19: One can see a computable reduction as the construction of a P-solver using a Q-solver, with only computable manipulations. Note that the original instance X of P can be used in the computation of the solution.

When the problems P and Q can be formulated as a second-order sentences, a reduction $\mathsf{P} \leq_c \mathsf{Q}$ can be seen as an implication $\mathsf{Q} \to \mathsf{P}$ over ω-models, in which only one application of Q is allowed.

2.3 Effective forcing

The framework of forcing was originally introduced by Paul Cohen to prove independence results in set theory. It is a central tool in computability theory to build sets of integers with specific computational properties, and can be seen as an elaboration of the finite extension method. The simplicity of its use in computability theory makes the setting ideal for a gentle introduction to forcing.

Definition 2.3.1. A *notion of forcing* is a partial order $(\mathbb{P}, \leq)$ together with an interpretation function $[\cdot] : \mathbb{P} \to \mathscr{P}(2^{\mathbb{N}})$ such that if $p \leq q$, then $[p] \subseteq [q]$. ◊

Elements of $\mathbb{P}$ are called *conditions*. If $p \leq q$, then p is an *extension*[20] of q. Informally, a condition p is a partial approximation of the constructed object G, and $[p]$ is the class of all "candidate" objects. If $q \leq p$, then the approximation q is "more precise" than p, hence has less candidates.

20: The term "extension" suggests that p carries more information than q, thus the decreasing order might be confusing. It might be helpful to think of p and q in terms of interpretation. Then the decreasing order represents the decreasing in candidates.

Example 2.3.2. The following are notions of forcing

- Cohen forcing: $2^{<\mathbb{N}}$ with $\tau \leq \sigma$ if σ is a prefix of τ. The interpretation of σ is $[\sigma] = \{X \in 2^{\mathbb{N}} : \sigma \prec X\}$.
- Jockusch-Soare forcing: $\mathbb{P}$ is the partial order of computable infinite binary trees, ordered by inclusion. The interpretation of T is the class of its paths $[T]$.

2.3.1 Filter and genericity

Infinite objects are usually constructed by successive refinement of approximations. In the forcing setting, this would correspond to the construction of an infinite, decreasing sequence of conditions.

Definition 2.3.3. A *filter* on $(\mathbb{P}, \leq)$ is a non-empty class $\mathcal{F} \subseteq \mathbb{P}$ satisfying:

1. upward-closure: $\forall p \in \mathcal{F} \forall q \in \mathbb{P}\ (p \leq q \to q \in \mathcal{F})$
2. compatibility: $\forall p, q \in \mathcal{F} \exists r \in \mathcal{F}\ (r \leq p, q)$. ◇

Filters are a generalization of decreasing sequences of conditions[21], in that every sequence $p_0 \geq p_1 \geq \dots$ induces a filter $\mathcal{F} = \{q \in \mathbb{P} : \exists n\ p_n \leq q\}$. When the filter is appropriately chosen, there is a unique element $G_{\mathcal{F}} \in \bigcap_{p \in \mathcal{F}}[p]$, which is the object constructed by the filter.

21: The distinction between the two notions is not relevant in computability theory, and one might think of a filter as an infinite decreasing sequence of conditions.

Definition 2.3.4. A class $\mathcal{D} \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$, there is some $q \leq p$ in $\mathcal{D}$. ◇

Intuitively, a class is dense if, when defining an infinite decreasing sequence of conditions, it is never too late to intersect $\mathcal{D}$. Indeed, at any point p_n of the construction, there exists an extension $p_{n+1} \leq p_n$ in $\mathcal{D}$.

Definition 2.3.5. A filter $\mathcal{F}$ is *generic* for a family of classes $\{\mathcal{D}_i\}_{i \in I}$ if $\mathcal{F} \cap \mathcal{D}_i \neq \emptyset$ for every $i \in I$. ◇

One can easily see by a greedy construction of an infinite decreasing sequence of conditions that every countable family of dense classes admits a generic filter. Given a notion of forcing $(\mathbb{P}, \leq)$ and a property $\varphi(G)$, the statement "Every sufficiently generic[22] set satisfies $\varphi(G)$" means that there exists a countable sequence of dense classes $\{D_n\}_{n \in \mathbb{N}}$ such that, for every $\{D_n\}_{n \in \mathbb{N}}$-generic filter $\mathcal{F}$, $\varphi(G_{\mathcal{F}})$ holds.

22: The concept of "sufficient genericity" alone does not exist, and always depends on a property $\varphi(G)$. We shall however sometimes say "Let $\mathcal{F}$ be a sufficiently generic filter" to mean that its level of genericity will be determined by the future properties we want $G_{\mathcal{F}}$ to satisfy.

All the notions of forcing we shall consider satisfy the following property:

(†) For every $n \in \mathbb{N}$, the following class is dense:

$$\mathcal{D}_n = \{p \in \mathbb{P} : \exists \sigma \in 2^n\ [p] \subseteq [\sigma]\}$$

In particular, for every $\{D_n\}_{n \in \mathbb{N}}$-generic filter $\mathcal{F}$, the intersection $\bigcap_{p \in \mathcal{F}}[p]$ will be a singleton.

2.3.2 Forcing relation

The core feature of forcing is the ability, given only an approximation $p \in \mathbb{P}$ of the object under construction, to already determine some properties the set will satisfy, no matter the remainder of the construction. Surprisingly, a very large class of properties can be determined in advance by approximations.

Definition 2.3.6. A condition $p \in \mathbb{P}$ *forces*[23] a property $\varphi(G)$ if for every sufficiently generic filter $\mathcal{F}$ containing p, $\varphi(G_{\mathcal{F}})$ holds. ◇

23: The naive approach would be to say that a condition p forces a property $\varphi(G)$ if it holds for every $G \in [p]$. This relation is too strong and does not enjoy the desirable properties of a forcing relation.

The above definition shall be referred to as a *semantic* definition. From a definitional viewpoint, the semantic definition is very complicated, as it requires to quantify over filters, which are higher-order objects. Thankfully, there exists an inductive syntactic definition of the forcing relation with much better definitional features.

In our setting, we shall be interested only in arithmetic properties.

Proposition 2.3.7. Let $(\mathbb{P}, \leq)$ be a notion of forcing satisfying (†) and $\varphi(G)$ be an arithmetic formula.

1. If p forces $\varphi(G)$ and $q \leq p$, then q forces $\varphi(G)$.
2. The class $\{p \in \mathbb{P} : p \text{ forces } \varphi(G) \text{ or } p \text{ forces } \neg\varphi(G)\}$ is dense. ★

This last property is essential, as it says that every arithmetic property can be decided by some condition. In particular, for every sufficiently generic filter $\mathcal{F}$, and every arithmetic formula $\varphi(G)$, then $\varphi(G_{\mathcal{F}})$ holds iff there is a condition $p \in \mathcal{F}$ forcing $\varphi(G)$.

First jump control

Cone avoidance 3

The appellation *first-jump control*[1] encompasses the set of techniques to build a set G while controlling its $\Sigma^0_1(G)$ properties. An immediate application is the construction of sets of low degree whenever the process is Δ^0_2. With the development of reverse mathematics, the subject gained a whole lot of interest, as being the main tool to prove separations over RCA_0. We shall see a variety of preservation properties (cone avoidance, PA avoidance, ...) motivated by specific subsystems of second-order arithmetic, such as ACA_0 and WKL_0. Nowadays, these techniques are part of the mandatory toolbox of a researcher in reverse mathematics.

Prerequisites: Chapter 2

1: The name might be confusing at first, since the technique is about computation and not jump computation. Actually, by deciding $\Sigma^0_1(G)$ properties, the first-jump control determines what the jump G' *is*, not what it *computes*. Moreover, since the predicate $\Phi^G_e(x)\downarrow$ is $\Sigma^0_1(G)$, the first-jump control enables to decide G-computation.

The general setting is the following: One wants to build a set G satisfying some structural properties (being a path through a tree, being homogeneous for a coloring, or more generally being a solution to an instance of a mathematical problem), while preserving some computational weakness properties (not computing a fixed set, not being of PA degree, being of low degree). There is a tension between the computational strength induced by the structural properties, and the desired computational weakness. As it turns out, all these proofs have a common denominator: the design of a so-called forcing question with good definitional properties. The study of the relation between the forcing question and iterated jump control constitutes the main subject of this textbook.

The first weakness property that we shall consider is called *cone avoidance*. Proofs of cone avoidance are good examples of the use of the forcing question, and they do not require to make the whole construction effective, as in proofs of lowness.

3.1 Context and motivation

Consider a mathematical problem P, formulated in term of *instances* and *solutions*.[2] The computability-theoretic study of P consists in identifying, given a (computable) instance X of P, the computational power of computing a solution to X. For this, one proves lower bounds, of the form "There exists a (computable) instance of P such that every solution is computationally strong" and upper bounds of the form "Every (computable) instance of P admits a computationally weak solution".

2: For example, *weak König's lemma* is the problem whose instances are infinite binary trees, and whose solutions are infinite paths

One of the first questions to ask about the strength of a problem is its ability to *encode* a Turing degree. More precisely, given a set C, is there a computable instance of P such that every solution computes C? This question is about the *computational strength* of P. One can ask the same question with no computable restriction to the instance of P. It is then about the *combinatorial strength* of P. The notion of cone avoidance is a strong negative answer to the first question.

Definition 3.1.1. A problem P admits *cone avoidance* if for every set Z and every non-Z-computable set C, every Z-computable instance X of P admits a solution Y such that C is not $Z \oplus Y$-computable. ◊

It might be simpler to think of its unrelativized version, where $Z = \emptyset$. Every known natural problem which satisfies the unrelativized version also satisfies the general statement. However, one can create artificial problems which do not.

Informally, if a problem admits cone avoidance, then it is not able to encode any non-computable Turing degree. If one drops the restriction by replacing "every Z-computable instance X of P" with "every instance X of P", one obtains the notion of *strong cone avoidance*.

A proof of cone avoidance of a problem P is an interesting statement in its own right, but it also has useful consequences in reverse mathematics. Recall that ACA_0 is the base system RCA_0 augmented with the comprehension axiom for arithmetical formulas with parameters. Since the halting set $\emptyset'$ is Σ^0_1-definable, every ω-model of ACA_0 contains the halting set. [3]

3: By the same argument, every ω-model of ACA_0 is closed under the Turing jump. Actually, there exists a smallest ω-model of ACA_0 whose second-order part is exactly the arithmetical sets.

On the other hand, if a Π^1_2 problem P admits cone avoidance[4], then it admits an ω-model which avoids the halting set, hence is not a model of ACA_0.

4: A problem P is Π^1_2 if if the relations $X \in \operatorname{dom} \mathsf{P}$ and $Y \in \mathsf{P}(X)$ are both arithmetically definable. Then, $\mathcal{M} \models \mathcal{P}$ if

$$\mathcal{M} \models \forall X \in \operatorname{dom} \mathsf{P}\ \exists Y \in \mathsf{P}(X)$$

Proposition 3.1.2. Fix a non-computable set C. Let P be a Π^1_2 problem which admits cone avoidance. There exists an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ which does not contain C. ★

Proof. Recall that an ω-model is fully characterized by its second-order part, and that it satisfies RCA_0 iff its second-order part is a Turing ideal. Also recall that $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ is Cantor's pairing function.

We are going to define a sequence of sets $Z_0 \leq_T Z_1 \leq_T \dots$ such that for all $n \in \mathbb{N}$,

(1) if $n = \langle e, s \rangle$ and $\Phi_e^{Z_s}$ is a P-instance X, then Z_{n+1} computes a solution to X;
(2) $C \not\leq_T Z_n$.

$Z_0 = \emptyset$. Suppose we have defined Z_n and say $n = \langle e, s \rangle$. If $\Phi_e^{Z_s}$ is not a P-instance, then let $Z_{n+1} = Z_n$. Otherwise, by cone avoidance of P relativized to Z_n, there is a solution Y to $\Phi_e^{Z_s}$ such that $C \not\leq_T Z_n \oplus Y$. Let $Z_{n+1} = Z_n \oplus Y$.

Let $\mathcal{I} = \{X \in 2^{\mathbb{N}} : \exists n\ X \leq_T Z_n\}$. By construction, the class $\mathcal{I}$ is a Turing ideal. Moreover, by (1), every P-instance $X \in \mathcal{I}$ admits a solution in $\mathcal{I}$. Last, by (2), $C \notin \mathcal{I}$. ∎

3.2 First examples

Before starting the development of an abstract framework to prove cone avoidance, let us start with a few basic proofs, in order to see some emerging patterns.

The most basic example of cone avoidance is Cohen genericity. Indeed, this notion of forcing enjoys very nice computability-theoretic features: the partial order is computable, with a computable domain. Recall that Cohen forcing is the notion of forcing whose conditions are finite strings, partially ordered by the suffix relation.

Theorem 3.2.1
Let C be a non-computable set. For every sufficiently Cohen generic set G, $C \not\leq_T G$.

Proof. It suffices to prove the following lemma, where $\Phi_e^G \neq C$ is a shorthand for $\exists x \Phi_e^G(x)\uparrow \vee \exists x \Phi_e^G(x)\downarrow \neq C(x)$.

Lemma 3.2.2. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^G \neq C$. ★

Proof. Fix a condition σ. Consider the following set[5]

$$U = \{(x, v) \in \mathbb{N} \times 2 : \exists \tau \geq \sigma\ \Phi_e^\tau(x)\downarrow= v\}$$

Note that the set U is Σ_1^0. There are three cases:[6]

- Case 1: $(x, 1 - C(x)) \in U$ for some $x \in \mathbb{N}$. Let $\tau \geq \sigma$ witness $(x, 1 - C(x)) \in U$, that is, let $\tau \geq \sigma$ be such that $\Phi_e^\tau(x)\downarrow= 1 - C(x)$. Then τ forces $\Phi_e^G \neq C$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. We claim that σ already forces $\Phi_e^G \neq C$. Indeed, if for some $Z \in [\sigma]$, $\Phi_e^Z = C$, then by the use property, these is some $\tau \preceq Z$ such that $\Phi_e^\tau(x)\downarrow= C(x)$, and by choosing τ long enough, it would witness $(x, C(x)) \in U$, contradiction.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_1^0 graph of the characteristic function of C, hence C is computable. This contradicts our hypothesis.[7] ■

5: In other words, U is a set of pairs (input/-value) such that one can find an extension forcing $\Phi_e^G(x)$ to halt and output v. This set will be recurrent in the proofs of cone avoidance, with the 3-case analysis pattern.

6: The idea is the following: the set U claims to be a nice (Σ_1^0) description of a set C which is hard to describe (not computable). Thus, either U gives only partial information about C (Case 2) or it gives some wrong information (Case 1).

7: We assume here that the functional Φ_e is $\{0, 1\}$-valued.

We are now ready to prove Theorem 3.2.1. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions τ forcing $\Phi_e^G \neq C$. It follows from Lemma 3.2.2 that every $\mathcal{D}_e$ is dense, hence every $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic set G satisfies $C \nleq_T G$. ■

Theorem 3.2.1 can be used to prove the existence of incomparable Turing degrees, as shows the following exercise:

Exercise 3.2.3.

1. Fix a set C. Show that for every sufficiently Cohen generic set G, C does not compute G.
2. Use Theorem 3.2.1 and the previous question to deduce the existence of incomparable Turing degrees. ★

The following example shows that every set A admits a Δ_2^0 description which avoids a cone. It is a fundamental bridge between computational weaknesses and combinatorial weaknesses of theorems, as we shall see later.

Theorem 3.2.4
Fix a set A and a non-computable set C. There exists a set G such that $G' \geq_T A$ and $G \ngeq_T C$.

Proof. By Shoenfield's limit lemma [8], $G' \geq_T A$ iff there is a G-computable function $f : \mathbb{N}^2 \to 2$ such that for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists and equals $A(x)$. We are therefore going to build directly the function f by forcing, and let G be the graph of f. The forcing conditions are pairs (g, n), such that

- $g \subseteq \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ is a partial function[8] with two parameters whose domain is finite, representing an initial segment of the function f that we are building.
- m is an integer "locking" the m first columns of f to the m first bits of A, meaning that from now on, when we extend the domain of g with a new pair (x, y), if $x < m$ then $g(x, y) = A(x)$.

8: The notation $f \subseteq A \to B$ is used for partial functions from A to B.

In other words the first m columns of the function f have already reached their limit behavior, which is $A\upharpoonright_m$. The *interpretation* $[g, m]$ of a condition (g, m) is the class of all partial or total functions $h \subseteq \mathbb{N}^2 \to 2$ such that

(1) $g \subseteq h$, i.e. $\operatorname{dom} g \subseteq \operatorname{dom} h$ and for all $(x, y) \in \operatorname{dom} g$, $g(x, y) = h(x, y)$;
(2) for all $(x, y) \in \operatorname{dom} h \setminus \operatorname{dom} g$, if $x < m$, then $h(x, y) = A(x)$.

Note that set of conditions is computable, but unlike Cohen forcing, the partial order is not. Thankfully, for a fixed condition (g, n), the set of all conditions extending (g, n) is computable. Indeed, it suffices to "hard code" the initial segment $A\upharpoonright_n$ in the algorithm, which is a finite piece of information.

A condition (h, n) *extends* (g, m) (denoted $(h, n) \leq (g, m)$) if $n \geq m$ and $h \in [g, m]$. Every filter $\mathcal{F}$ for this notion of forcing induces a function $f_{\mathcal{F}} = \bigcup\{g : (g, n) \in \mathcal{F}\}$. In particular, $f_{\mathcal{F}} \in \bigcap\{[g, n] : (g, n) \in \mathcal{F}\}$. Moreover, if $\mathcal{F}$ is sufficiently generic, then $f_{\mathcal{F}}$ is total, and $\lim_x f_{\mathcal{F}}(x, y) = A(x)$.

Lemma 3.2.5. For every condition (g, n) and every Turing index $e \in \mathbb{N}$, there is an extension $(h, n) \leq (g, n)$ forcing $\Phi_e^f \neq C$. ⋆

Proof. Fix a condition (g, n). Consider the following set

This is the second appearance of the set U of all pairs (input/value) such that one can find an extension forcing $\Phi_e^f(x)$ to halt and output v.

$$U = \{(x, v) \in \mathbb{N} \times 2 : \exists h \in [g, n]\ \Phi_e^h(x)\downarrow = v\}$$

Note that the set U is Σ_1^0 since by the use property, the existential quantifier is first-order. There are three cases:

We have the same 3-case analysis as in the proof Lemma 3.2.2, and which is characteristic of proofs of cone avoidance.

- Case 1: $(x, 1 - C(x)) \in U$ for some $x \in \mathbb{N}$. Let $h \in [g, n]$ witness $(x, 1-C(x)) \in U$, that is, let $h \in [g, n]$ be such that $\Phi_e^h(x)\downarrow = 1-C(x)$. Then (h, n) forces $\Phi_e^f \neq C$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. We claim that (g, n) already forces $\Phi_e^f \neq C$. Indeed, if for some $f \in [g, n]$, $\Phi_e^f = C$, then by the use property, these is some finite $h \subseteq f$ such that $\Phi_e^h(x)\downarrow = C(x)$, and by choosing $\operatorname{dom} h \supseteq \operatorname{dom} g$, it would witness $(x, C(x)) \in U$, contradiction.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_1^0 graph of the characteristic function of C, hence C is computable. This contradicts our hypothesis. ■

We are now ready to prove Theorem 3.2.4. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing, and let $f = f_{\mathcal{F}}$. The set of conditions (g, n) such that $x \in \operatorname{dom} g$ is dense, thus f is total. Moreover, for every $k \in \mathbb{N}$, the set of conditions (g, n) such that $n \geq k$ is also dense, so for every $x \in \mathbb{N}$, $\lim_y f(x, y) = A(x)$. Last, by Lemma 3.2.5, $f \not\geq_T C$. This completes the proof of Theorem 3.2.4. ■

Recall that a set G is of *high* degree if $G' \geq_T \emptyset''$. It follows from Theorem 3.2.4 that if C is a non-computable set, there exists a set G of high degree such that $C \not\leq_T G$.

Our last example is the famous *cone avoidance* Π_1^0 *basis theorem*. It says that if every path of an infinite computable binary tree computes a single set, then this set is computable. This will be our first example of the use of an over-approximation because the natural formula does not have the desired complexity.

Theorem 3.2.6 (Jockusch and Soare [9])
Fix a non-computable set C and a non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$. There exists a member $G \in \mathcal{P}$ such that $G \not\geq_T C$.

Proof. Jockusch-Soare forcing is the notion of forcing whose conditions are infinite computable binary trees $T \subseteq 2^{<\mathbb{N}}$, partially ordered by the subset relation. The *interpretation* $[T]$ of a tree T is the class of its paths. Every sufficiently filter $\mathcal{F}$ for this notion of forcing induces a path $G_{\mathcal{F}}$ which is the unique element of $\bigcap\{[T] : T \in \mathcal{F}\}$.

Lemma 3.2.7. For every condition T and every Turing index $e \in \mathbb{N}$, there is an extension $S \subseteq T$ forcing $\Phi^G_e \neq C$. ⋆

A natural first attempt would be to define $\mathcal{U}$ as the set

$$\{(x, v) : \exists\sigma \text{ extendible in } T\ \Phi^\sigma_e(x)\downarrow= v\}$$

However, being extendible is a Π^0_1 predicate, hence U would be Σ^0_2. The third case would then yield that C is $\emptyset'$-computable, which does not contradict our hypothesis.

The over-approximation is the following: at every length, at least one node must be extendible in T, so it suffices to ask the property to hold for every nodes of a given length.

Proof. Fix a condition T. Consider the following set

$$U = \{(x, v) \in \mathbb{N} \times 2 : \exists\ell \in \mathbb{N}\forall\sigma \in 2^\ell \cap T\ \Phi^\sigma_e(x)\downarrow= v\}$$

Note that the set U is Σ^0_1. There are three cases:

- Case 1: $(x, 1 - C(x)) \in U$ for some $x \in \mathbb{N}$. We claim that T already forces $\Phi^G_e \neq C$. Indeed, for every $G \in [T]$, letting $\sigma = G\upharpoonright_\ell$, where ℓ witnesses $(x, 1 - C(x)) \in U$, we have $\sigma \in 2^\ell \cap T$, hence $\Phi^\sigma_e(x)\downarrow= 1 - C(x)$. By the use property, $\Phi^G_e(x)\downarrow= 1 - C(x)$
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. Let

$$S = \{\sigma \in T : \forall s < |\sigma|\ \Phi^\sigma_e(x)[s]\uparrow \vee \Phi^\sigma_e(x)[s]\downarrow\neq C(x)\}$$

  Since $(x, C(x)) \notin U$, S contains a string of every length. Moreover, S is closed under prefix, so it is an infinite binary subtree of T. Again, by the use property, S forces $\Phi^G_e \neq C$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_1 graph of the characteristic function of C, hence C is computable. This contradicts our hypothesis. ■

We still have the same 3-case analysis as in the proof Lemma 3.2.2, but the situation is slightly different: instead of taking a proper extension in Case 1 and already forcing the property in Case 2, the situation is inverted.

We are now ready to prove Theorem 3.2.6. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing, and let $G = G_{\mathcal{F}}$. By Lemma 3.2.7, $G \not\geq_T C$. This completes the proof of Theorem 3.2.6. ■

Exercise 3.2.8. A *(computable) Mathias condition* is a pair (σ, X) where $\sigma \in 2^{<\mathbb{N}}$ and $X \subseteq \mathbb{N}$ is an infinite (computable) set with $|\sigma| < \min X$. The *interpretation* $[\sigma, X]$ of a (computable) Mathias condition is the class $\{Y \in 2^{\mathbb{N}} : \sigma \subseteq Y \subseteq \sigma \cup X\}$, identifying σ with the finite set $\{n < |\sigma| : \sigma(n) = 1\}$. Intuitively, σ is the initial segment of the set that we construct, and X is an infinite reservoir which restricts the futur elements of the set.

A condition (τ, Y) *extends* a condition (σ, X) if $\tau \succeq \sigma$, $Y \subseteq X$ and $\tau \setminus \sigma \subseteq X$. Every filter $\mathcal{F}$ for this notion of forcing induces a set $G_{\mathcal{F}} = \bigcup\{\sigma : (\sigma, X) \in \mathcal{F}\}$.

Prove that if C is a non-computable set, then for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$. ⋆

3.3 Forcing question

One can easily see an emerging pattern in all the previous proofs of cone avoidance. In every case, given a condition p, one defines a set U of pairs

(x, v) such that such that there is an extension forcing $\Phi_e^G(x)\downarrow = v$. Moreover, for every pair (x, v) outside U, there is an extension forcing the opposite. This motivates the following definition:

Definition 3.3.1. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a *forcing question* is a relation $?\vdash : \mathbb{P} \times \Gamma$ such that, for every $p \in \mathbb{P}$ and $\varphi(G) \in \Gamma$,

1. If $p \mathrel{?\vdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\varphi(G)$;
2. If $p \mathrel{?\nvdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\neg\varphi(G)$. ◊

One can see a forcing question as a completion of the forcing relation. Intuitively, given a formula $\varphi(G) \in \Gamma$, one can divide the conditions in $\mathbb{P}$ into three categories: the ones which force $\varphi(G)$, those which force $\neg\varphi(G)$, and the ones which do not decide $\varphi(G)$. A forcing question has no degree of freedom when considering conditions of the first two categories: it must give the appropriate answer. On the other hand, a condition belonging to the third category has extensions forcing $\varphi(G)$ and other extensions forcing $\neg\varphi(G)$. A forcing question draws a dividing line within this category.

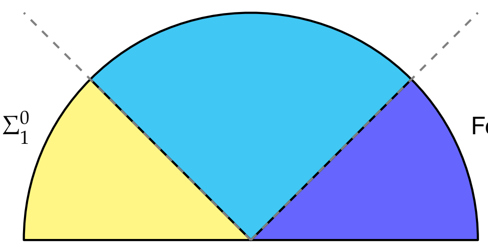

Figure 3.1: The yellow part and the dark blue part represent the conditions forcing a fixed Σ_1^0 and its negation, respectively. The light blue part represent the conditions of the third category. In the proof of Theorem 3.2.6, the dividing line is at the left-most position, while for Cohen forcing, the dividing line is at the opposite position.

Exercise 3.3.2. Show that a relation $?\vdash : \mathbb{P} \times \Gamma$ is a forcing question for Γ iff it satisfies the following properties:

1. If p forces $\varphi(G)$, then $p \mathrel{?\vdash} \varphi(G)$;
2. If p forces $\neg\varphi(G)$, then $p \mathrel{?\nvdash} \varphi(G)$. ★

In each cone avoidance proof, one then considers the following set:

$$U = \{(x, v) \in \mathbb{N} \times 2 : p \mathrel{?\vdash} \Phi_e^G(x)\downarrow = v\}$$

By definition of a forcing question, the two first cases can be handled abstractly. On the other hand, the contradiction of the third case lies on the complexity of the set U. This is our last ingredient of the proof.

Definition 3.3.3. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*preserving* if for every $p \in \mathbb{P}$ and every formula $\varphi(G, x) \in \Gamma$, the relation $p \mathrel{?\vdash} \varphi(G, x)$ is in Γ uniformly in x. ◊

We are now ready to prove our abstract theorem of cone avoidance.

Theorem 3.3.4
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ_1^0-preserving forcing question.

For every non-computable set C and every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$.

Proof. It suffices to prove the following lemma:

Lemma 3.3.5. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi_e^G \neq C$. ⋆

Proof. Consider the following set

$$U = \{(x, v) \in \mathbb{N} \times 2 : p \mathrel{?{\vdash}} \Phi_e^G(x){\downarrow}= v\}$$

Since the forcing question is Σ_1^0-preserving, the set U is Σ_1^0. There are three cases:

- Case 1: $(x, 1-C(x)) \in U$ for some $x \in \mathbb{N}$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^G(x){\downarrow}= 1 - C(x)$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. By Property (2) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^G(x){\uparrow}$ or $\Phi_e^G(x){\downarrow}\neq C(x)$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_1^0 graph of the characteristic function of C, hence C is computable. This contradicts our hypothesis. ■

We are now ready to prove Theorem 3.3.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi_e^G \neq C$.. It follows from Lemma 3.3.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $C \not\leq_T G_{\mathcal{F}}$. This completes the proof of Theorem 3.3.4. ■

By the abstract theorem above, the question whether a problem admits cone avoidance is reduced to the question whether one can construct solutions using a notion of forcing which admits a forcing question with the right definitional property.

We can revisit the previous proofs in terms of forcing questions.

Exercise 3.3.6. Given a string $\sigma \in 2^{<\mathbb{N}}$ and a Σ_1^0 formula $\varphi(G)$, define $\sigma \mathrel{?{\vdash}} \varphi(G)$ to hold if there is some $\tau \succeq \sigma$ such that $\varphi(\tau)$ holds. Prove that the relation is a Σ_1^0-preserving forcing question for Cohen forcing. ⋆

Exercise 3.3.7. Given a computable infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ and a Σ_1^0 formula $\varphi(G)$, define $T \mathrel{?{\vdash}} \varphi(G)$ to hold if there is some level $\ell \in \mathbb{N}$ such that $\varphi(\sigma)$ holds for every node σ at level ℓ in T. Prove that the relation is a Σ_1^0-preserving forcing question for Jockusch-Soare forcing. ⋆

The notion of forcing question is more useful as a unifying terminology than as a formal notion. We shall see in the next section a disjunctive notion of forcing building two generic sets simultaneously. Although the concept of forcing question will need some adaptation to the current setting, the similarity of terminology will help emphasize the common features with the previous proofs of cone avoidance.

3.4 Seetapun's theorem

In short, Seetapun's theorem states that Ramsey's theorem for pairs admits cone avoidance. It is one of the most celebrated theorems of reverse mathematics. Given a set $X \subseteq \mathbb{N}$, we let $[X]^n$ denote the set of all n-element subsets of X.[9] A set $H \subseteq \mathbb{N}$ is *homogeneous* for a coloring $f : [\mathbb{N}]^n \to k$ if f is monochromatic on $[H]^n$. Ramsey's theorem for n-tuples and k colors is the problem RT^n_k whose instances are colorings $f : [\mathbb{N}]^n \to k$ and whose solutions are infinite f-homogeneous sets.[10]

9: We shall often identify $[X]^n$ with the set of increasing ordered n-tuples, and write $f(x_0, \dots, x_{n-1})$ rather than $f(\{x_0, \dots, x_{n-1}\})$, assuming $x_0 < \dots < x_{n-1}$.

10: Ramsey's theorem is formulated in terms of colorings of $[\mathbb{N}]^n$. However, it is a set-theoretic statement, and it still holds when replacing $\mathbb{N}$ with any infinite set. One can prove prove this stronger statement as a blackbox: Given an infinite set $X \subseteq \mathbb{N}$ and a coloring $f : [X]^n \to k$, define the coloring $g : [\mathbb{N}]^n \to k$ by $g(F) = f(\iota[F])$, where $\iota : \mathbb{N} \to X$ is the canonical bijection. For any infinite g-homogeneous set $H \subseteq \mathbb{N}$, the set $\iota[H]$ is an infinite f-homogeneous subset of X.

When using the stronger statement, one must take into account the computational strength of the set X, as the f-homogeneous set is $H \oplus X$-computable.

In particular, RT^1_k is the infinite pigeonhole principle[11], while the statement RT^2_k states that if the edges of an infinite clique is k-colored, then there is an infinite subset of vertices whose induced subgraph is monochromatic. The question whether Ramsey's theorem for pairs implies ACA_0 over RCA_0 was open for a decade, before Seetapun [10] answered it negatively by proving that RT^2_2 admits cone avoidance. Since then, the original proof was simplified [11] and extended to other preservation properties [12]. We will present the simplified version and leave the original one as an exercise.

11: It might be useful to consider sets $A \in 2^{\mathbb{N}}$ as instances of RT^1_2. A solution to A is then an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$.

The modern version of Seetapun's theorem is divided into two steps, based on the decomposition of Ramsey's theorem for pairs into the cohesiveness and the pigeonhole principles. An infinite set $C \subseteq \mathbb{N}$ is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R}_n$, where $\subseteq^*$ means "included up to finite changes". The *cohesiveness principle* is the problem COH whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets.

We start with a proof of Ramsey's theorem for pairs using the cohesiveness principle and the pigeonhole principle, with no computability-theoretic consideration.

Theorem 3.4.1 (Ramsey)
Every coloring $f : [\mathbb{N}]^2 \to 2$ admits an infinite f-homogeneous set.

Proof. The proof is divided into three steps.

Cohesive step: Let $\vec{R} = R_0, R_1, \dots$ be the sequence of sets defined for every $x \in \mathbb{N}$ by $R_x = \{y \in \mathbb{N} : f(x, y) = 1\}$. By COH, there is an infinite $\vec{R}$-cohesive set $X \subseteq \mathbb{N}$. In particular, for every $x \in X$, $\lim_{y \in X} f(x, y)$ exists.

From a computability-theoretic perspective, the sequence $\vec{R}$ is f-computable, the coloring $\hat{f}$ is $\Delta^0_2(f \oplus X)$, and the set H is $f \oplus X \oplus Y$-computable.

Pigeonhole step: Let $\hat{f} : X \to 2$ be the limit coloring of f, that is, $\hat{f}(x) = \lim_{y \in X} f(x, y)$. By RT^1_2, there is an infinite $\hat{f}$-homogeneous set $Y \subseteq X$ for some color $i < 2$.

Post-processing: Since for every $x \in Y$, $\lim_{y \in Y} f(x, y) = i$, one can thin out the set Y to obtain an infinite f-homogeneous subset $H \subseteq Y$. ∎

Seetapun's theorem will therefore be proven by combining cone avoidance of the cohesiveness principle and strong cone avoidance of the pigeonhole principle. There exists a simple proof of cone avoidance of COH using computable Mathias forcing.

Theorem 3.4.2
Let C be a non-computable set. For every uniformly computable sequence of sets $R_0, R_1, \dots$, there is an infinite $\vec{R}$-cohesive set G such that $C \not\leq_T G$.

PROOF. Recall the notion of computable Mathias forcing[12] from Exercise 3.2.8. Given a condition (σ, X) and a Σ^0_1 formula $\varphi(G)$, one can define a Σ^0_1-preserving forcing question $(\sigma, X) ?\vdash \varphi(G)$ which holds if there is some $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. Thus, for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$. We now show that $G_{\mathcal{F}}$ is $\vec{R}$-cohesive.

Given some $n \in \mathbb{N}$, let $\mathcal{D}_n$ be the set of all conditions (σ, X) such that either $X \subseteq R_n$, or $X \subseteq \overline{R}_n$. The set $\mathcal{D}_n$ is dense, since given a computable Mathias condition (σ, X), either $X \cap R_n$ is infinite, or $X \cap \overline{R}_n$ is infinite (say the former case holds), in which case $(\sigma, X \cap R_n) \in \mathcal{D}_n$. Thus, if $\mathcal{F}$ is $\{\mathcal{D}_n\}_{n \in \mathbb{N}}$-generic, then $G_{\mathcal{F}}$ is $\vec{R}$-cohesive. ■

12: One could have used a variant of Mathias forcing where conditions are pairs (σ, X) such that $C \not\leq_T X$. In general, one requires the reservoirs to satisfy the desired property of the theorem.

Actually, the exact computational strength of the cohesiveness principle is well-understood: given a uniformly computable sequence of sets $\vec{R} = R_0, R_1, \dots$, and $\sigma \in 2^{<\mathbb{N}}$, one can define the set R_σ as follows:

$$R_\sigma = \bigcap_{\sigma(n)=0} \overline{R}_n \bigcap_{\sigma(n)=1} R_n$$

Then, let $\mathscr{C}(\vec{R})$ be the $\Pi^0_1(\emptyset')$ class of all $P \in 2^{\mathbb{N}}$ such that for every $\sigma \prec P$, R_σ is infinite.

The natural proof of COH consists in deciding which one of R_0 or $\overline{R}_0$ is infinite (say R_0), then picking an element $x_0 \in R_0$, then deciding which one of $R_0 \cap R_1$ or $R_0 \cap \overline{R}_1$ is infinite (say $R_0 \cap \overline{R}_1$), then picking an element $x_1 \in R_0 \cap \overline{R}_1$, and so on. The class $\mathscr{C}(\vec{R})$ represents the collection of all "valid" decisions, that is, choices which will not yield a finite set.

Exercise 3.4.3 (Jockusch and Stephan [13]).

1. Fix a uniformly computable sequence of sets $\vec{R} = R_0, R_1, \dots$. Show that the degrees of the $\vec{R}$-cohesive sets are exactly the degrees whose jump computes a a member of $\mathscr{C}(\vec{R})$.
2. Show that for every $\Pi^0_1(\emptyset')$ class $\mathscr{P} \subseteq 2^{\mathbb{N}}$, there exists a uniformly computable sequence of sets $\vec{R} = R_0, R_1, \dots$ such that $\mathscr{C}(\vec{R}) = \mathscr{P}$. ★

It follows from Exercise 3.4.3 that the computability-theoretic study of COH is inherited from the study of Π^0_1 classes. In particular, since there exists a universal Π^0_1 class whose members are of PA degree, there exists a maximally difficult sequence of uniformly computable sets $\vec{R} = R_0, R_1, \dots$ such that the jump of every $\vec{R}$-cohesive set is of PA degree over $\emptyset'$.

Exercise 3.4.4. Combine Exercise 3.4.3 and Theorem 3.2.4 to give an alternative proof of Theorem 3.4.2. ★

Exercise 3.4.5 (Patey [14]). Use Exercise 3.4.3 to prove that if a computable instance of COH admits a solution of low degree, then it admits a computable solution. ★

The last component of our proof of Seetapun's theorem is strong cone avoidance of the pigeonhole principle.[13]

Theorem 3.4.6 (Dzhafarov and Jockusch [11])
Let C be a non-computable set. For every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that $C \not\leq_T H$.

13: The proof of Ramsey's theorem involves only Δ^0_2 instances of the pigeonhole principle. Thus, at first sight, it seems too strong to consider arbitrary instances. However, by Theorem 3.2.4, every instance of RT^1_2 is Δ^0_2 relative to a cone avoiding degree, so considering arbitrary instances or Δ^0_2 instances is equivalent.

Proof. Fix C and A. The first difficulty of this theorem is the disjunctive nature of the statement. One does not know in advance what side of A is more suitable to build an infinite subset. This is why we are going to build two sets G_0, G_1 simultaneously, with $G_0 \subseteq A$ and $G_1 \subseteq \overline{A}$. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

The two sets will be constructed through a variant of Mathias forcing, whose *conditions* are triples (σ_0, σ_1, X) where

There is an easy way to see that at least one of the two initial segments is extendible into an infinite solution: Given a condition (σ_0, σ_1, X), there is some $i < 2$ such that $X \cap A_i$ is infinite. Thus, $\sigma_i \cup (X \cap A_i)$ is an infinite subset of A_i.

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_i \subseteq A_i$;
3. $C \not\leq_T X$.

One must really think of a condition as a pair of Mathias conditions which share a same reservoir. The *interpretation* $[\sigma_0, \sigma_1, X]$ of a condition (σ_0, σ_1, X) is the class

Note that throughout the proof, the only manipulations of the reservoir are finite truncation and splitting based on a Π^0_1 class of 2-colorings. Thus, the whole argument would work by fixing a Scott ideal $\mathcal{M}$ such that $C \notin \mathcal{M}$ and requiring $X \in \mathcal{M}$.

$$[\sigma_0, \sigma_1, X] = \{(G_0, G_1) : \forall i < 2\ \sigma_i \preceq G_i \subseteq \sigma_i \cup X\}$$

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X) for each $i < 2$. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$. Note that $(G_{\mathcal{F},0}, G_{\mathcal{F},1}) \in \bigcap\{[\sigma_0, \sigma_1, X] : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$.

The goal is therefore to build two infinite sets G_0, G_1, satisfying the following requirements for every $e_0, e_1 \in \mathbb{N}$: [14]

$$\mathcal{R}_{e_0,e_1} : \Phi^{G_0}_{e_0} \neq C \vee \Phi^{G_1}_{e_1} \neq C$$

14: One could use Posner's trick, saying that if G_0 and G_1 both compute C, then there is a single Turing functional Φ_e such that $\Phi^{G_0}_e = \Phi^{G_1}_e = C$. Then, the requirement becomes $\mathcal{R}_e : \Phi^{G_0}_e \neq C \vee \Phi^{G_1}_e \neq C$.

If every requirement is satisfied, then an easy *pairing argument*[15] shows that either $C \not\leq_T G_0$, or $C \not\leq_T G_1$. However, in general, it is not possible to ensure that G_0 and G_1 are both infinite. For example, A could be finite or co-finite. Thankfully, in any of these cases, there is a simple computable solution. More generally, we make the following assumption:

15: A *pairing argument* says that if for every $(a, b) \in \mathbb{N}^2$, either $a \in A$ or $b \in B$, then either $A = \mathbb{N}$ or $B = \mathbb{N}$.

There is no infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that $C \not\leq_T H$. (H1)

Under this assumption, one can prove that if $\mathcal{F}$ is sufficiently generic, then both $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ are infinite.

Lemma 3.4.7. Suppose (H1). Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$. There is an extension (τ_0, τ_1, Y) of p and some $n > |\sigma_i|$ such that $n \in \tau_i$. ★

Proof. If $X \cap A^i$ is empty, then $X \subseteq A^{1-i}$, but $C \not\leq_T X$, which contradicts (H1). Thus, there is $n \in X \cap A^i$. Let $\tau_i = \sigma_i \cup \{n\}$, and $\tau_{1-i} = \sigma_{1-i}$. Then, $(\tau_0, \tau_1, X \setminus \{0, \dots, n-1\})$ is an extension of p such that $n \in \tau_i$. ∎

We will now prove the core lemma.

Lemma 3.4.8. Let $p = (\sigma_0, \sigma_1, X)$ be a condition, and $e_0, e_1 \in \mathbb{N}$. There is an extension (τ_0, τ_1, Y) of p forcing $\mathcal{R}_{e_0,e_1}$. ★

Proof. Consider the following set[16]

$$U = \{(x, v) \in \mathbb{N} \times 2 : \forall Z_0 \sqcup Z_1 = X\ \exists i < 2\ \exists \rho \subseteq Z_i\ \Phi^{\sigma_i \cup \rho}_{e_i}(x){\downarrow} = v\}$$

16: The naïve set to consider would be $U = \{(x, v) : \exists i < 2\ \exists \rho \subseteq X \cap A_i\ \Phi^{\sigma_i \cup \rho}_{e_i}(x){\downarrow} = v\}$. It would yield valid forcing question, but with a bad definitional complexity: the set U is $\Sigma^0_1(X \oplus A)$. The third case would yield that $C \leq_T X \oplus A$, which is not a contradiction.

One must get rid of the set A which is arbitrary complex. For this, we use an over-approximation by considering *all* instances of RT^1_2. Since the class of all instances of RT^1_2 is efffectively closed in Cantor space, hence effectively compact, this over-approximation yields a $\Sigma^0_1(X)$ set.

At first sight, this set seems computationally very strong, as it contains a universal second-order quantification. However, by a compactness argument[17],

17: Consider the tree of finite 2-partitions of initial segments of $\mathbb{N}$.

the set can be equivalently defined as

$$\{(x, v) \in \mathbb{N} \times 2 : \exists \ell \in \mathbb{N} \forall Z_0 \sqcup Z_1 = X \upharpoonright_\ell \exists i < 2 \, \exists \rho \subseteq Z_i \, \Phi_{e_i}^{\sigma_i \cup \rho}(x)\downarrow = v\}$$

Thus, the set U is $\Sigma^0_1(X)$. There are three cases:

- Case 1: $(x, 1 - C(x)) \in U$ for some $x \in \mathbb{N}$. Letting $Z_0 = A_0 \cap X$ and $Z_1 = A_1 \cap X$, there is some $i < 2$ and some $\rho \subseteq Z_i$ such that $\Phi_{e_i}^{\sigma_i \cup \rho}(x) \downarrow = 1 - C(x)$. Letting $\tau_i = \sigma_i \cup \rho$ and $\tau_{1-i} = \sigma_{1-i}$, the condition $(\tau_0, \tau_1, X \setminus \{0, \dots, \max \rho\})$ is an extension of p forcing $\Phi_{e_i}^{G_i}(x) \downarrow \neq C(x)$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. Consider the class $\mathcal{P}$ of all sets $B \in 2^{\mathbb{N}}$ such that, letting $B_0 = B$ and $B_1 = \overline{B}$, for every $i < 2$, and every $\rho \subseteq X \cap B_i$, $\Phi_{e_i}^{\sigma_i \cup \rho}(x)\uparrow$ or $\Phi_{e_i}^{\sigma_i \cup \rho}(x)\downarrow \neq C(x)$. The class $\mathcal{P}$ is $\Pi^0_1(X)$, so by the cone avoidance basis theorem (Theorem 3.2.6), there is some $B \in \mathcal{P}$ such that $C \not\leq_T X \oplus B$. Since X is infinite, there is some $i < 2$ such that $X \cap B_i$ is infinite. The condition $(\sigma_0, \sigma_1, X \cap B_i)$ is an extension of p forcing $\Phi_{e_i}^{G_i}(x)\uparrow \vee \Phi_{e_i}^{G_i}(x)\downarrow \neq C(x)$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a $\Sigma^0_1(X)$ graph of the characteristic function of C, hence C is X-computable. This contradicts our hypothesis. ■

Because of the use of an over-approximation, in Case 2, the instance B of RT^1_2 witnessing the negation has nothing to do with the original instance A. The instance B is chosen so that every solution to it will satisfy the Π^0_1 fact. By committing to be simultaneously a solution to A and B, one can create a solution to A which forces the Π^0_1 fact. This ability to be simultaneously a solution to multiple instances is a feature of Ramsey-type statements.

We are now ready to prove Theorem 3.4.6. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing, and for each $i < 2$, let $G_i = G_{\mathcal{F},i}$. By Lemma 3.4.7, both sets are infinite. Moreover, by Lemma 3.4.8, either $C \not\leq_T G_0$ or $C \not\leq_T G_1$. Letting H be this set, it satisfies the statement of Theorem 3.4.6. ■

One can formulate the proof of Theorem 3.4.6 in terms of forcing question, with the appropriate disjunctive definition.

Definition 3.4.9. Given a disjunctive notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a *forcing question* is a relation $?\vdash : \mathbb{P} \times \Gamma$ such that, for every $p \in \mathbb{P}$ and every pair of formulas $\varphi_0(G), \varphi_1(G) \in \Gamma$,

1. If $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension $q \leq p$ forcing $\varphi_i(G_i)$ for some $i < 2$;
2. If $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension $q \leq p$ forcing $\neg\varphi_i(G_i)$ for some $i < 2$. ◇

Note that if $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, one does not force $\neg\varphi_0(G_0) \wedge \neg\varphi_1(G_1)$, but their disjunction.

Exercise 3.4.10. Fix a non-computable set C, a set A, and consider the notion of forcing of Theorem 3.4.6. Given a condition $p = (\sigma_0, \sigma_1, X)$ and two Σ^0_1 formulas $\varphi_0(G), \varphi_1(G)$, define $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ to hold if for every 2-partition $Z_0 \sqcup Z_1 = X$, there is some $i < 2$ and a finite set $\rho \subseteq Z_i$ such that $\varphi(\sigma_i \cup \rho)$ holds.

1. Show that the relation $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ is $\Sigma^0_1(X)$.
2. Prove that it is a forcing question in the sense of Definition 3.4.9. ★

We now have all the necessary ingredients to prove Seetapun's theorem.

Theorem 3.4.11 (Seetapun [10])
Let C be a non-computable set. For every computable coloring $f : [\mathbb{N}]^2 \to \mathbb{N}$, there is an infinite f-homogeneous set H such that $C \not\leq_T H$.

Proof. The proof follows the one of Theorem 3.4.1, using cone avoidance of COH (Theorem 3.4.2) and strong cone avoidance of RT^1_2 (Theorem 3.4.6).

Fix C and f. Let $\vec{R} = R_0, R_1, \dots$ be the computable sequence of sets defined for every $x \in \mathbb{N}$ by $R_x = \{y \in \mathbb{N} : f(x,y) = 1\}$. By Theorem 3.4.2, there is an infinite $\vec{R}$-cohesive set $X \subseteq \mathbb{N}$ such that $C \not\leq_T X$. In particular, for every $x \in X$, $\lim_{y \in X} f(x,y)$ exists. Let $\hat{f} : X \to 2$ be the limit coloring of f, that is, $\hat{f}(x) = \lim_{y\in X} f(x,y)$. By Theorem 3.4.6, there is an infinite $\hat{f}$-homogeneous set $Y \subseteq X$ for some color $i < 2$ such that $C \not\leq_T Y \oplus X$. Since for every $x \in Y$, $\lim_{y \in Y} f(x,y) = i$, one can thin out the set Y to obtain an infinite f-homogeneous subset $H \subseteq Y$. ■

The original proof of Seetapun's theorem [10] was more direct, using a notion of forcing to build homogeneous sets for colorings of pairs. We leave it as an exercise.

Exercise 3.4.12 (Seetapun and Slaman [10]). Fix a computable coloring $f : [\mathbb{N}]^2 \to 2$ and a non-computable set C. Consider the notion of forcing whose conditions[18] are 3-tuples (σ_0, σ_1, X) such that for every $i < 2$,

1. (σ_i, X) is a Mathias condition ;
2. For every $x \in X$, $\sigma_i \cup \{x\}$ is f-homogeneous for color i ;
3. $C \not\leq_T X$.

The extension relation is the same as in the proof of Theorem 3.4.6. Given a condition $p = (\sigma_0, \sigma_1, X)$ and two Σ^0_1 formulas $\varphi_0(G)$ and $\varphi_1(G)$, let $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ iff for every 2-partition $Z_0 \sqcup Z_1 = X$, there is some $i < 2$ and a finite f-homogeneous set $\rho \subseteq Z_i$ for color i such that $\varphi_i(\sigma_i \cup \rho)$ holds.[19][20]

1. Prove that the relation $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ is $\Sigma^0_1(X)$.
2. Show that it is a forcing question in the sense of Definition 3.4.9.
3. Prove Seetapun's theorem using this notion of forcing. ★

18: One can apply the same trick as in Theorem 3.4.6 to see that one of the initial segments is extendible. Given a condition (σ_0, σ_1, X), apply Ramsey's theorem for pairs to $f \upharpoonright [X]^2$ to obtain an infinite f-homogeneous subset $H \subseteq X$ for some color $i < 2$. The properties of the condition are designed to ensure that $\sigma_i \cup H$ is f-homogeneous.

19: Notice the strong similarity of this forcing question with the one in Theorem 3.4.6. The only difference is that one requires ρ to be f-homogeneous as well.

20: If the coloring f is stable, that is, $\lim_y f(x,y)$ always exists, then the interpretation of the 2-partition $Z_0 \sqcup Z_1 = X$ is clear: it is the limit coloring of f. This forcing question might be more confusing in the general case, since f has no limit behavior. This is where compactness comes into play: find a bound to quantify over finite 2-partitions, then "stabilize" the behavior of f over this finite initial segment, by thinning out the remaining reservoir. This limit behavior induces a 2-partition of the initial segment.

It is sometimes useful to think of instances of COH as countably many instances of RT^1_2, where a solution is an infinite set which is simultaneously homogeneous for all instances of RT^1_2, up to finite changes. With this intuition in mind, one can strengthen Theorem 3.4.2 to prove that it holds even when considering arbitrary instances of COH.

Exercise 3.4.13 (Wang [15]). Fix a non-computable set C and an arbitrary countable sequence $\vec{R} = R_0, R_1, \dots$ of sets, with no effectiveness restriction whatsoever. Consider the variant of Mathias forcing, whose conditions[21] are pairs (σ, X) where $C \not\leq_T X$.

1. Use Theorem 3.4.6 to show that the set $\mathscr{D}_n = \{(\sigma, X) : X \subseteq R_n \vee X \subseteq \overline{R_n}\}$ is dense.
2. Deduce the existence of an infinite $\vec{R}$-cohesive set G such that $C \not\leq_T G$. ★

21: Note that contrary to the proof of cone avoidance of COH, one needs to use Mathias conditions (σ, X) where $C \not\leq_T X$ instead of computable Mathias conditions.

Cone avoidance fails when considering computable colorings of 3-tuples. The reason is that one can create computable coloring $f : [\mathbb{N}]^3 \to 2$ such that every infinite homogeneous set H is so sparse, that its principal function p_H is very fast-growing, and dominates the modulus of $\emptyset'$. Recall that the *principal function* p_X of an infinite set $X = \{x_0 < x_1 < \dots\}$ is defined by $p_X(n) = x_n$.

Exercise 3.4.14 (Jockusch [16]).

1. Show that for every function $g : \mathbb{N} \to \mathbb{N}$, there is a g-computable coloring $f : [\mathbb{N}]^2 \to 2$ such that for every infinite f-homogeneous set H, the principal function p_H dominates g.
2. Show that for every $\emptyset'$-computable coloring $f : [\mathbb{N}]^2 \to 2$, there is a computable coloring $h : [\mathbb{N}]^3 \to 2$ such that every infinite h-homogeneous set is f-homogeneous.
3. Deduce the existence of a computable coloring $h : [\mathbb{N}]^3 \to 2$ such that every infinite h-homogeneous set computes $\emptyset'$. ★

One can actually go one step further, and construct a computable coloring $f : [\mathbb{N}]^3 \to 2$ such that every infinite homogeneous set is of PA degree over $\emptyset'$.

Exercise 3.4.15 (Hirschfeldt and Jockusch [17]).
A set $P \subseteq \mathbb{N}$ is *pre-homogeneous* for a coloring $f : [\mathbb{N}]^{n+1} \to 2$ if for every $F \in [P]^n$ and every $x, y \in P$ with $\max F < x, y$, then $f(F \cup \{x\}) = f(F \cup \{y\})$. Construct a computable coloring $f : [\mathbb{N}]^3 \to 2$ such that every infinite pre-homogeneous set is of PA degree over $\emptyset'$. ★

3.5 Preserving definitions

The existence of a notion of forcing with a Σ^0_1-preserving forcing question enables to prove abstractly some stronger weakness properties, such as preservation of one non-Σ^0_1 definition. Some sets such as $\emptyset'$ can be used to "simplify" the definition of other sets in the arithmetic hierarchy. For example, any Σ^0_2 set is $\Sigma^0_1(\emptyset')$. The notion of preservation of 1 non-Σ^0_1-definition reflects the unability of a problem to simplify the description of a non-Σ^0_1 set to make it Σ^0_1 relative to a solution.

Definition 3.5.1. A problem P admits *preservation of 1 non-Σ^0_1 definition* if for every set Z and every non-$\Sigma^0_1(Z)$ set C, every Z-computable instance X of P admits a solution Y such that C is not $\Sigma^0_1(Z \oplus Y)$. ◊

Thanks to Post's theorem, preservation of 1 non-Σ^0_1 definition implies cone avoidance:

Exercise 3.5.2. Prove that if a problem P admits preservation of 1 non-Σ^0_1 definition, then it admits cone avoidance. ★

The proof of Theorem 3.3.4 can be strengthened to prove an abstract theorem about preservation of 1 non-Σ^0_1 definition.[22]

22: The proof of preservation of non-Σ^0_1 definitions is simpler and arguably more natural than the one of cone avoidance. This naturality comes from the fact that, in some sense, Σ^0_1 sets are more natural than computable ones, as they form a syntactic family and thus have a better behavior.

Theorem 3.5.3
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving forcing question. For every non-Σ^0_1 set C and every sufficiently generic filter $\mathcal{F}$, C is not $\Sigma^0_1(G_\mathcal{F})$.

Proof. It suffices to prove the following lemma:

Lemma 3.5.4. For every condition $p \in \mathbb{P}$ and every Turing index e, there is an extension $q \leq p$ forcing $C \neq W^G_e$. ★

Proof. Consider the following set

$$U = \{x \in \mathbb{N} : p \mathrel{?}\!\vdash x \in W_e^G\}$$

Since the forcing question is Σ^0_1-preserving, the set U is Σ^0_1. There are three cases:

- Case 1: there is some $x \in U \setminus C$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $x \in W_e^G$.
- Case 2: there is some $x \in C \setminus U$. By Property (2) of the forcing question, there is an extension $q \leq p$ forcing $x \notin W_e^G$.
- Case 3: $U = C$. Then C is Σ^0_1, contradiction.

In the first two cases, the extension q forces $W_e^G \neq C$. ■

We are now ready to prove Theorem 3.5.3. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $W_e^G \neq C$. It follows from Lemma 3.5.4 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so C is not $\Sigma^0_1(G_{\mathcal{F}})$. This completes the proof of Theorem 3.5.3. ■

It follows from Theorem 3.5.3 that the proofs of cone avoidance for Cohen genericity and Π^0_1 classes have a straightforward adaptation to prove preservation of 1 non-Σ^0_1 definition. We leave these adaptations as an exercise:

Exercise 3.5.5. Let C be a non-Σ^0_1 set. Prove that for every sufficiently Cohen generic set G, C is not $\Sigma^0_1(G)$. ★

Exercise 3.5.6. Let C be a non-Σ^0_1 set. Prove that for every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there is a member $G \in \mathcal{P}$ such that C is not $\Sigma^0_1(G)$. ★

It is natural to wonder whether some problems admit cone avoidance but not preservation of 1 non-Σ^0_1 definition. Actually, this happens not to be the case, thanks to the relativized formulation of both notions.[23]

23: The proof of Exercise 3.5.2 also holds when considering non-relativized versions of cone avoidance of preservation of 1 non-Σ^0_1 definitions. On the other hand, the reverse direction uses a different set Z. One can construct artificial problems which admit non-relativized cone avoidance but not non-relativized preservation of 1 non-definition.

Theorem 3.5.7 (Downey et al. [18])
Let C be a non-Σ^0_1 set. There is a set Z and a set $D \not\leq_T Z$ such that for every set G such that C is $\Sigma^0_1(G \oplus Z)$, $D \leq_T G \oplus Z$.

The proof of Theorem 3.5.7 is quite technical and outside the scope of this book.

Corollary 3.5.8 (Downey et al. [18])
A problem P *admits preservation of 1 non-Σ^0_1 definition iff it admits cone avoidance.*[24]

24: Given the simplicity of the forward direction, the technicality of the reciprocal, and the naturality of the proof of preservation of 1 non-Σ^0_1 definition using a Σ^0_1-preserving forcing question, it is preferable to directly prove preservation of 1 non-Σ^0_1 definition when the result is needed.

Proof. The forward direction is Exercise 3.5.2. Let us prove reciprocal. Suppose P admits cone avoidance. Fix a set Z and a non-$\Sigma^0_1(Z)$ set C and let $X \leq_T Z$ be an instance of P. By Theorem 3.5.7 relativized to Z, there is a set Z_1 and a set $D \not\leq_T Z \oplus Z_1$ such that for every set G such that C is $\Sigma^0_1(G \oplus Z \oplus Z_1)$, $D \leq_T G \oplus Z \oplus Z_1$. By cone avoidance of P relativized to $Z \oplus Z_1$, there is a solution Y to X such that $D \not\leq_T Y \oplus Z \oplus Z_1$. By choice of Z_1 and D, it follows that C is not $\Sigma^0_1(Y \oplus Z \oplus Z_1)$. In particular, C is not $\Sigma^0_1(Y \oplus Z)$. ■

3.6 Preserving hyperimmunities

There exists a well-known duality between computing sets and computing fast-growing functions. The simplest example is the correspondence between the halting set $\emptyset'$, and the halting time function $\mu_{\emptyset'} : \mathbb{N} \to \mathbb{N}$ which to e associates the smallest time t such that $\Phi_e(e)[t]\downarrow$, if it exists, and equals 0 otherwise. The function μ is $\emptyset'$-computable, and every function dominating $\mu_{\emptyset'}$ computes $\emptyset'$. More generally, a function $f : \mathbb{N} \to \mathbb{N}$ is a *modulus* of a set X if every function dominating f computes X. If furthermore f is X-computable, then it is a *self-modulus*. By Solovay [19], the sets admitting a modulus are exactly the Δ^1_1 sets, or equivalently the hyperarithmetic sets. On the other hand, there exist Δ^0_3 sets with no self-modulus.

Proposition 3.6.1 (Martin and Miller [20]). Every Δ^0_2 set admits a self-modulus. ⋆

Proof. Let A be a Δ^0_2 set, with Δ^0_2 approximation $A_0, A_1, \ldots$ The *computation function* $c_A : \mathbb{N} \to \mathbb{N}$ maps x to the smaller integer $n \geq x$ such that $A_n \upharpoonright_x = A \upharpoonright_x$. Let f be a function dominating c_A. Let $h(x)$ be the largest $y \leq x$ such that for all $x \leq t \leq f(x)$, $A_t \upharpoonright_y = A_{f(x)} \upharpoonright_y$. The function h is total f-computable. Moreover, h tends towards $+\infty$, because the approximation of A being Δ^0_2, it will stabilize on increasingly larger initial segments. Finally, as $x \leq c_A(x) \leq f(x)$, then if $h(x) = y$, $A_x \upharpoonright_y = A_{c_A(x)} \upharpoonright_y = A \upharpoonright_y$. Then, to decide if $n \in A$, it suffices to find an integer x such that $h(x) > n$, then test if $n \in A_x$. This procedure is f-computable. ■

Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *hyperimmune* if it is not dominated by any computable function. In particular, if a function f is a modulus of a non-computable set C, then it is hyperimmune. Moreover, if it is a self-modulus, then avoiding the cone above C is equivalent to preserving the hyperimmunity of the function f. This motivates the following definition:

Definition 3.6.2. A problem P admits *preservation of 1 hyperimmunity* if for every set Z and every Z-hyperimmune function f, every Z-computable instance X of P admits a solution Y such that f is $Z \oplus Y$-hyperimmune. ◊

At first sight, the sole existence of a Σ^0_1-preserving forcing question does not seem to be sufficient to prove preservation of 1 hyperimmunity. One furthermore needs the forcing question to satisfy some kind of compactness as follows:

Definition 3.6.3. Given a notion of forcing $(\mathbb{P}, \leq)$, a forcing question is Σ^0_n-*compact* if for every $p \in \mathbb{P}$ and every Σ^0_n formula $\varphi(G, x)$, if $p \mathrel{?\vdash} \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ such that $p \mathrel{?\vdash} \exists x \in F\ \varphi(G, x)$. ◊

All the forcing questions seen in this chapter are Σ^0_1-compact. Thanks to this compactness property, one can prove preservation of 1 hyperimmunity.

Theorem 3.6.4
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-compact, Σ^0_1-preserving forcing question. For every hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and every sufficiently generic filter $\mathcal{F}$, f is $G_{\mathcal{F}}$-hyperimmune.

Proof. It suffices to prove the following lemma:

Lemma 3.6.5. For every condition $p \in \mathbb{P}$ and every Turing index e, there is an extension $q \leq p$ forcing Φ_e^G not to dominate f.[25] ⋆

25: By this, we mean forcing either Φ_e^G to be partial, or $\Phi_e^G(x) < f(x)$ for some $x \in \mathbb{N}$.

Proof. Suppose first that $p \mathrel{?\nvdash} \exists v \Phi_e^G(x)\downarrow = v$ for some $x \in \mathbb{N}$. Then by Property (2) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^G(x)\uparrow$, and we are done. Suppose now that for every $x \in \mathbb{N}$, $p \mathrel{?\vdash} \exists v \Phi_e^G(x)\downarrow = v$. By Σ_1^0-compactness of the forcing question, for every $x \in \mathbb{N}$, there is a finite set $F_x \subseteq \mathbb{N}$ such that $p \mathrel{?\vdash} \exists v \in F_x\, \Phi_e^G(x)\downarrow = v$. Let $h : \mathbb{N} \to \mathbb{N}$ be the function which on input x, looks for some finite set F_x such that $p \mathrel{?\vdash} \exists v \in F_x\, \Phi_e^G(x)\downarrow = v$ and outputs $\max F_x$. Such a function is total by hypothesis, and computable by Σ_1^0-preservation of the forcing question. Since f is hyperimmune, $h(x) < f(x)$ for some $x \in \mathbb{N}$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\exists v \in F_x \Phi_e^G(x)\downarrow = v$. Since $f(x) > \max F_x$, q forces $\Phi_e^G(x)\downarrow < f(x)$. ■

We are now ready to prove Theorem 3.6.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing Φ_e^G not to dominate f. It follows from Lemma 3.5.4 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so f is $G_{\mathcal{F}}$-hyperimmune. This completes the proof of Theorem 3.6.4. ■

Contrary to preservation of 1 non-Σ_1^0 definition, there is no immediate link between preservation of 1 hyperimmunity and cone avoidance. Furthermore, preservation of 1 hyperimmunity seems to require an extra property which may not always be satisfied. However, the two notions turn out again to be equivalent in their relativized form. Recall Theorem 3.2.4 which informally says that every set can become Δ_2^0 while avoiding a cone.

Theorem 3.6.6 (Downey et al. [18])
If a problem P *admits preservation of 1 hyperimmunity, then it admits cone avoidance.*

Proof. Fix a set Z, a set $C \not\leq_T Z$ and an instance $X \leq_T Z$ of P. By Theorem 3.2.4, there is a set Z_1 such that $C \not\leq_T Z \oplus Z_1$ and $C \leq_T (Z \oplus Z_1)'$. By Proposition 3.6.1 relative to $Z \oplus Z_1$, there is a $C \oplus Z \oplus Z_1$-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every function g dominating f, $C \leq_T g \oplus Z \oplus Z_1$. In particular, f is $Z \oplus Z_1$-hyperimmune. Since P admits preservation of 1 hyperimmunity, there is a solution Y to X such that f is $Y \oplus Z \oplus Z_1$-hyperimmune. It follows that $C \not\leq_T Y \oplus Z \oplus Z_1$. ■

The reverse direction also holds, using the following theorem which says that every non-decreasing hyperimmune function is a modulus of some set in a relativized setting.

Theorem 3.6.7 (Downey et al. [18])
Fix a non-decreasing hyperimmune function $f : \mathbb{N} \to \mathbb{N}$. There is a set Z and a set $C \not\leq_T Z \oplus G$ such that f is a Z-modulus for C.

Here again, the proof of Theorem 3.6.7 is out of the scope of this book.

Corollary 3.6.8 (Downey et al. [18])
A problem P *admits preservation of 1 hyperimmunity iff it admits cone avoidance.*

PROOF. The forward direction is Theorem 3.6.6. Let us prove reciprocal. Suppose P admits cone avoidance. Fix a set Z, a Z-hyperimmune function $f : \mathbb{N} \to \mathbb{N}$, and let $X \leq_T Z$ be an instance of P. By Theorem 3.6.7 relativized to Z, there is a set Z_1 and a set $C \not\leq_T Z \oplus Z_1$ such that f is a Z-modulus for C. By cone avoidance of P relativized to $Z \oplus Z_1$, there is a solution Y to X such that $C \not\leq_T Y \oplus Z \oplus Z_1$. By choice of Z_1 and C, it follows that f is $Y \oplus Z \oplus Z_1$-hyperimmune. In particular, f is not $Y \oplus Z$-hyperimmune. ∎

Lowness 4

Prerequisites: Chapters 2 and 3

Recall that a set X is *low* if $X' \leq_T \emptyset'$. Constructing sets of low degree given a notion of forcing with a Σ^0_1-preserving forcing question is not a huge conceptual step from cone avoidance. It simply consists in effectivizing[1] the construction of a generic set with an appropriate representation of forcing conditions and a refined analysis of the properties of the forcing question.

Effectivization of a forcing construction first requires to fix a coding of forcing conditions. Whenever a condition is a finite object, any reasonable coding, such as a Gödel numbering, is sufficient. For any such numbering, one can switch from one representation to the other computably, and this does not affect the complexity of the overall construction. In most cases however, forcing conditions are naturally defined as infinitary mathematical objects, and one must use an appropriate finitary representation of their effective version.

1: Effectiveness is a concept more general than computability. Any construction requiring some amount of computability, such as being c.e., or arithmetic, or even involving some higher computational models, is considered as effective. On the other hand, a forcing construction is not considered as effective, even if its forcing conditions are computable, as the construction of the generic filter does not have any computability restriction.

4.1 Motivation

One of the main motivation of the development of a framework of iterated jump control is reverse mathematics. To prove the existence of an ω-model of a problem P which is not a model of Q, one needs to find an invariant property preserved by P but not by Q. These invariant properties can be divided into two big families: genericity properties, and effectiveness properties.

- A *genericity property* is a property which may locally involve some computability-theoretic features, but does not require the overall construction to be effective. Such properties can be satisfied by every sufficiently generic set for the appropriate notion of forcing. Cone avoidance, preservation of hyperimmunity, or preservation of 1 non-Σ^0_1 definition are examples of such properties.
- An *effectiveness property* is a property which requires the overall construction to satisfy some amount of computability. Being c.e., arithmetic, or of low degree, are examples of such effectiveness properties. Usually, only countably many sets satisfy these properties.

Effectiveness properties are arguably more complex to satisfy than genericity properties, as one usually needs to resort to coding to represent forcing conditions, and the proofs of density require to satisfy some amount of uniformity. This is why genericity properties are preferably used when one only cares about proving a separation from a problem to another in reverse mathematics. On the other hand, effectiveness properties are closer to the original motivation of computability-theory in general, and of reverse mathematics in particular: identifying the right amount of computability needed to find a solution to a problem. From this perspective, the existence of a low solution is very informative.

Definition 4.1.1. A problem P admits a *low basis* if for every set Z and every Z-computable instance X of P, there is a solution Y to X such that $(Y \oplus Z)' \leq_T Z'$. ◊

Besides the intrinsic interest of proving that a problem admits a low basis, such a notion has two technical applications. First, lowness is a natural class of Δ^0_2 sets which is closed under relativization:

Exercise 4.1.2. A set X is *low over* Y if $(X \oplus Y)' \leq_T Y$. Show that if X is low over Y and Y is low, then X is low. ⋆

It follows that if a problem admits a low basis, then it admits a model with only sets of low degree, and therefore a model with only Δ^0_2 sets.[2]

2: A problem P admits a Δ^0_2 basis if for every set Z and every Z-computable instance X of P, there is a $\Delta^0_2(Z)$ solution Y to X. The Turing jump problem, which to any instance X associates a unique solution X', admits a Δ^0_2 basis, but one easily sees that any ω-model of it contains all the arithmetic sets.

Proposition 4.1.3. Let P be a Π^1_2 problem which admits a low basis. There exists an ω-model of RCA_0 + P with only low sets. ⋆

Proof. Recall that an ω-model is fully characterized by its second-order part, and that it satisfies RCA_0 iff its second-order part is a Turing ideal. Also recall that $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ is Cantor's pairing function.

We are going to define a sequence of sets $Z_0 \leq_T Z_1 \leq_T \dots$ such that for all $n \in \mathbb{N}$,

(1) if $n = \langle e, s \rangle$ and $\Phi^{Z_s}_e$ is a P-instance X, then Z_{n+1} computes a solution to X;
(2) Z_n is of low degree.

$Z_0 = \emptyset$. Suppose we have defined Z_n and say $n = \langle e, s \rangle$. If $\Phi^{Z_s}_e$ is not a P-instance, then let $Z_{n+1} = Z_n$. Otherwise, since P admits a low basis, there is a solution Y to $\Phi^{Z_s}_e$ such that $(Y \oplus Z_n)' \leq_T Z'_n \leq_T \emptyset'$. Let $Z_{n+1} = Z_n \oplus Y$.

Let $\mathcal{I} = \{X \in 2^{\mathbb{N}} : \exists n\ X \leq_T Z_n\}$. By construction, the class $\mathcal{I}$ is a Turing ideal. Moreover, by (1), every P-instance $X \in \mathcal{I}$ admits a solution in $\mathcal{I}$. Last, by (2), every set in $\mathcal{I}$ is of low degree. ■

As an immediate consequence, if a Π^1_2 problem admits a low basis, then it does not imply ACA_0 over RCA_0. Indeed, every ω-model of ACA_0 contains all arithmetic sets by the arithmetic comprehension axiom, thus the model of Proposition 4.1.3 does not satisfy ACA_0. However, as mentioned above, effectiveness properties are harder to satisfy than genericity properties, so since cone avoidance is enough to prove a separation from ACA_0, one usually prefers to prove the latter.

Some other problems, such as Ramsey's theorem for pairs, admit cone avoidance, but not a low basis.[3]

3: The Chain-AntiChain principle (CAC) is the problem whose instances are infinite partial orders, and whose solutions are either infinite chains, or infinite antichains. By Herrmann [21], there is a computable linear order with no Δ^0_2 infinite chains or antichains. Thus, CAC does not admit a Δ^0_2 basis.

The Ascending Descending Sequence principle (ADS) is the problem whose instances are infinite linear orders, and whose solutions are either infinite ascending or descending sequences. By Manaster (see Downey [22]), ADS admits a Δ^0_2 basis, but by Hirschfeldt and Shore [23], there is a computable infinite linear ordering with no low infinite ascending or descending sequence.

It follows that if a Π^1_2 problem admits a low basis, then it implies neither CAC, nor ADS over RCA_0.

Exercise 4.1.4 (Jockusch [16]). Construct a computable coloring $f : [\mathbb{N}]^2 \to 2$ with no Δ^0_2 infinite homogeneous set. ⋆

Thus, proving that a Π^1_2 problem admits a low basis is a way to separating it from Ramsey's theorem for pairs.

The second technical advantage of the low basis theorem concerns iterated jump control. As we shall see in Chapter 9, iterated jump is much more difficult to control than first jump. On the other hand, if a set G is of low degree, then by Post's theorem, every $\Sigma^0_2(G)$ property is $\Sigma^0_1(G')$, so by lowness is $\Sigma^0_1(\emptyset')$, and again by Post's theorem is Σ^0_2. Thus, if a problem admits a low basis, it satisfies every weakness property at the second jump and higher jump levels.

Exercise 4.1.5. Suppose that a problem P admits a low basis. Let C be a non-Δ^0_2 set, and X be a computable instance of P. Show that there is a solution Y to X such that C is not $\Delta^0_2(Y)$. ★

One will therefore rather prove the existence of a low basis than control higher jump if possible.

4.2 Indices

Consider a finite set $F \subseteq \mathbb{N}$. There exists multiple unequivalent ways to represent it by an integer, depending on whether it is considered as finite, computable, c.e., among others. Depending on the representation, some functions such as the cardinality, or the maximum, are not uniformly computable. We explore some natural representations and their limitations.

Definition 4.2.1. The *canonical index* of a finite set $F \subseteq \mathbb{N}$ is the integer $\sum_{x\in F} 2^x$. ◊

The canonical index of a finite set keeps the full information about it. One can list all its elements, compute the size of the set, and decide whether an element belongs to it or not.

Definition 4.2.2. A Δ^0_1*-index*[4] of a computable set $X \subseteq \mathbb{N}$ is an integer $e \in \mathbb{N}$ such that Φ_e is the characteristic function of X. ◊

4: One could as well have considered to code computable sets X by pairs $\langle e, i\rangle$ such that e and i are Σ^0_1-indices of X and $\overline{X}$, respectively. However, one can switch from one representation to the other computably.

Given a Δ^0_1-index e of a computable set $X \subseteq \mathbb{N}$, one can decide uniformly whether an element belongs to it or not. However, one cannot uniformly find a canonical index of a finite set from a Δ^0_1-index:

Lemma 4.2.3 (Soare [3]). There is no partial computable function Φ_e such that for every $n \in \mathbb{N}$, if Φ_n is the characteristic function of a finite set F, then $\Phi_e(n)\downarrow$ and equals the canonical index of F. ★

Proof. Suppose Φ_e exists. Using Kleene's fixpoint theorem, define the following total computable function Φ_n, knowing n in advance. $\Phi_n(x)\downarrow = 1$ if x is the least stage such that $\Phi_e(n)[x]\downarrow$, and $\Phi_n(x)\downarrow = 0$ otherwise. By construction, Φ_n is the characteristic function of either the empty set, or a singleton x, thus $\Phi_e(n)\downarrow$ and x is defined. By convention, if $\Phi_e(n)[x]\downarrow$, then $\Phi_e(n)[x] < x$, so $\Phi_e(n)$ is not the canonical index of $\{x\}$. ■

Using a Δ^0_1-index of a finite set F and its cardinality, one can compute the canonical index of F. Therefore, the cardinality function is not uniformly computable from a Δ^0_1-index.

Definition 4.2.4. A Σ^0_1*-index* of a c.e. set $X \subseteq \mathbb{N}$ is an integer $e \in \mathbb{N}$ such that $W_e = X$. ◊

From a Σ^0_1-index of a c.e. set X, one can list exhaustively all its elements over time, but not in order. Furthermore, if X is computable, one cannot uniformly compute a Δ^0_1-index of X.

Lemma 4.2.5 (Soare [3]). There is no partial computable function Φ_e such that for every $n \in \mathbb{N}$, if W_n is computable, then $\Phi_e(n)\downarrow$ and equals a Δ^0_1-index of W_n. ⋆

Proof. Suppose Φ_e exists. Using Kleene's fixpoint theorem, define the following partial computable function Φ_n, knowing n in advance. Let $\Phi_n(0)\downarrow$ if $\Phi_e(n)\downarrow = y$ and $\Phi_y(0)\downarrow = 0$. For every $x > 0$, $\Phi_n(x)\uparrow$. Thus, W_n is either empty, or the singleton 0, so $\Phi_e(n)\downarrow = y$ for some $y \in \mathbb{N}$ such that Φ_y is total. By construction of Φ_n, $\Phi_y(0)\downarrow = 0$, iff $0 \in W_n$, so Φ_y is not the characteristic function of W_n. ■

One can generalize the previous definitions to every level of the arithmetic hierarchy, either using the representation of sets by formulas, or using Post's theorem, by iterations of the Turing jump. Both representations are equivalent, as one can switch from one to another computably.

As we have seen, when using a representation of a mathematical object as part of a larger family of objects, one might loose some information. It is therefore important to choose the most precise representation as possible, given the provided information. For instance, consider a low set X. It is in particular Δ^0_2, so one could use a Δ^0_2-index, that is, an integer e such that $\Phi^{\emptyset'}_e$ is the characteristic function of X. However, this would loose the lowness information of X. It is therefore preferable to represent it by a Δ^0_2-index of X', that is, an integer e such that $\Phi^{\emptyset'}_e$ is the characteristic function of X'.

Definition 4.2.6. A *lowness index* of a low set $X \subseteq \mathbb{N}$ is an integer $e \in \mathbb{N}$ such that $\Phi^{\emptyset'}_e$ is the characteristic function of X'. ◊

Exercise 4.2.7. Show that is no partial computable function Φ_e such that for every $n \in \mathbb{N}$, if $\Phi^{\emptyset'}_n$ is the characteristic function of a low set X, then $\Phi_e(n)\downarrow$ and is a lowness index of X. ⋆

4.3 Coding ideals

Recall that a Turing ideal is a class of sets $\mathcal{M} \subseteq 2^{\mathbb{N}}$ closed under the effective join, and downward-closed under the Turing reduction. Turing ideals are exactly the second-order parts of ω-models of RCA_0.[5]

5: The class of all the computable sets, and the class of all the arithmetic sets are two basic examples of Turing ideals. More generally, given a set X, the class of all X-computable sets is a Turing ideal. On the other hand, the class of all low sets is downward-closed under the Turing reduction, but not closed under the effective join: There exist two low c.e. sets A and B such that $A \cup B = \emptyset'$.

Coding Turing ideals plays an important role in effectivization of forcing constructions, as some combinatorial notions of forcing such as Mathias forcing can be effectivized by restricting their conditions to ω-models of some appropriate theory. For example, solutions to COH can be produced using Mathias forcing over ω-models of RCA_0, in other words, over Turing ideals. Solutions to arbitrary instances of RT^1_2 or computable instances of RT^2_2 can be obtained using a variant of Mathias forcing over ω-models of WKL_0. The second-order part of ω-models of WKL_0 are precisely Scott ideals, that is, Turing ideals which are closed under the existence of PA degrees.

There exist multiple natural ways to code members of countable Turing ideals. The infinite effective join of an infinite sequence $Z_0, Z_1, \dots$ is the set $\bigoplus_i Z_i = \{\langle i, x\rangle : x \in Z_i\}$.

Definition 4.3.1. A set M *codes* a family $\mathcal{M} = \{Z_0, Z_1, \dots\}$ if $M = \bigoplus_i Z_i$. An M-*index* of a set $X \in \mathcal{M}$ is an integer $i \in \mathbb{N}$ such that $X = Z_i$. ◊

By an immediate diagonalization argument, no Turing ideal contains its own code. Therefore, it requires more computational power to compute the code of a Turing ideal than to compute its members. On the other hand, Scott ideals are particularly interesting, as any PA degree computes the code of a Scott ideal. In other words, it does not require more computational power to compute the code of a Scott ideal than to compute its members. Fix an enumeration of all the primitive recursive functionals $T_0, T_1, \dots$ such that for every $X \in 2^{\mathbb{N}}$, T_e^X is an infinite binary tree.[6]

6: Such an enumeration exists, as given a primitive recursive tree functional S_e, one can define a primitive recursive tree functional T_e which, if at some level, sees all the nodes of S_e die, keeps in T_e the last node alive. Thus, given $X \in 2^{\mathbb{N}}$, if S_e^X is infinite, then $T_e^X = S_e^X$, and otherwise, T_e^X is any infinite binary tree.

Theorem 4.3.2 (Scott [24])
The following class is Π^0_1 and non-empty:

$$\mathscr{C} = \left\{ \bigoplus_i Z_i : \forall a \forall b \forall c \; Z_{\langle a,b,c \rangle} \in [T_c^{Z_a \oplus Z_b}] \right\}$$

Moreover, every member of $\mathscr{C}$ codes a Scott ideal.[7]

7: Note that with an appropriate numbering of the listing $T_0, T_1, \dots$, the resulting code M admits some stronger properties: one can computably obtain M-indices of sets witnessing downward-closure, effective join and PA closure. For example, there exists a total computable function which, given an M-index a and a Turing index e such that $\Phi_e^{Z_a}$ is total, outputs an M-index b such that $Z_b = \Phi_e^{Z_a}$.

Proof. The class $\mathscr{C}$ is clearly Π^0_1 and non-empty by choice of $T_0, T_1, \dots$ Let $\bigoplus_i Z_i \in \mathscr{C}$ and say $\mathcal{M} = \{Z_0, Z_1, \dots\}$. We claim that $\mathcal{M}$ is a Scott ideal.

- Downward-closure: Suppose that $Z_a \in \mathcal{M}$ and $Y \leq_T Z_a$. Say $\Phi_e^{Z_a} = Y$ for some $e \in \mathbb{N}$. Then, the primitive recursive tree functional T_b defined by[8]
$$T_c^{A \oplus B} = \{\sigma \in 2^{<\mathbb{N}} : \sigma \text{ and } \Phi_e^A[|\sigma|] \text{ are compatible }\}$$
is such that $[T_c^{Z_a \oplus Z_b}] = \{Y\}$, so $Z_{\langle a,b,c \rangle} = Y \in \mathcal{M}$.
- Effective join: Suppose that $Z_a, Z_b \in \mathcal{M}$. Then the primitive recursive tree functional T_c defined by
$$T_c^A = \{\sigma \in 2^{<\mathbb{N}} : \sigma \prec A\}$$
is such that $[T_c^{Z_a \oplus Z_b}] = \{Z_a \oplus Z_b\}$, so $Z_{\langle a,b,c \rangle} = Z_a \oplus Z_b \in \mathcal{M}$.
- PA closure: Suppose that $Z_a \in \mathcal{M}$. Then the primitive recursive tree functional T_c defined by
$$T_c^{A \oplus B} = \{\sigma \in 2^{<\mathbb{N}} : \forall e < |\sigma| \; \Phi_e^A(e)[|\sigma|]\uparrow \vee \downarrow \neq \sigma(e)\}$$
is such that $[T_c^{Z_a \oplus Z_b}]$ is the class of all $\{0,1\}$-valued DNC functions relative to Z_a. Thus $Z_{\langle a,b,c \rangle}$ is PA over Z_a and in $\mathcal{M}$. ■

8: By "compatible", we mean that for every $x < |\sigma|$, if $\Phi_e^A(x)[|\sigma|]\downarrow$, then the value equals $\sigma(x)$.

In particular, there exists a computable infinite binary tree such that every path codes a Scott ideal.[9]

9: By an immediate relativization, for every set X, there exists an X-computable infinite binary tree such that every path codes a Scott ideal containing X.

Exercise 4.3.3. Let T be a computable tree functional such that for every $X \in 2^{\mathbb{N}}$, $[T^X]$ is the class of all $\{0,1\}$-valued DNC functions relative to X.

1. Show that the class $\{X \oplus Y : X \in T^{\emptyset} \wedge Y \in T^X\}$ is Π^0_1 and non-empty.
2. Deduce that for every PA degree $\mathbf{a}$, there is a PA degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ is PA over $\mathbf{b}$. ★

Given a Turing ideal $\mathcal{M}$, a set A $\mathcal{M}$-*computes* B if there is some $X \in \mathcal{M}$ such that $B \leq_T A \oplus X$. A Turing ideal $\mathcal{M}$ is *topped by* X if $\mathcal{M} = \{Z \in 2^{\mathbb{N}} : Z \leq_T X\}$.

Computation over Turing ideals can be seen as a generalization of regular computation. Indeed, computation over a topped Turing ideal is nothing but relativized computation. Interesting behaviors happen when working with non-topped Turing ideals, such as Scott ideals. By definition, when a Turing ideal is not topped, it cannot be represented as the collection of sets computable by a single set X. However, Spector [25] proved that every countable Turing ideal can be represented by two sets A and B.

Definition 4.3.4. A pair of sets A, B forms an *exact pair* for a countable Turing ideal $\mathcal{M}$ if $\mathcal{M} = \{Z \in 2^{\mathbb{N}} : Z \leq_T A \wedge Z \leq_T B\}$. ◊

Theorem 4.3.5 (Spector [25])
Every countable Turing ideal $\mathcal{M}$ admits an exact pair.

Proof. Say $\mathcal{M} = \{Z_0, Z_1, \dots\}$. The idea is to construct two sets $G_0 = \bigoplus_n X_n^0$ and $G_1 = \bigoplus_n X_n^1$ such that each column X_n^i for $i \in \{0, 1\}$ is equal to the set Z_n, except for a finite number of bits. It is then clear that every set in $\mathcal{M}$ is computable both by G_0 and G_1. However, one must build the sets G_0 and G_1 so that they satisfy the following requirements:[10]

$$\mathcal{R}_{e_0,e_1} : \Phi_{e_0}^{G_0} = \Phi_{e_1}^{G_1} \longrightarrow \Phi_{e_0}^{G_0} \in \mathcal{M}$$

Consider the notion of forcing whose conditions are 3-tuples (σ_0, σ_1, n) where $\sigma_0, \sigma_1 \in 2^{<\mathbb{N}}$ and $n \in \mathbb{N}$. The parameter n is used to "lock" the n first columns of G_0 and G_1, meaning that from now on, these columns will coincide with the n first sets of $\mathcal{M}$. [11] The *interpretation* of a condition (σ_0, σ_1, n) is the class of all pairs of finite or infinite sequences[12] (G_0, G_1) such that

- $\sigma_i \preceq G_i$;
- for every $k < n$ and every $\langle k, a\rangle$ such that $|\sigma_i| \leq \langle k, a\rangle < |G_i|$, $G_i(\langle k, a\rangle) = Z_k(a)$.

A condition (τ_0, τ_1, m) *extends* (σ_0, σ_1, n) if $n \leq m$ and $(\tau_0, \tau_1) \in [\sigma_0, \sigma_1, n]$. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$, defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, n) \in \mathcal{F}\}$. Note that $(G_{\mathcal{F},0}, G_{\mathcal{F},1}) \in \bigcap\{[\sigma_0, \sigma_1, n] : (\sigma_0, \sigma_1, n) \in \mathcal{F}\}$. We now prove the core lemma:

Lemma 4.3.6. Let $p = (\sigma_0, \sigma_1, n)$ be a condition and $e_0, e_1 \in \mathbb{N}$. There is an extension (τ_0, τ_1, n) of p forcing $\mathcal{R}_{e_0,e_1}$. ★

Proof. There are three cases:

- Case 1: there is some $x \in \mathbb{N}$ and some finite pair $(\tau_0, \tau_1) \in [\sigma_0, \sigma_1, n]$ such that $\Phi_{e_0}^{\tau_0}(x)\downarrow \neq \Phi_{e_1}^{\tau_1}(x)\downarrow$. Then (τ_0, τ_1, n) is an extension of p forcing $\mathcal{R}_{e_0,e_1}$.
- Case 2: there is some $x \in \mathbb{N}$ and some $i < 2$ such that for every finite pair $(\tau_0, \tau_1) \in [\sigma_0, \sigma_1, n]$, $\Phi_{e_i}^{\tau_i}(x)\uparrow$. Then the condition p already forces $\mathcal{R}_{e_0,e_1}$.
- Case 3: none of Case 1 and Case 2 holds. We claim that p forces $\Phi_{e_0}^{G_0}$ to be either partial, or $Z_0 \oplus \dots \oplus Z_{n-1}$-computable, hence to be in $\mathcal{M}$. Indeed, define the partial $Z_0 \oplus \dots \oplus Z_{n-1}$-computable function h by searching on every input $x \in \mathbb{N}$ for some finite pair $(\tau_0, \tau_1) \in [\sigma_0, \sigma_1, n]$ such that $\Phi_{e_1}^{\tau_1}(x)\downarrow$, and return the output. By negation of Case 2, the function h is total. Moreover, by negation of Case 1, p forces $\Phi_{e_0}^{G_0}$ to be either partial, or equal to h. ■

10: There are three ways to satisfy this requirement: either force partiality of $\Phi_{e_i}^{G_i}$ for some $i < 2$, or force $\Phi_{e_0}^{G_0}$ and $\Phi_{e_1}^{G_1}$ to both halt on a same value and disagree, or force $\Phi_{e_0}^{G_0} \in \mathcal{M}$.

11: This notion of forcing has a similar flavor as the one used in Theorem 3.2.4. In particular, both have a lock playing the same role.

12: More formally, $G_i \in 2^{\leq\mathbb{N}}$, and we let $|G_i| \in \mathbb{N} \cup \{\mathbb{N}\}$ be the length of this sequence.

We are now ready to prove Theorem 4.3.5. Let $\mathcal{F}$ be a sufficiently generic filter for this notion for forcing. For each $i < 2$, let $G_i = G_{\mathcal{F},i}$. For every $k \in \mathbb{N}$, the set of conditions (σ_0, σ_1, n) such that $\min(|\sigma_0|, |\sigma_1|, n) \geq k$ is dense, so if $\mathcal{F}$ is sufficiently generic, then $(G_{\mathcal{F},0}, G_{\mathcal{F},1})$ is a pair of infinite sequences and the set $\{n \in \mathbb{N} : (\sigma_0, \sigma_1, n) \in \mathcal{F}\}$ is infinite. It follows that eventually, the kth column of $G_{\mathcal{F},0}$ will be equal to Z_k, except for a finite number of bits. Thus, every set in $\mathcal{M}$ is both G_0 and G_1-computable. Moreover, by Lemma 4.3.6, if $G_0 \geq_T X$ and $G_1 \geq_T X$, then $X \in \mathcal{M}$. Thus, G_0, G_1 is an exact pair for $\mathcal{M}$. This completes the proof of Theorem 4.3.5. ■

This notion was introduced by Spector to give an alternative proof that the Turing degrees do not form a lattice.

Exercise 4.3.7 (Kleene and Post [26]). Show that for every ascending sequence of sets $X_0 <_T X_1 <_T \dots$, the family $\mathcal{M} = \{Z \in 2^{\mathbb{N}} : \exists n\ Z \leq_T X_n\}$ is a countable Turing ideal. Deduce from Theorem 4.3.5 that there exists two Turing degrees with no greatest lower bound. ⋆

4.4 Basic constructions

As mentioned, low sets are typically obtained by effectivizing the construction of a generic set for a notion of forcing with a Σ^0_1-preserving forcing question. For any reasonable notion of forcing, and any fixed set A, the set of conditions forcing $G \neq A$ is dense. Hence, for any sufficiently generic filter $\mathcal{F}$, the set $G_{\mathcal{F}}$ will not belong to the arithmetic hierarchy or more generally to any fixed countable collection of sets. Thus, effectivizing the construction of a filter restricts its amount of genericity. In particular, for the construction of low sets, 1-genericity is the appropriate amount of genericity.

Definition 4.4.1. A condition p *decides* a formula $\varphi(G)$ if p forces $\varphi(G)$ or its negation. A filter $\mathcal{F}$ *decides* a formula if it contains a condition deciding it. A filter $\mathcal{F}$ is n*-generic*[13] if it decides every Σ^0_n formula. ◇

13: The definition is slightly different for Cohen forcing, but they coincide if one considers an appropriate forcing relation.

When effectivizing forcing constructions, we shall work with infinite decreasing sequences of conditions rather than with actual filters. Recall that any decreasing sequence of conditions $p_0 \geq p_1 \geq \dots$ induces a filter $\mathcal{F} = \{q \in \mathbb{P} : \exists n\ p_n \leq q\}$. By extension, we call such a decreasing sequence n*-generic* if its induced filter is n-generic. In many situations, the partial order will not be computable, and therefore the induced filter will be less computable than the decreasing sequence.

The most basic example of effectivization of a forcing construction is the proof of the existence of a non-computable set of low degree using Cohen forcing.

Theorem 4.4.2
There exists a non-computable set of low degree.

Proof. We shall construct a 1-generic decreasing sequence of Cohen conditions[14] computably in $\emptyset'$. As a byproduct of our decision procedure for 1-genericity, the resulting set G will not be computable. However, for the sake of simplicity, we shall explicitly satisfy the non-computability requirements. We therefore prove two lemmas which will ensure 1-genericity and non-computability, respectively.

14: Cohen conditions are finite objects, and therefore don't need any specific coding.

Lemma 4.4.3. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ deciding $\Phi_e^G(e)\downarrow$. Furthermore, the extension τ and the decision can be obtained $\emptyset'$-computably uniformly in σ and e. ⋆

Proof. The oracle $\emptyset'$ can decide whether there is some $\tau \geq \sigma$ such that $\Phi_e^\tau(e)\downarrow$.[15] In the former case, such a τ can be found computably in σ and e while in the latter case, σ already forces $\Phi_e^G(e)\uparrow$. ■

15: Recall that for a Σ_1^0 formula $\varphi(G)$, $\sigma \mathrel{?\vdash} \varphi(G)$ is defined as $\exists \tau \geq \sigma\ \varphi(\tau)$. Since this is a Σ_1^0-preserving forcing question, $\emptyset'$ can decide whether it holds or not. Furthermore, in either case, the extension witnessing it can be found $\emptyset'$-computably.

Lemma 4.4.4. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing $G \neq \Phi_e$.[16] Furthermore, the extension τ can be obtained $\emptyset'$-computably uniformly in σ and e. ⋆

16: Here, $G \neq \Phi_e$ is a notation for

$$\exists x \Phi_e(x)\uparrow \vee \exists x \Phi_e(x)\downarrow \neq G(x)$$

Proof. Letting $x = |\sigma|$, the oracle $\emptyset'$ can decide whether $\Phi_e(x)\downarrow$ or not. In the former case, let $\tau = \sigma \cdot (1 - \Phi_e(x))$, so that τ forces $G \neq \Phi_e$. In the latter case, σ already forces $G \neq \Phi_e$, so let $\tau = \sigma$. In either case, τ can be found $\emptyset'$-computably uniformly in σ and e. ■

We are now ready to prove Theorem 4.4.2. Thanks to Lemma 4.4.3 and Lemma 4.4.4, define a $\emptyset'$-computable infinite decreasing sequence of Cohen conditions $\sigma_0 \prec \sigma_1 \prec \dots$ such that for every $e \in \mathbb{N}$, σ_{2e+1} decides $\Phi_e^G(e)\downarrow$ and σ_{2e+2} forces $G \neq \Phi_e$. Moreover, for every e, we can ensure that $|\sigma_e| \geq e$, so that $\bigcap_e [\sigma_e]$ is a singleton G. Note that $G = G_{\mathcal{F}}$ where $\mathcal{F}$ is the induced filter for this sequence. By construction, $G' \leq_T \emptyset'$ and G is not computable. This completes the proof of Theorem 4.4.2. ■

Exercise 4.4.5. Every non-computable set of low degree is of hyperimmune degree, so Theorem 4.4.2 implies the existence of a hyperimmune set of low degree. Adapt the proof of Theorem 4.4.2 to directly construct such a set. ⋆

The next example is known as the low basis theorem, and is arguably one of the most useful theorems of computability theory.

Theorem 4.4.6 (Jockusch and Soare [9])
Fix a non-empty Π_1^0 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$. There exists a member $G \in \mathcal{P}$ of low degree.

Proof. Consider the Jockusch-Soare forcing defined in Theorem 3.2.6, that is, the notion of forcing whose conditions are computable infinite binary trees, partially ordered by the inclusion relation. A condition $T \subseteq 2^{<\mathbb{N}}$ can be coded by a Δ_1^0-index, that is, some Turing index b such that $\Phi_b = T$. We shall construct an infinite $\emptyset'$-computable sequence of Δ_1^0-indices $b_0, b_1, \dots$ of a 1-generic decreasing sequence of conditions $T_0 \supseteq T_1 \supseteq \dots$ The following lemma ensures that 1-genericity can be obtained $\emptyset'$-uniformly.

Lemma 4.4.7. For every condition $T \subseteq 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $S \subseteq T$ deciding $\Phi_e^G(e)\downarrow$. Furthermore, a Δ_1^0-index of S and the decision can be obtained $\emptyset'$-computably uniformly in e and a Δ_1^0-index of T. ⋆

Proof. The oracle $\emptyset'$ can decide whether there exists a level $\ell \in \mathbb{N}$ in the tree such that for every $\sigma \in T$ of length ℓ, $\Phi_e^\sigma(e)\downarrow$.[17] In the former case, T already forces $\Phi_e^G(e)\downarrow$. In the latter case, the tree $S = \{\sigma \in T : \Phi_e^\sigma(e)\uparrow\}$ is an extension of T forcing $\Phi_e^G(e)\uparrow$. In both cases, the witness can be found $\emptyset'$-computably. ■

17: Here again, recall that for a Σ_1^0 formula $\varphi(G)$, $T \mathrel{?\vdash} \varphi(G)$ is defined as $\forall P \in [T]\ \varphi(P)$, or equivalently by compactness $(\exists \ell)(\forall \sigma \in T \cap 2^\ell)\varphi(\sigma)$. Since this is a Σ_1^0-preserving forcing question, $\emptyset'$ can decide whether it holds or not. This lemma shows that in either case, the witnessing extension can be found $\emptyset'$-computably.

We are now ready to prove Theorem 4.4.6. Thanks to Lemma 4.4.7, define a $\emptyset'$-computable infinite sequence of Δ^0_1-indices $b_0, b_1, \dots$ of a decreasing sequence of conditions $T_0 \supseteq T_1 \supseteq \dots$ starting with $[T_0] = \mathscr{P}$ and such that for every $e \in \mathbb{N}$, T_{e+1} decides $\Phi^G_e(e)\downarrow$. Note that $\bigcap_e [T_e]$ is a singleton G, as for every $n \in \mathbb{N}$, there is a Turing functional Φ_e such that $\Phi^G_e(e)\downarrow$ iff $G(n) = 1$. Note again that $G = G_{\mathscr{F}}$ where $\mathscr{F}$ is the induced filter for this sequence. By definition of a condition, $G \in [T_0] = \mathscr{P}$, and by construction $G' \leq_T \emptyset'$. This completes the proof of Theorem 4.4.6. ■

In summary, both constructions were obtained by constructing an infinite $\emptyset'$-computable sequence of codes of a 1-generic decreasing sequence of conditions. For Cohen forcing, the situation was slightly simpler as conditions were identified with their own code. In any case, such a sequence was obtained by proving the existence of a Σ^0_1-preserving forcing question such that the codes of their witnessing extensions were obtained $\emptyset'$-computably uniformly in codes of the conditions.

4.5 Weak preservation

Contrary to cone avoidance, it is not necessary to have a Σ^0_1-preserving forcing question to produce a set of low degree. It is sufficient to have a Δ^0_2 forcing question for Σ^0_1 formulas[18], uniformly in its parameters (including the condition, under the appropriate coding). This is in particular the case of the following theorem, stating the existence of an infinite subset of low degree.

18: As mentioned in Section 3.5, Σ^0_n sets are arguably more natural than Δ^0_n sets, as the former class is syntactic, while the latter is semantic. As a consequence, when proving a theorem with a purely combinatorial hypothesis through forcing, the forcing question for Σ^0_1 formulas will naturally be either Σ^0_1-preserving, or not even Δ^0_2. In other words, all constructions in this section will exploit some computational distorsion of the combinatorics. In Theorem 4.5.2, the co-hyperimmunity hypothesis is computability-theoretic and is responsible of this distorsion.

What is a sufficient largeness condition for a Σ^0_2 set to have an infinite subset of low degree? Being infinite is not sufficient, as there exists infinite Δ^0_2 sets such that every infinite subset computes $\emptyset'$: consider the set of all initial segments of the halting set $A = \{\sigma \in 2^{<\mathbb{N}} : \sigma \prec \emptyset'\}$. Recall that an *array* is a sequence of pairwise disjoint finite sets $\{F_n\}_{n\in\mathbb{N}}$. An array $\{F_n\}_{n\in\mathbb{N}}$ is c.e. if there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is the canonical code of F_n. Last, an infinite set A is *hyperimmune* if for every c.e. array $\{F_n\}_{n\in\mathbb{N}}$, there is some $n \in \mathbb{N}$ such that $A \cap F_n = \emptyset$.

Exercise 4.5.1. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is hyperimmune if it is not dominated by any computable function. The *principal function* of an infinite set $A = \{x_0 < x_1 < \dots\}$ is the function $p_A : \mathbb{N} \to \mathbb{N}$ defined by $p_A(n) = x_n$. Show that an infinite set A is hyperimmune iff its principal function is hyperimmune. ⋆

Informally, if A is hyperimmune, then $\overline{A}$ contains a lot of elements. Therefore, co-hyperimmunity is a notion of largeness.

Theorem 4.5.2
For every Σ^0_2 co-hyperimmune set A, there is an infinite set $H \subseteq A$ of low degree.

Proof. Consider a variant of Cohen forcing where conditions $\sigma \in 2^{<\mathbb{N}}$ are subsets of A, that is, $\forall x < |\sigma|\ \sigma(x) = 1 \to x \in A$. To avoid confusion, we shall write $\tau \leq \sigma$ for condition extension and keep $\preceq$ for the usual strings extension. Therefore, $\tau \leq \sigma$ iff $\sigma \preceq \tau$ and $\tau \subseteq A$. The interpretation[19] of a condition σ is $[\sigma] = \{Z \in 2^{\mathbb{N}} : \sigma \prec Z\}$. We shall construct a 1-generic

19: One could have defined $[\sigma]$ as

$$\{Z \in 2^{\mathbb{N}} : \sigma \prec Z \wedge Z \subseteq A\}$$

decreasing sequence of conditions computably in $\emptyset'$. The core of the argument lies in the following lemma.

Lemma 4.5.3. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau > \sigma$ deciding $\Phi_e^G(e)\downarrow$. Furthermore, the extension τ and the decision can be obtained $\emptyset'$-computably uniformly in σ and e. ★

Proof. Let 0^n denote the string of length n with only 0's. Given a condition σ, we claim that at least one of the following two Σ_2^0 statements is true:

(1) There is some $\tau \geq \sigma$ with $\tau \subseteq A$ such that $\Phi_e^\tau(e)\downarrow$.
(2) There is some $n \in \mathbb{N}$ such that, letting $\tau = \sigma \cdot 0^n$, for every $\mu \geq \tau$, $\Phi_e^\mu(e)\uparrow$.

Suppose not. Then, by negation of (2) for every $n \in \mathbb{N}$, there is some $\mu_n \geq \sigma \cdot 0^n$ such that $\Phi_e^{\mu_n}(e)\downarrow$. For every $n \in \mathbb{N}$, let $F_n = \{x > |\sigma| + n : \mu_n(x) = 1\}$. By negation of (1), $F_n \cap \overline{A} \neq \emptyset$ for every n. By considering a pairwise disjoint computable sub-collection of sets to obtain a c.e. array, we contradict hypermmunity of $\overline{A}$.

Thus, since both statements are Σ_2^0, search $\emptyset'$-computably for some τ witnessing either case.[20] ■

20: Because of the combinatorial distorsion induced by the co-hyperimmunity assumption, the statement of the forcing question is not natural: Given a Σ_1^0 formula $\varphi(G)$, let $\sigma \mathrel{?\vdash} \varphi(G)$ hold if the first witness found in the $\emptyset'$-computable search belongs to the first case.

We are now ready to prove Theorem 4.5.2. Thanks to Lemma 4.5.3, define a $\emptyset'$-computable infinite decreasing sequence of conditions $\sigma_0 \geq \sigma_1 \geq \dots$ such that for every $e \in \mathbb{N}$, σ_{e+1} decides $\Phi_e^G(e)\downarrow$. Moreover, since A is co-hyperimmune, it is infinite, so for every e, we can ensure that $\operatorname{card} \sigma_e = \{n : \sigma_e(n) = 1\} \geq e$ by waiting $\emptyset'$-computably for some new elements of A to be enumerated. As a consequence, $\bigcap_e [\sigma_e]$ is a singleton G. Note that $G = G_{\mathscr{F}}$ where $\mathscr{F}$ is the induced filter for this sequence. By construction, $G' \leq_T \emptyset'$ and G is an infinite subset of A. This completes the proof of Theorem 4.5.2. ■

Theorem 4.5.2 has some interesting consequences for the computable analysis of partial and linear orders. Let ω be the order type of $(\mathbb{N}, <)$. Given two order types α, β, let α^* be the reverse order, and $\alpha + \beta$ be the order type such that every element of α is smaller than every element of β. A linear order $\mathscr{L} = (\mathbb{N}, <_{\mathscr{L}})$ is *stable* if it is of order type $\omega + \omega^*$, that is, for every element $x \in \mathbb{N}$, either $\forall^\infty y(x <_{\mathscr{L}} y)$ or $\forall^\infty y(x >_{\mathscr{L}} y)$. Here, the notation $\forall^\infty$ means "for all but finitely many".

Exercise 4.5.4 (Hirschfeldt and Shore [23]). Let $\mathscr{L} = (\mathbb{N}, <_{\mathscr{L}})$ be a computable stable linear order. Let $A = \{x : \forall^\infty y\ (x <_{\mathscr{L}} y\}$ and $A^* = \{x : \forall^\infty y\ (y <_{\mathscr{L}} x\}$.

1. Show that $A \sqcup A^* = \mathbb{N}$ and A is Δ_2^0.
2. Show that A and A^* are immune iff they are hyperimmune.[21]
3. Use Theorem 4.5.2 to prove that $\mathscr{L}$ admits an infinite ascending or descending sequence of low degree. ★

21: An infinite set A is *immune* if it has no infinite computable subset, or equivalently no infinite c.e. subset.

4.6 Beyond $\emptyset'$

Some problems do not admit a low basis, but always have a solution which is close to being low, in the sense that every PA degree over $\emptyset'$ computes the jump of a solution. The various basis theorems for Π^0_1 classes show that PA degrees share many features of the **0** degree: the computably dominated and the cone avoidance basis theorems say that the existence of a PA degree does not help computing fast-growing functions[22], or computing fixed non-computable sets. By relativization over $\emptyset'$, having the jump of a solution computed by any PA degree over 0' is close to having a the jump of a solution computed by $\emptyset'$, in other words to having a solution of low degree.

22: In the sense that a non-decreasing hyperimmune function is growing so fast that no computable function dominates it.

Definition 4.6.1. A problem P admits a *weakly low basis* if for every set Z and every PA degree P over Z', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)' \leq_T P$. ◇

At first sight, Definition 4.6.1 does not yield an invariant property, as one would require P to be PA over $(Y \oplus Z)'$ instead of only computing $(Y \oplus Z)'$. However, based on the density properties of PA degrees, Definition 4.6.1 is actually equivalent to the stronger statement.

Exercise 4.6.2. Use Exercise 4.3.3 to prove that if a problem P admits a weakly low basis, then for every set Z and every PA degree P over Z', every Z-computable instance X of P admits a solution Y such that P is of PA degree over $(Y \oplus Z)'$. ★

A set X is of *low*$_2$ degree if $X'' \leq_T \emptyset''$. If a problem admits a weakly low basis, then it always admits solutions of low$_2$ degree, by choosing an appropriate PA degree.

Exercise 4.6.3. A problem P admits a *low*$_2$ *basis* if for every set Z and every Z-computable instance X of P, there is a solution Y to X such that $(Y \oplus Z)'' \leq_T Z''$. Use the low basis theorem for Π^0_1 classes (Theorem 4.4.6) to show that if P admits a weakly low basis, then it admits a low$_2$ basis. ★

As for sets of low degree, if a set G is of low$_2$ degree, then by Post's theorem, every $\Sigma^0_3(G)$ property is Σ^0_3. Thus, if a problem admits a low$_2$ basis, then it satisfies every weakness property at the third and higher jump levels. Some weakness properties at the second jump level are also preserved, depending on the existence of the appropriate basis theorem for Π^0_1 classes.

Exercise 4.6.4. Suppose that a problem P admits a weakly low basis. Let C be a non-Δ^0_2 set, and X be a computable instance of P. Use the cone avoidance basis theorem for Π^0_1 classes (Theorem 3.2.6) to show that there is a solution Y to X such that C is not $\Delta^0_2(Y)$. ★

There is a well-known correspondence between computability and definability. By Post's theorem, Δ^0_n sets are exactly the $\emptyset^{(n-1)}$-computable ones. Historically, the Turing jump of a set X is defined as $X' = \{e : \Phi^X_e(e)\downarrow\}$, but it could be equivalently defined as the set of codes of true $\Sigma^0_1(X)$ formulas. PA degrees also admit a characterization in terms of decidability of formulas:

Exercise 4.6.5. Let $\varphi_0, \varphi_1, \dots$ be an effective enumeration of all $\Pi^0_1(X)$ sentences. Show that any PA degree over X computes a total function $f : \mathbb{N}^2 \to 2$ such that for every $(a, b) \in \mathbb{N}^2$ for which at least one of φ_a, φ_b is true, if $f(a, b) = 0$ then φ_a is true, and if $A(n) = 1$ then φ_b is true.[23] ★

23: If φ_a and φ_b have the same truth value, then $f(a, b)$ can be either 0 or 1 but must output a value anyway. The careful reader will have recognized the behavior of $\{0, 1\}$-valued DNC functions.

By Post's theorem, any PA degree over $\emptyset'$ is able to choose, given a sequence of pairs of Π^0_2 formulas such that for every pair at least one is true, a sequence of true formulas. Among the natural Π^0_2 formulas, we shall be particularly interested in infinity of a computable set.

Exercise 4.6.6. Let $X_0, X_1, \dots$ a uniformly computable sequence of sets. Use Exercise 4.6.5 to show that any PA degree over $\emptyset'$ computes a sequence $A \in 2^{\mathbb{N}}$ such that for every n, if $A(n) = 0$ then X_n is infinite, and if $A(n) = 1$, then $\overline{X}_n$ is infinite. ⋆

4.7 Ramsey's theorem for pairs

The main application of the previous section will be the proof by Cholak, Jockusch and Slaman [27] that Ramsey's theorem for pairs admits a weakly low basis. The *jump*[24] of a problem P is the problem P′ whose instances are Δ^0_2 approximations of an instance X of P, in other words, stable functions $f : \mathbb{N}^2 \to 2$ whose limit is X, and whose solutions are P-solutions to X. Following Theorem 3.4.1, RT^2_2 can be obtained by applying the cohesiveness principle (COH), and then the pigeonhole principle for Δ^0_2 instances ($\mathsf{RT}^{1\prime}_2$).[25] Thanks to Exercise 4.6.2, it suffices to independently prove that COH and $\mathsf{RT}^{1\prime}_2$ admit a weakly low basis to obtain the same conclusion for RT^2_2.

24: The notion of jump of a problem comes from Weihrauch complexity.

25: The problem $\mathsf{RT}^{1\prime}_2$ is also known as D^2_2 in the literature. More generally, D^n_k is the statement "For every Δ^0_n k-partition $A_0 \sqcup \dots \sqcup A_{k-1} = \mathbb{N}$, there is some $i < k$ and an infinite set $H \subseteq A_i$". The practice shows that it is more convenient to think of it as the jump of the pigeonhole principle.

Recall that by Exercise 3.4.3, for every uniformly computable sequence of sets $\vec{R} = R_0, R_1, \dots$, there is a non-empty $\Pi^0_1(\emptyset')$ class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ such that the degrees computing an $\vec{R}$-cohesive set are exactly those whose jump compute a member of $\mathcal{P}$.

Exercise 4.7.1. Use Exercise 3.4.3 to prove that COH admits a weakly low basis, but does not admit a low basis. ⋆

We will now give an alternative direct proof that COH admits a weakly low basis using an effectivization of computable Mathias genericity. This will serve as a warm-up to the proof that $\mathsf{RT}^{1\prime}_2$ admit a weakly low basis.[26]

26: This proof, due to Cholak, Jockusch and Slaman [27], is actually very close to the original proof of Jockusch and Stephan [13], except we decide the jump of an $\vec{R}$-cohesive set C in a set P of PA degree over $\emptyset'$, while the original proof used a Δ^0_2 approximation of P to construct C. In both proofs, there is a "delay" in the satisfaction of cohesiveness: in our case, this is due to the genericity requirements, while in the original proof, the Δ^0_2 approximation of P may take some time to converge to a right answer.

Theorem 4.7.2 (Jockusch and Stephan [13])
Let $\vec{R} = R_0, R_1, \dots$ be an infinite uniformly computable sequence of sets and let P be of PA degree over $\emptyset'$. There exists an infinite $\vec{R}$-cohesive set C such that $C' \leq_T P$.

Proof. Recall that a computable Mathias condition is a Mathias condition (σ, X) whose reservoir X is computable. Any computable Mathias condition (σ, X) can therefore be coded by a pair $\langle \sigma, b \rangle$ such that b is a Δ^0_1-index of X. We shall construct an infinite P-computable sequence of codes $\langle \sigma_0, b_0 \rangle, \langle \sigma_1, b_1 \rangle, \dots$ representing a 1-generic decreasing sequence of computable Mathias conditions $(\sigma_0, X_0) \geq (\sigma_1, X_1) \geq \dots$. The following lemma shows that such a sequence can be obtained $\emptyset'$-computably:

Lemma 4.7.3. For every condition (σ, X) and every Turing index $e \in \mathbb{N}$, there is an extension (τ, Y) deciding $\Phi^G_e(e)\downarrow$. Furthermore, a code for (τ, Y) and the decision can be obtained $\emptyset'$-computably uniformly in a code for (σ, X) and e. ⋆

Proof. The oracle $\emptyset'$ can decide whether there exists a finite string $\rho \subseteq X$ such that $\Phi_e^{\sigma\cup\rho}(e)\downarrow$. If so, then $(\sigma\cup\rho, X\setminus\{0,\dots,|\rho|\})$ is an extension forcing $\Phi_e^G(e)\downarrow$. Otherwise, (σ, X) already forces $\Phi_e^G(e)\uparrow$. Note that a Δ_1^0-index of $X\setminus\{0,\dots,|\rho|\}$ can be computably found in a Δ_1^0-index of X and ρ. Therefore, a code for the extension can be obtained $\emptyset'$-computably uniformly in a code for (σ, X) and e. ■

Lemma 4.7.3 only requires $\emptyset'$ instead of a PA degree over $\emptyset'$. Therefore, one can obtain a $\emptyset'$-computable 1-generic decreasing sequence of computable Mathias conditions. However, the resulting set will not be $\vec{R}$-cohesive. We need to interleave steps to satisfy cohesiveness for more and more sets. This is the purpose of the following lemma:

Lemma 4.7.4. For every condition (σ, X) and every computable set R, there is an extension (σ, Y) such that $Y \subseteq R$ or $Y \subseteq \overline{R}$. Furthermore, a code for (σ, Y) and the decision can be obtained P-computably uniformly in a code for (σ, X) and a Δ_1^0-index of R. ⋆

Proof. Fix an effective enumeration of all Π_2^0 sentences $\varphi_0, \varphi_1, \dots$ Let $f : \mathbb{N}^2 \to 2$ be the P-computable function satisfying Exercise 4.6.5. From Δ_1^0-indices of X and R, one can compute codes $a, b \in \mathbb{N}$ such that $\varphi_a \equiv \forall x \exists y (y > x \wedge y \in X \cap R)$ and $\varphi_b \equiv \forall x \exists y (y > x \wedge y \in X \cap \overline{R})$. Note that at least one of φ_a and φ_b is true. Thus, if $f(a,b) = 0$, $(\sigma, X\cap R)$ is a valid extension, and if $f(a,b) = 1$, $(\sigma, X \cap \overline{R})$ is a valid extension. In both cases, Δ_1^0-indices of $X\cap R$ and $X\cap\overline{R}$ can be obtained computably from Δ_1^0-indices of X and R, so a code for the extension can be obtained P-computably in a code for (σ, X) and a Δ_1^0-index of R. ■

We are now ready to prove Theorem 4.7.2. Thanks to Lemma 4.7.3 and Lemma 4.7.4, define a P-computable infinite sequence of codes

$$\langle \sigma_0, b_0\rangle, \langle \sigma_1, b_1\rangle, \dots$$

representing a decreasing sequence of computable Mathias conditions

$$(\sigma_0, X_0) \geq (\sigma_1, X_1) \geq \dots$$

such that for every $e \in \mathbb{N}$, $(\sigma_{2e+1}, X_{2e+1})$ decides $\Phi_e^G(e)\downarrow$ and either $X_{2e+2} \subseteq R_e$, or $X_{2e+2} \subseteq \overline{R_e}$. Moreover, for every e, we can ensure that $\operatorname{card} \sigma_e \geq e$, so that $G = \bigcup_e \sigma_e$ is an infinite set. By construction, $G' \leq_T P$ and G is $\vec{R}$-cohesive. This completes the proof of Theorem 4.7.2. ■

The previous example involved a Σ_1^0-preserving forcing question with the appropriate uniformity properties to build a set of low degree, but the additional requirements to produce a cohesive set used a PA degree over $\emptyset'$. In the following example, the Σ_1^0-preserving forcing question itself will require a PA degree over $\emptyset'$ to produce a code of an extension.

Theorem 4.7.5 (Cholak, Jockusch and Slaman [27])
Let A be a Δ_2^0 set and let P be of PA degree over $\emptyset'$. There exists an infinite set $G \subseteq A$ or $G \subseteq \overline{A}$ such that $G' \leq_T P$.

Proof. By the low basis theorem for Π^0_1 classes (Theorem 4.4.6) and Theorem 4.3.2, there exists a set $M = \bigoplus_n Z_n$ of low degree coding for a Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

As in the proof of Theorem 3.4.6, consider a variant of Mathias forcing, whose *conditions* are triples (σ_0, σ_1, X) where

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_i \subseteq A_i$;
3. $X \in \mathcal{M}$.

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X). Recall that an M-code of a set $X \in \mathcal{M}$ is an integer $a \in \mathbb{N}$ such that $X = Z_a$. A code for a condition (σ_0, σ_1, X) is therefore a 3-tuple $\langle \sigma_0, \sigma_1, a \rangle$ where a is an M-code for X.

Following the proof of Theorem 3.4.6, we shall make the following assumption to ensure that both sets G_0 and G_1 will be infinite:

$$\text{There is no infinite set } H \subseteq A \text{ or } H \subseteq \overline{A} \text{ such that } H \in \mathcal{M}. \qquad \text{(H1)}$$

Since $\mathcal{M}$ contains only sets of low degree, if the assumption is false, then the statement of the theorem holds, so suppose it is true.

Lemma 4.7.6. Suppose (H1). Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$. There is an extension (τ_0, τ_1, Y) of p and some $n > |\sigma_i|$ such that $n \in \tau_i$. Furthermore, a code for (τ_0, τ_1, Y) can be found $\emptyset'$-computably uniformly in a code for p and i. ⋆

Proof. If $X \cap A^i$ is empty, then $X \subseteq A^{1-i}$, but $X \in \mathcal{M}$, which contradicts (H1). Thus, there is some $n \in X \cap A^i$. Let $\tau_i = \sigma_i \cup \{n\}$, and $\tau_{1-i} = \sigma_{1-i}$. Then, $(\tau_0, \tau_1, X \setminus \{0, \dots, n\})$ is an extension of p such that $n \in \tau_i$. Moreover, since A is Δ^0_2, and $M' \leq_T \emptyset'$, the oracle $\emptyset'$ can find such an n from an M-code of X and $i < 2$. An M-code of $X \setminus \{0, \dots, n\}$ can be found computably from an M-code of X and n, so a code for (τ_0, τ_1, Y) can be found $\emptyset'$-computably uniformly in a code for p and i. ∎

Due to the disjunctive nature of the notion of forcing, we need to redefine what it means for a filter to be 1-generic. Recall that the interpretation of a Mathias condition (σ, X) is the class $[\sigma, X]$ of all sets G such that $\sigma \subseteq G \subseteq \sigma \cup X$. Each condition (σ_0, σ_1, X) has two interpretations, namely, $[\sigma_0, X]$ and $[\sigma_1, X]$, depending on the side.[27] A condition (σ_0, σ_1, X) *decides* $(\varphi_0(G_0), \varphi_1(G_1))$ if there is some $i < 2$ such that (σ_i, X) decides $\varphi_i(G)$. A filter $\mathcal{F}$ *decides* $(\varphi_0(G_0), \varphi_1(G_1))$ if there is a condition $p \in \mathcal{F}$ deciding $(\varphi_0(G_0), \varphi_1(G_1))$. A filter $\mathcal{F}$ is *1-generic* if it decides every pair of Σ^0_1 formulas.

27: This interpretation of a condition is different from the one in the proof of Theorem 3.4.6, where we considered a class of pairs of sets.

Lemma 4.7.7. For every condition $p = (\sigma_0, \sigma_1, X)$ and every pair of Turing indices $e_0, e_1 \in \mathbb{N}$, there is an extension $q = (\tau_0, \tau_1, Y)$ deciding $(\Phi^{G_0}_{e_0}(e_0)\downarrow, \Phi^{G_1}_{e_1}(e_1)\downarrow)$. Furthermore, a code for q and the decision can be obtained P-computably uniformly in a code for p and e_0, e_1. ⋆

Proof. Let $\mathcal{P}$ be the $\Pi^0_1(X)$ class of all $B \in 2^{\mathbb{N}}$ such that, letting $B_0 = B$ and $B_1 = \overline{B}$, for every $i < 2$ and every $\rho \subseteq X \cap B_i$, $\Phi^{\sigma_i \cup \rho}_{e_i}(e_i)\uparrow$. The oracle $\emptyset'$ can decide whether $\mathcal{P}$ is empty or not from an M-code of X, since M is of low degree.[28]

28: The careful reader will have recognized the disjunctive forcing question of Exercise 3.4.10.

- Suppose $\mathcal{P} = \emptyset$. Then, by compactness, there is a level $\ell \in \mathbb{N}$ such that for every set $\beta \in 2^\ell$, letting $\beta_0 = \beta$ and β_1 be the bitwise negation of β, there is some $i < 2$ and some $\rho \subseteq X \cap \beta_i$ such that $\Phi_{e_i}^{\sigma_i \cup \rho}(e_i)\downarrow$. Such an $\ell \in \mathbb{N}$ can be found M-computably from an M-code of X and e_0, e_1. Since A is Δ^0_2, the oracle $\emptyset'$ can find $\beta = A\upharpoonright_\ell$, and the associated $i < 2$ and ρ. Let $\tau_i = \sigma_i \cup \rho$ and $\tau_{1-i} = \sigma_{1-i}$. Then $q = (\tau_0, \tau_1, X \setminus \{0, \dots, |\rho|\})$ is an extension of p such that $(\tau_i, X \setminus \{0, \dots, |\rho|\})$ forces $\Phi_{e_i}^G(e_i)\downarrow$, hence q decides $(\Phi_{e_0}^{G_0}(e_0)\downarrow, \Phi_{e_1}^{G_1}(e_1)\downarrow)$. Moreover, an M-code for $X \setminus \{0, \dots, |\rho|\}$ can be computed from an M-code for X and ρ, so a code for q can be obtained $\emptyset'$-computably from a code for p.
- Suppose $\mathcal{P} \neq \emptyset$. Then one can obtain an M-code for some $B \in \mathcal{P} \cap \mathcal{M}$ computably from an M-code for X. Using Exercise 4.6.5, since P is of PA degree over M', P can find some $i < 2$ such that $X \cap B_i$ is infinite, and an M-code of $X \cap B_i$. The condition $q = (\sigma_0, \sigma_1, X \cap B_i)$ is an extension of p such that $(\sigma_i, X \cap B_i)$ forces $\Phi_{e_i}^G(e_i)\uparrow$, hence q decides $(\Phi_{e_0}^{G_0}(e_0)\downarrow, \Phi_{e_1}^{G_1}(e_1)\downarrow)$. Moreover, a code for q can be obtained P-computably from a code for p.[29] ■

29: Note that in this lemma, a PA degree over $\emptyset'$ is only used in the second case, to find a side of B whose intersection with X is infinite.

We are now ready to prove Theorem 4.7.5. As usual, thanks to Lemma 4.7.6 and Lemma 4.7.7 and we shall construct an infinite P-computable sequence of codes

$$\langle \sigma_{0,0}, \sigma_{1,0}, b_0 \rangle, \langle \sigma_{0,1}, \sigma_{1,1}, b_1 \rangle, \dots, \langle \sigma_{0,s}, \sigma_{1,s}, b_s \rangle, \dots$$

for a 1-generic decreasing sequence of conditions

$$(\sigma_{0,0}, \sigma_{1,0}, X_0) \geq (\sigma_{0,1}, \sigma_{1,1}, X_1) \geq \dots \geq (\sigma_{0,s}, \sigma_{1,s}, X_s) \geq \dots$$

such that for every $s \in \mathbb{N}$, letting $s = \langle e_0, e_1 \rangle$, $(\sigma_{0,s}, \sigma_{1,s}, X_s)$ decides $(\Phi_{e_0}^{G_0}(e_0)\downarrow, \Phi_{e_1}^{G_1}(e_1)\downarrow)$, and there is some $n_0, n_1 > s$ such that $n_i \in \sigma_{i,s}$. Moreover, P computes the side deciding each formula, and the decision. More precisely, P computes two functions $f, g : \mathbb{N}^2 \to 2$ such that for every $e_0, e_1 \in \mathbb{N}$, letting $s = \langle e_0, e_1 \rangle$ and $i = f(e_0, e_1)$, if $g(e_0, e_1) = 0$ then $(\sigma_{i,s}, X_s)$ forces $\Phi_{e_i}^G(e_i)\uparrow$, and if $g(e_0, e_1) = 1$, then $(\sigma_{i,s}, X_s)$ forces $\Phi_{e_i}^G(e_i)\downarrow$.

By the pigeonhole principle, there is a side $i < 2$ such that for every $e_i \in \mathbb{N}$, there is some $e_{1-i} \in \mathbb{N}$ such that $f(e_0, e_1) = i$. Let $G_i = \bigcup_s \sigma_{i,s}$. By definition of a condition, $G_i \subseteq A_i$, and by construction, G_i is infinite. Last, given $e_i \in \mathbb{N}$, to decide $e_i \in G_i'$, search P-computably for some $e_{1-i} \in \mathbb{N}$ such that $f(e_0, e_1) = i$, and output $g(e_0, e_1)$. Thus, $G_i' \leq_T P$. This completes the proof of Theorem 4.7.5. ■

By Exercise 4.7.1, COH admits a weakly low basis, but not low basis. Actually, every computable instance of COH with no computable solution admits no low solution. What about $\mathsf{RT}^{1\prime}_2$? Downey, Hirschfeldt, Lempp and Solomon [28] proved that $\mathsf{RT}^{1\prime}_2$ admits no low basis.

Theorem 4.7.8 (Downey et al [28])
There exists a Δ^0_2 set A with no low infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$.

First, notice that by Theorem 4.5.2, such an A can be neither hyperimmune or co-hyperimmune, as every Σ^0_2 co-hyperimmune set admits an infinite subset of low degree. The proof of Theorem 4.7.8 involves an infinite injury priority construction and is outside the scope of this book.[30]

30: Note that the proof of Theorem 4.7.8 is intrinsically complicated, as Chong, Slaman and Yang [29] constructed a non-standard model of $\mathsf{WKL}_0 + \mathsf{RT}^{1\prime}_2$ with only low sets. They exploited a failure of Σ^0_2-induction.

One can put together Theorem 4.7.2 and Theorem 4.7.5 to prove that Ramsey's theorem for pairs admits a weakly low basis.

Theorem 4.7.9 (Cholak, Jockusch and Slaman [27])
Let $f : [\mathbb{N}]^2 \to 2$ be a computable coloring and let P be of PA degree over $\emptyset'$. There exists an infinite f-homogeneous set G such that $G' \leq_T P$.

Proof. The proof follows the one of Theorem 3.4.1. Fix f and P. Let $\vec{R} = R_0, R_1, \dots$ be the computable sequence of sets defined for every $x \in \mathbb{N}$ by $R_x = \{y \in \mathbb{N} : f(x, y) = 1\}$. By Theorem 4.7.2 and Exercise 4.6.2, there is an infinite $\vec{R}$-cohesive set $X \subseteq \mathbb{N}$ such that P is PA over X'. In particular, for every $x \in X$, $\lim_{y \in X} f(x, y)$ exists. Let $\hat{f} : X \to 2$ be the limit coloring of f, that is, $\hat{f}(x) = \lim_{y \in X} f(x, y)$. By Theorem 4.7.5, there is an infinite $\hat{f}$-homogeneous set $Y \subseteq X$ for some color $i < 2$ such that $(Y \oplus X)' \leq_T P$. Since for every $x \in Y$, $\lim_{y \in Y} f(x, y) = i$, one can Y-computably thin out the set Y to obtain an infinite f-homogeneous subset $H \subseteq Y$. Since $H \leq_T Y$, $H' \leq_T P$. ■

Recall that Seetapun's theorem states that Ramsey's theorem for pairs admits cone avoidance. The modern proof goes through the decomposition into cohesiveness and the pigeonhole principle, but the original proof was direct and left as an exercise (Exercise 3.4.12).

Exercise 4.7.10. Adapt Exercise 3.4.12 to give a direct proof that Ramsey's theorem for pairs admits a weakly low basis. ★

5 Compactness avoidance

Prerequisites: Chapters 2 and 3

Compactness arguments form a central tool in mathematics in general and in topology in particular. From a reverse mathematical viewpoint, many ordinary theorems are equivalent to the Heine-Borel compactness theorem. Some other theorems contain weaker compactness arguments, and some are compactness-free. In this chapter, we study various levels of compactness, namely, weak König's lemma (PA degrees), weak weak König's lemma (random degrees), DNC degrees, and a Ramsey-type weak König's lemma. For the three former notions, we develop the tools to prove that some theorems lack compactness.

This chapter pushes further the correspondence between computability-theoretic features of a generic set and the existence of a forcing question with appropriate definability and combinatorial features. In particular, PA and DNC avoidance both result from the existence of a forcing question with the ability to find simultaneous answers to independent questions.

5.1 PA avoidance

PA degrees are one of the most important notions in computability-theory, both from a conceptual and a technical perspective. In particular, they form a natural Muchnik degree[1] of intermediate strength between $\mathbf{0}$ and $\mathbf{0}'$. In reverse mathematics, the existence of PA degrees is equivalent to the system WKL_0, which informally corresponds to compactness arguments. Many theorems, such as the Heine-Borel compactness theorem, or Gödel's completeness theorem, are equivalent to WKL_0. Thus, the notion of PA avoidance is not only a technical tool to separate a theorem from WKL_0 in reverse mathematics, but it also reflects the lack of compactness in the proof of the theorem, which is an interesting result in its own right.

1: Muchnik degrees are a generalization of Turing degrees. Many natural computational phenomena are better expressed as families of Turing degrees rather than individual degrees.

Definition 5.1.1. A problem P admits *PA avoidance*[2] if for every pair of sets Z and $D \leq_T Z$ such that Z is not of PA degree over D, every Z-computable instance X of P admits a solution Y such that $Y \oplus Z$ is not of PA degree over D. ◊

2: Here again, the unrelativized formulation with $Z = D = \emptyset$ is far more natural, but does not behave well with artificial problems.

Recall that a *Scott ideal* is a Turing ideal $\mathcal{M}$ such that for every $X \in \mathcal{M}$, there is a set $Y \in \mathcal{M}$ of PA degree over X. Equivalently, a Scott ideal is a Turing ideal such that for every infinite binary tree $T \in \mathcal{M}$, there is an infinite path $P \in [T]$ in $\mathcal{M}$. In reverse mathematics, Turing ideals and Scott ideals are exactly the second-order parts of ω-models of RCA_0 and WKL_0, respectively.

Exercise 5.1.2. Let P be a Π^1_2 problem which admits PA avoidance. Show the existence of an ω-model of RCA_0 + P which does not contain any set of PA degree. ⋆

Let us start with a concrete example of a proof of PA avoidance. As usual, Cohen forcing is the best behaving notion of forcing, as its partial order is computable. In all our proofs of PA avoidance, we shall use $\{0, 1\}$-valued DNC functions. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is diagonally non-computable (DNC) if for every $e \in \mathbb{N}$, $f(e) \neq \Phi_e(e)$. A degree is PA iff it computes a $\{0,1\}$-valued DNC function.

Theorem 5.1.3
For every sufficiently Cohen generic set G, G is not of PA degree.

Proof. It suffices to prove the following lemma, where "Φ_e^G is not a DNC$_2$ function" is a shorthand for $\exists x \Phi_e^G(x)\uparrow \vee \exists x \Phi_e^G(x)\downarrow = \Phi_x(x)$. We shall assume as usual that every Turing functional is $\{0,1\}$-valued.

Lemma 5.1.4. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing Φ_e^G not to be a DNC$_2$ function. ⋆

Proof. Fix a condition σ. Consider the following set[3]

3: Notice that this set is the same as in Lemma 3.2.2.

$$U = \{(x, v) \in \mathbb{N} \times 2 : \exists \tau \geq \sigma \ \Phi_e^\tau(x)\downarrow = v\}$$

Note that the set U is Σ_1^0. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. Let $\tau \geq \sigma$ witness $(x, \Phi_x(x)) \in U$, that is, let $\tau \geq \sigma$ be such that $\Phi_e^\tau(x)\downarrow = \Phi_x(x)$. Then τ forces Φ_e^G not to be a DNC$_2$ function.
- Case 2: $(x,0),(x,1) \notin U$ for some $x \in \mathbb{N}$. We claim that σ already forces $\Phi_e^G(x)\uparrow$.[4] Indeed, if for some $Z \in [\sigma]$, $\Phi_e^Z(x)\downarrow$, then by the use property, there is some $\tau \preceq Z$ such that $\Phi_e^\tau(x)\downarrow$, and by choosing τ long enough, it would witness $(x, v) \in U$ for $v = \Phi_e^\tau(x)$, contradiction.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_1^0 graph of a $\{0,1\}$-valued DNC function. This contradicts the fact that the degree $\mathbf{0}$ is not PA. ■

4: Note that we exploit the assumption that the functionals are $\{0,1\}$-valued to force divergence. Indeed, the contradiction comes from the fact that $v \in \{0,1\}$.

We are now ready to prove Theorem 5.1.3. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions τ forcing Φ_e^G not to be a DNC$_2$ function. It follows from Lemma 5.1.4 that every $\mathcal{D}_e$ is dense, hence for every $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic set G, G is not of PA degree. ■

Exercise 5.1.5. Adapt the proof of Theorem 3.2.4 to show that for any set A, there exists a set G such that $G' \geq_T A$ and G is not of PA degree. ⋆

On the other hand, one cannot adapt the proof of Theorem 3.2.6 to show that WKL admits PA avoidance. Indeed, the class of $\{0,1\}$-valued DNC functions is Π_1^0.

Exercise 5.1.6. Try to adapt the proof of Theorem 3.2.6 to show that any non-empty Π_1^0 class admits a member of non-PA degree. Identify the point of failure. ⋆

The main structural difference between the cone avoidance proof of Theorem 3.2.1 and the PA avoidance proof of Theorem 5.1.3 is in Case 2: Assuming the forcing question gives a negative answer independently to $p \operatorname{?\vdash} \Phi_e^G(x)\downarrow = 0$ and $p \operatorname{?\vdash} \Phi_e^G(x)\downarrow = 1$, we use the existence of a single extension (which in the proof of Theorem 5.1.3 is p itself) forcing simultaneously $\neg(\Phi_e^G(x)\downarrow = 0)$ and $\neg(\Phi_e^G(x)\downarrow = 1)$. Assuming the functional is $\{0,1\}$-valued, then the extension forces divergence. This ability to give a single extension witnessing simultaneously two independent negative answers is the core feature of PA avoidance.

Definition 5.1.7. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*merging* if for every $p \in \mathbb{P}$ and every pair of Γ-formulas $\varphi_0(G), \varphi_1(G)$, if $p \ {?}{\vdash} \ \varphi_0(G)$ and $p \ {?}{\vdash} \ \varphi_1(G)$ both hold, then there is an extension $q \leq p$ forcing $\varphi_0(G) \wedge \varphi_1(G)$. ◊

Note that a forcing question for Σ^0_n formulas induces a forcing question for Π^0_n formulas by considering the complement. Thus, by extension, we say that a forcing question for Σ^0_n formulas is Π^0_n-merging if, whenever $p \ {?}{\nvdash} \ \varphi_0(G)$ and $p \ {?}{\nvdash} \ \varphi_1(G)$, there is an extension forcing $\neg\varphi_0(G) \wedge \neg\varphi_1(G)$.

Remark 5.1.8. In Figure 3.1, the forcing questions at the left-most position are Σ^0_1-merging, and the ones at the right-most position are Π^0_1-merging. We shall see examples of Π^0_1 forcing questions at intermediary positions. ★

We have the necessary ingredients to prove our abstract theorem on PA avoidance.

Theorem 5.1.9
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving Π^0_1-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is not of PA degree.

Proof. It suffices to prove the following lemma:

Lemma 5.1.10. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing Φ^G_e not to be a DNC_2 function. ★

Proof. Consider the following set

$$U = \{(x, v) \in \mathbb{N} \times 2 : p \ {?}{\vdash} \ \Phi^G_e(x)\downarrow = v\}$$

Since the forcing question is Σ^0_1-preserving, the set U is Σ^0_1. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi^G_e(x)\downarrow = \Phi_x(x)$.
- Case 2: $(x, 0), (x, 1) \notin U$ for some $x \in \mathbb{N}$. Since the forcing question is Π^0_1-merging, there is an extension $q \leq p$ forcing $\neg(\Phi^G_e(x)\downarrow = 0) \wedge \neg(\Phi^G_e(x)\downarrow = 1)$, hence forcing Φ^G_e not to be a DNC_2 function.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_1 graph of a $\{0, 1\}$-valued DNC function. This contradicts the fact that **0** is not PA. ■

We are now ready to prove Theorem 5.1.9. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing Φ^G_e not to be a DNC_2 function. It follows from Lemma 5.1.10 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathcal{F}}$ is not of PA degree. This completes the proof of Theorem 5.1.9. ■

5.2 Weak merging

In some cases, such as with disjunctive notions of forcing with Σ^0_1-preserving disjunctive forcing questions, the forcing question is not Π^0_1-merging simply because given a pair of Π^0_1 formulas $\varphi_0(G)$ and $\varphi_1(G)$ the extension might force $\varphi_0(G_0)$ on the left side, and $\varphi_1(G_1)$ on the right side. If however one considers three Π^0_1 formulas, by the pigeonhole principle, two of them must be forced on the same side. We will later consider tree-like notions of forcing whose number of disjunctive clauses might increase over extension, thus requiring a larger number of formulas to find an extension forcing two of them simultaneously. This motivates the following definition.

Definition 5.2.1. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *weakly* Γ*-merging*[5] if for every $p \in \mathbb{P}$, there is some $k \in \mathbb{N}$ such that for every k-tuple of Γ-formulas $\varphi_0(G), \dots, \varphi_{k-1}(G)$, if $p \operatorname{?\vdash} \varphi_i(G)$ for each $i < k$, then there is an extension $q \leq p$ and two indices $i < j < k$ such that q forces $\varphi_i(G) \wedge \varphi_j(G)$. ◊

5: Note that in the definition of a weakly Γ-merging forcing question, the parameter k might depend on the condition p.

The following exercise shows that the forcing question of the Dzhafarov-Jockusch theorem is weakly Π^0_1-merging, with the appropriate adaptation to disjunctive forcing notions.

Exercise 5.2.2. Consider the question of forcing of Exercise 3.4.10. Let $\{\varphi^j_0(G), \varphi^j_1(G) : j < 3\}$ be a family of Σ^0_1 formulas. Show that if for each $j < 3$, $p \operatorname{?\nvdash} \varphi^j_0(G_0) \vee \varphi^j_1(G_1)$, then there is an extension $q \leq p$, a side $i < 2$ and two indices $a < b < 3$ such that q forces $\neg\varphi^a_i(G_i) \wedge \neg\varphi^b_i(G_i)$. ⋆

As for every avoidance or preservation notion, the key diagonalization lemma is based on a 3-case analysis. The first case says that the Turing functional outputs some erroneous description of an object, while the second case ensures that the Turing functional is partial. The two first cases are not mutually exclusive. The third case, which consists of the negation of Case 1 and Case 2, cannot happen, because otherwise, there will be an effective description of some uncomputable object. For cone avoidance, preservation of 1 hyperimmunity, or preservation of 1 non-Σ^0_1 definition, the third case was trivial. Working with weakly merging forcing questions yields the first non-trivial case analysis. Let us first introduce some terminology.

A *valuation*[6] is a partial $\{0,1\}$-valued function $h \subseteq \mathbb{N} \to 2$. A valuation is finite if it has finite support, that is, $\operatorname{dom} h$ is finite. A valuation h is *correct* if for every $n \in \operatorname{dom} h$, $\Phi_n(n)\downarrow \neq h(n)$. Two valuations f and h are *compatible* if for every $n \in \operatorname{dom} f \cap \operatorname{dom} h$, $f(n) = h(n)$.

6: The idea is the following: We considered so far only valuations with a singleton domain, thus there were at most 2 incompatible such valuations. Considering valuations with finite domain is a way to obtain more pairwise incompatible valuations.

Lemma 5.2.3 (Liu [12]). Let U be a c.e. set of finite valuations. Either U contains a correct valuation, or for every $k \in \mathbb{N}$, there are k pairwise incompatible finite valuations outside of U. ⋆

Proof. Suppose U contains no correct valuation, otherwise we are done. Let S be the set of finite sets $F \subseteq \mathbb{N}$ such that for each $n \notin F$, either $\Phi_n(n)\downarrow$, or there is a valuation $h \in U$ such that $F \cup \{n\} \subseteq \operatorname{dom} h$ and for every $m \in \operatorname{dom} h \setminus (F \cup \{n\})$, $\Phi_m(m)\downarrow \neq h(m)$. Note that if $F \notin S$, this is witnessed by some $n \notin F$.

Claim 1: $\emptyset \notin S$. Indeed, otherwise, for each $n \in \mathbb{N}$, one of the two Σ^0_1 cases holds:

1. $\Phi_n(n)\downarrow$;
2. there is a finite valuation $h \in U$ such that $n \in \operatorname{dom} h$ and for every $m \neq n$, $\Phi_m(m)\downarrow \neq h(m)$.

Then one can compute a $\{0,1\}$-valued DNC function by waiting on input n for either case to occur. Then output $1-\Phi_n(n)$ in the former case, and $1-h(n)$ in the latter case. Since U contains no correct valuation, $h(n)=\Phi_n(n)$.

Claim 2: For any set $F \notin S$ and w witnessing this fact, $F \cup \{w\} \notin S$. Indeed, otherwise, for each $n \notin F \cup \{w\}$, one of the two Σ^0_1 cases holds:

1. $\Phi_n(n)\downarrow$;
2. there is a finite valuation $h \in U$ such that $F \cup \{w, n\} \subseteq \operatorname{dom} h$ and for every $m \notin F \cup \{w, n\}$, $\Phi_m(m)\downarrow \neq h(m)$.

Here again, one can compute a $\{0,1\}$-valued DNC function by hardcoding the appropriate values on $F \cup \{w\}$, and for any $n \notin F \cup \{w\}$, waiting for either case to occur. In the first case, output $1-\Phi_m(m)$, and in the second case, output $1-h(n)$. We cannot have $\Phi_n(n)\downarrow \neq h(n)$, otherwise h would be a counter-example to the fact that w is a witness of $F \notin S$.

Using Claim 1 and Claim 2, one can define for any k an infinite sequence $n_0, n_1, \dots$ such that for any $i \in \mathbb{N}$, n_i witnesses that $\{n_j : j < i\} \notin S$. There are 2^{i+1} many pairwise incompatible valuations with domain $\{n_j : j \leq i\}$, and none of them can be in U, as it would contradict the fact that n_i is a witness of $\{n_j : j < i\} \notin S$. ∎

We can prove the following abstract PA avoidance theorem using Liu's lemma. [12]

Theorem 5.2.4
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving weakly Π^0_1-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is not of PA degree.

Proof. It suffices to prove the following diagonalization lemma.

Lemma 5.2.5. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing Φ^G_e not to be a DNC_2 function. ⋆

Proof. Let $k \in \mathbb{N}$ witness that the forcing question is weakly Π^0_1-merging for p. Consider the following set

$$U = \{h \text{ finite valuation } : p \operatorname{?\vdash} \Phi^G_e \text{ is incompatible with } h\}$$

Note that being incompatible is a Σ^0_1 statement, so since the forcing question is Σ^0_1-preserving, the set U is Σ^0_1. There are three cases:

- Case 1: U contains a correct valuation h. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing Φ^G_e to be incompatible with h. In particular, q forces Φ^G_e not to be a DNC_2 function.
- Case 2: there are k pairwise incompatible finite valuations $h_0, \dots, h_{k-1}$ outside of U. Since the forcing question is Π^0_1-merging, there is an extension $q \leq p$ and two indices $a < b < k$ such that q forces Φ^G_e to be compatible simultaneously with h_a and h_b. Since h_a and h_b are incompatible, then q forces Φ^G_e to be partial.
- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.2.3. ∎

We are now ready to prove Theorem 5.2.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing Φ^G_e not to be a DNC_2 function. It follows from

Lemma 5.2.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathcal{F}}$ is not of PA degree. This completes the proof of Theorem 5.2.4. ■

5.3 Ramsey-type WKL

Both the original proof and the modern proof of Seetapun's theorem involve Π^0_1 classes of instances of RT^1_2, and thus make use of compactness. It is natural to ask whether this use is necessary. Liu's theorem states that Ramsey's theorem for pairs admits PA avoidance. However, PA avoidance only means that full compactness is not needed, but does not rule out the presence of some weak form of compactness. As it turns out, Ramsey's theorem for pairs implies a weak form of compactness called the Ramsey-type weak König's lemma (RWKL). Informally, RWKL states that for every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists some infinite set H which is homogeneous for one of the members $X \in \mathcal{P}$ seen as an instance of RT^1_2. However, the exact formulation requires more technicality not to imply the existence of X.

Definition 5.3.1. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. A finite set $F \subseteq \mathbb{N}$ is *homogeneous* for T if $\{\sigma \in T : (\forall x \in F)\sigma(x) = i\}$ is infinite for some $i < 2$. An infinite set $H \subseteq \mathbb{N}$ is *homogeneous* for T if every finite subset of it is homogeneous for T. ◊

By extension, we say that an infinite set H is homogeneous for a Π^0_1 class $\mathcal{P}$ if it is homogeneous for a tree T such that $\mathcal{P} = [T]$. The Ramsey-type weak König's lemma (RWKL)[7] is the statement "Every infinite binary tree admits an infinite homogeneous set."

7: The statement was originally introduced by Flood [30] under the name Ramsey-type König's lemma (RKL). It was later renamed for consistency.

Proposition 5.3.2 (Flood [30]). RT^2_2 implies RWKL over RCA_0. ★

Proof. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. Define $f : [\mathbb{N}]^2 \to 2$ by $f(x, y) = \sigma_y(x)$, where σ_y is the left-most element of T of length y. Any infinite homogeneous set for f is homogeneous for T. ■

The remainder of this section is devoted to the proof that RWKL admits PA avoidance, hence is strictly weaker than WKL_0.[8]

8: There exists an alternative simpler proof [31] of this theorem which exploits the fact that the class of $\{0, 1\}$-valued DNC functions is Π^0_1 and not simply closed in Cantor space. The proof given in this book, although more complex, is morally the "true" proof, in that its combinatorics extend to stronger theorems, such as Liu [32].

Theorem 5.3.3 (Liu [12])
Let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be a non-empty Π^0_1 class. There is an infinite homogeneous set H for $\mathcal{P}$ of non-PA degree.

Proof. Let $\mathbb{P}$ be the notion of forcing whose conditions are tuples $(k, \vec{\sigma}, \mathcal{A})$ where

1. $k \in \mathbb{N}$ is the number of parts ;
2. $\vec{\sigma} = \langle \sigma_0, \dots, \sigma_{k-1} \rangle$ is a k-tuple of binary strings ;
3. $\mathcal{A} \subseteq k^{\omega}$ is a non-empty Π^0_1 class of k-partitions.

One can see a condition $p = (k, \vec{\sigma}, \mathcal{A})$ as a k-tuple of families of Mathias preconditions[9] $(\sigma_i, X^{-1}(i) \setminus \{0, \dots, |\sigma|\})$ for any $X \in \mathcal{A}$. We say that *part i of p is acceptable* if there exists some $X \in \mathcal{A}$ such that $X^{-1}(i)$ is infinite.

9: A *Mathias precondition* is a pair (σ, X) such that $\forall x \in X\ x > |\sigma|$, but X might be finite or empty.

The intended initial condition is $(2, \langle \emptyset, \emptyset \rangle, \mathscr{P})$. The *interpretation* of a condition $(k, \vec{\sigma}, \mathscr{A})$ is

$$[k, \vec{\sigma}, \mathscr{A}] = \{(G_0, \dots, G_{k-1}) : \exists X \in \mathscr{A}\ \forall i < k\ \sigma_i \subseteq G_i \subseteq \sigma_i \cup X^{-1}(i)\}$$

A condition $q = (\ell, \vec{\tau}, \mathscr{B})$ *extends* $p = (k, \vec{\sigma}, \mathscr{A})$ if $\ell \geq k$ and there is a map[10] $f : \ell \to k$ such that for every $Y \in \mathscr{B}$, there is some $X \in \mathscr{A}$ such that for every $i < \ell$, $(\tau_i, Y^{-1}(i))$ Mathias extends $(\sigma_i, X^{-1}(i))$, that is, $Y^{-1}(i) \subseteq X^{-1}(i)$ and $\sigma_i \subseteq \tau_i \subseteq \sigma_i \cup X^{-1}(i)$. We say that *part i of q refines part $f(i)$ of p*.

10: Over extension, some parts of a condition might be splitting. The map keeps track of which part refines which one. This map may not be unique, but it does not matter.

Given a condition $p = (k, \vec{\sigma}, \mathscr{A})$, we shall construct actually only two kinds of extensions:

- A condition $q = (\ell, \vec{\tau}, \mathscr{B})$ is a *part i extension* of p if $\ell = k$, the extension map f is the identity function, and $\tau_j = \sigma_j$ for all $j \neq i$.
- A condition $q = (\ell, \vec{\tau}, \mathscr{B})$ is a *splitting extension* of p if, letting f be the map witnessing the extension, for every $i < \ell$, $\tau_i = \sigma_{f(i)}$.

Given a condition $p = (k, \vec{\sigma}, \mathscr{A})$, and some Turing index e, let $I_e(p) \subseteq k$ be the set of acceptable parts i of p which do not already force Φ_e^G not to be a DNC_2 function.

Lemma 5.3.4. For every condition $p = (k, \vec{\sigma}, \mathscr{A})$ and every Turing index e such that $I_e(p) \neq \emptyset$, there is an extension $q \leq p$ such that $I_e(q) \subsetneq I_e(p)$. ⋆

Proof. We will either find a part i extension $q \leq p$ for some $i \in I_e(p)$ such that q which will force Φ_e^G not to be a DNC_2 function on part i, in which case $I_e(q) = I_e(p) \setminus \{i\}$, or a splitting extension forcing Φ_e^G not to be a DNC_2 function on every part, in which case $I_e(q) = \emptyset$.

Recall the notion of valuation from Theorem 5.2.4. Consider the following set:[11]

11: The set U plays the same role as in Lemma 5.2.5.

$$U = \left\{ h \text{ finite valuation } : \begin{array}{l} \forall X \in \mathscr{A}\ \exists i \in I_e(p)\ \exists \rho \subseteq X^{-1}(i) \\ \Phi_e^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

Note that by effective compactness, letting $T \subseteq k^{<\mathbb{N}}$ be a computable tree such that $[T] = \mathscr{A}$, the set U can equivalently be defined as

$$U = \left\{ h \text{ finite valuation } : \begin{array}{l} \exists n \forall \tau \in T \cap k^n\ \exists i \in I_e(p)\ \exists \rho \subseteq \tau^{-1}(i) \\ \Phi_e^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

Thus, the set U is Σ^0_1. There are three cases.

- Case 1: U contains a correct valuation h. Fix some $X \in \mathscr{A}$, and let $i \in I_e(p)$ and $\rho \subseteq X^{-1}(i)$ be such that $\Phi_e^{\sigma_i \cup \rho}$ is incompatible with h. Letting $\mathscr{B} = \{Y \in \mathscr{A} : \rho \subseteq Y^{-1}(i)\}$, $\tau_i = \sigma_i \cup \rho$ and $\tau_j = \sigma_j$ otherwise, the condition $(k, \vec{\tau}, \mathscr{B})$ is a part i extension of p forcing Φ_e^G to be incompatible with h on part i, hence forcing Φ_e^G not to be a DNC_2 function on part i.
- Case 2: there are $k+1$ pairwise incompatible finite valuations $h_0, \dots, h_k$ outside of U. For each $s \leq k$, let $\mathscr{B}_s \subseteq k^{\mathbb{N}}$ be the Π^0_1 class of all $X \in \mathscr{A}$ such that for every $i \in I_e(p)$ and every $\rho \subseteq X^{-1}(i)$, $\Phi_e^{\sigma_i \cup \rho}$ is compatible with h_s. By assumption, $\mathscr{B}_s \neq \emptyset$ for every $s \leq k$. We say that $Y \in (k^{k+1})^\omega$ is the *refined partition* of $(X_0, \dots, X_k) \in \mathscr{B}_0 \times \cdots \times \mathscr{B}_k$ if for every $\nu < k^{k+1}$ interpreted as a k-ary string of length $k+1$, $Y^{-1}(\nu) = \bigcap_{s \leq k} X_s^{-1}(\nu(s))$. Let $\mathscr{B} \subseteq (k^{k+1})^\omega$ be the class of all refined partitions

of members of $\mathcal{B}_0 \times \cdots \times \mathcal{B}_k$. By the pigeonhole principle, for every $\nu \in k^{k+1}$, there is some $i_\nu \in k$ and some $s < t \leq k$ such that $\nu(s) = \nu(t) = i_\nu$. Let $f : k^{k+1} \to k$ be the defined by $f(\nu) = i_\nu$. For each $\nu \in k^{k+1}$, let $\tau_\nu = \sigma_{f(\nu)}$. The condition $q = (k^{k+1}, \vec{\tau}, \mathcal{B})$ is a splitting extension of p. Moreover, every part ν of q refining some part $i \in I_e(p)$ of p forces Φ_e^G to be compatible with h_s and h_t, for $s < t \leq k$ such that $\nu(s) = \nu(t) = f(\nu)$. Since h_s and h_t are incompatible, such part ν of q forces Φ_e^G to be partial, hence $\nu \notin I_e(q)$. Last, if part ν of q refines some part $i \notin I_e(p)$ of p, then $\nu \notin I_e(q)$, so $I_e(q) = \emptyset$.

- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.2.3. ■

Consider an infinite, sufficiently generic decreasing sequence of conditions $p_0 \geq p_1 \geq \dots$ with $p_s = (k_s, \vec{\sigma}_s, \mathcal{A}_s)$, together with the refinement maps $f_s : k_{s+1} \to k_s$ witnessing the extensions. Note that each condition has an acceptable part, and if part i of p_{s+1} is acceptable, then so is part $f_s(i)$ of p_s. Thus, by König's lemma, there exists a sequence $P \in \omega^\omega$ such that for every s, part $P(s)$ of p_s is acceptable, and part $P(s+1)$ of p_{s+1} refines part $P(s)$ of p_s, that is, $f_s(P(s+1)) = P(s)$. This induces a set G_P defined by $G = \bigcup_s \sigma_{s,P(s)}$. By genericity of the sequence, G_P is infinite. Moreover, by Lemma 5.3.4, G_P is not of PA degree. This completes the proof of Theorem 5.3.3. ■

5.4 Liu's theorem

Liu's theorem states that Ramsey's theorem for pairs admits PA avoidance. Recall that the modern proof of Seetapun's theorem (Theorem 3.4.11) was divided into a proof of cone avoidance of COH and a proof of strong cone avoidance of RT^1_2. The proof of Liu's theorem follows the same structure.

Recall that an infinite set C is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$. The cohesiveness principle (COH) is the problem whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets.

Exercise 5.4.1. Combine Exercise 3.4.3 and Exercise 5.1.5 to prove that COH admits PA avoidance. ★

Exercise 5.4.2. Recall the notion of computable Mathias forcing from Exercise 3.2.8. Given a condition (σ, X) and a Σ^0_1 formula $\varphi(G)$, let $(\sigma, X) \mathrel{?\vdash} \varphi(G)$ hold if there is some $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds.

1. Show that this is a Σ^0_1-preserving, Π^0_1-merging forcing question.
2. Deduce that COH admits PA avoidance. ★

Our last step consists in proving that RT^1_2 admits strong PA avoidance.[12]

12: The original proof of Liu's theorem was also using the decomposition into COH and RT^1_2. However, it directly proved that RT^1_2 admits strong PA avoidance without using PA avoidance of RWKL. Proving first PA avoidance of RWKL enables to reduce the complexity of each forcing, by separating the compactness from the disjunction issues.

Theorem 5.4.3 (Liu [12])
For every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ of non-PA degree.[13]

13: From many viewpoints, the proof of this theorem will be similar to the proof of Theorem 3.4.6. It is strongly advised to have a good understanding of the latter proof.

Proof. Fix A. As in Theorem 3.4.6, we shall build two sets G_0, G_1 simultaneously, with $G_0 \subseteq A$ and $G_1 \subseteq \overline{A}$. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

The two sets will be constructed through a variant of Mathias forcing, whose *conditions* are triples (σ_0, σ_1, X) where

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_i \subseteq A_i$;
3. X is not of PA degree[14].

14: This is the only difference with the notion of forcing of Theorem 3.4.6.

The *interpretation* $[\sigma_0, \sigma_1, X]$ of a condition (σ_0, σ_1, X) is the class

$$[\sigma_0, \sigma_1, X] = \{(G_0, G_1) : \forall i < 2\ \sigma_i \preceq G_i \subseteq \sigma_i \cup X\}$$

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X) for each $i < 2$. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$. Note that $(G_{\mathcal{F},0}, G_{\mathcal{F},1}) \in \bigcap\{[\sigma_0, \sigma_1, X] : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$.

The goal is therefore to build two infinite sets G_0, G_1, satisfying the following requirements for every $e_0, e_1 \in \mathbb{N}$:

$$\mathcal{R}_{e_0,e_1} : \Phi_{e_0}^{G_0} \text{ is not DNC}_2 \vee \Phi_{e_1}^{G_1} \text{ is not DNC}_2$$

If every requirement is satisfied, then a pairing argument shows that either G_0, or G_1 is not of PA degree. We make the following assumption:

$$\text{There is no infinite set } H \subseteq A \text{ or } H \subseteq \overline{A} \text{ of non-PA degree.} \tag{H1}$$

Under this assumption, one can prove that if $\mathcal{F}$ is sufficiently generic, then both $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ are infinite.

Lemma 5.4.4. Suppose (H1). Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$. There is an extension (τ_0, τ_1, Y) of p and some $n > |\sigma_i|$ such that $n \in \tau_i$. ⋆

Proof. If $X \cap A^i$ is empty, then $X \subseteq A^{1-i}$, but X is of non-PA degree, which contradicts (H1). Thus, there is $n \in X \cap A^i$. Let $\tau_i = \sigma_i \cup \{n\}$, and $\tau_{1-i} = \sigma_{1-i}$. Then, $(\tau_0, \tau_1, X \setminus \{0, \dots, n-1\})$ is an extension of p such that $n \in \tau_i$. ■

We will now prove the core lemma.

Lemma 5.4.5. Let $p = (\sigma_0, \sigma_1, X)$ be a condition, and $e_0, e_1 \in \mathbb{N}$. There is an extension (τ_0, τ_1, Y) of p forcing $\mathcal{R}_{e_0,e_1}$. ⋆

15: The set U is a combination of the forcing question of Theorem 3.4.6, but working with valuations due to the disjunctive nature of the forcing question.

Proof. Consider the following set[15]

$$U = \left\{ h \text{ finite valuation } : \begin{array}{l} \forall Z_0 \sqcup Z_1 = X\ \exists i < 2\ \exists \rho \subseteq Z_i \\ \Phi_{e_i}^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

Here again, the previous set is $\Sigma^0_1(X)$, as it can be equivalently defined as

$$\left\{ h \text{ finite valuation } : \begin{array}{l} \exists \ell \in \mathbb{N}\forall Z_0 \sqcup Z_1 = X \upharpoonright_\ell \exists i < 2\ \exists \rho \subseteq Z_i \\ \Phi_{e_i}^{\sigma_i \cup \rho} \text{ is incompatible with } h \end{array} \right\}$$

There are three cases:

- Case 1: U contains a correct valuation h. Letting $Z_0 = A_0 \cap X$ and $Z_1 = A_1 \cap X$, there is some $i < 2$ and some $\rho \subseteq Z_i$ such that $\Phi_{e_i}^{\sigma_i \cup \rho}$ is incompatible with h. Letting $\tau_i = \sigma_i \cup \rho$ and $\tau_{1-i} = \sigma_{1-i}$, the condition $(\tau_0, \tau_1, X \setminus \{0, \dots, \max \rho\})$ is an extension of p forcing $\Phi_{e_i}^{G_i}$ to be incompatible with h, hence not being a DNC_2 function.
- Case 2: there are 3 pairwise incompatible finite valuations h_0, h_1, h_2 outside of U. For each $s < 3$, let $\mathcal{P}_s \subseteq 2^{\mathbb{N}}$ be the Π^0_1 class of all Y_s such that, letting $Y_{s,0} = Y_s$ and $Y_{s,1} = \overline{Y}_s$, for every $i < 2$ and every $\rho \subseteq Y_{s,i} \cap X$, $\Phi_{e_i}^{\sigma_i \cup \rho}$ is compatible with h_s. By assumption, $\mathcal{P}_s \neq \emptyset$ for every $s < 3$. Since RWKL admits PA avoidance (Theorem 5.3.3), there is a decreasing sequence of sets $X \supseteq Y_0 \supseteq Y_1 \supseteq Y_2$ such that Y_s is homogeneous for $\mathcal{P}_s$ for some color $i_s < 2$, and $Y_2 \oplus Y_1 \oplus Y_0 \oplus X$ is not of PA degree. By the pigeonhole principle, there exist some $s < t < 3$ and some color $i < 2$ such that $i = i_s = i_t$. The condition $(\sigma_0, \sigma_1, Y_2)$ is an extension of p forcing $\Phi_{e_i}^{G_i}$ to be compatible with h_s and h_t, hence forcing $\Phi_{e_i}^{G_i}$ to be partial.
- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.2.3. ■

As in the proof of strong cone avoidance, we are getting a Π^0_1 class of instances of RT^1_2. In the proof of strong cone avoidance, we simply picked a member of this class using the cone avoidance basis theorem. Here, since we need to avoid PA degrees, we cannot pick a member, so we use RWKL instead of WKL. The true complexity of this construction is hidden in the proof that RWKL admits PA avoidance.

We are now ready to prove Theorem 5.4.3. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing, and for each $i < 2$, let $G_i = G_{\mathcal{F},i}$. By Lemma 5.4.4, both sets are infinite. Moreover, by Lemma 5.4.5, either G_0 or G_1 is not of PA degree. Letting H be this set, it satisfies the statement of Theorem 5.4.3. ■

We can now prove Liu's theorem by combining PA avoidance of COH and strong PA avoidance of RT^1_2.

Theorem 5.4.6 (Liu [12])
Every computable coloring $f : [\mathbb{N}]^2 \to 2$ has an infinite f-homogeneous set of non-PA degree.

Proof. The proof follows the one of Theorem 3.4.1. Fix f. Let $\vec{R} = R_0, R_1, \dots$ be the computable sequence of sets defined for every $x \in \mathbb{N}$ by $R_x = \{y \in \mathbb{N} : f(x,y) = 1\}$. By Exercise 5.4.1, there is an infinite $\vec{R}$-cohesive set $X \subseteq \mathbb{N}$ of non-PA degree. In particular, for every $x \in X$, $\lim_{y \in X} f(x,y)$ exists. Let $\hat{f} : X \to 2$ be the limit coloring of f, that is, $\hat{f}(x) = \lim_{y \in X} f(x,y)$. By Theorem 5.4.3, there is an infinite $\hat{f}$-homogeneous set $Y \subseteq X$ for some color $i < 2$ such that $Y \oplus X$ is of non-PA degree. Since for every $x \in Y$, $\lim_{y \in Y} f(x,y) = i$, one can thin out the set Y to obtain an infinite f-homogeneous subset $H \subseteq Y$. ■

5.5 Randomness

Algorithmic randomness is a sub-field of computability theory studying the amount of randomness contained in binary sequences taken individually. Contrary to the notion of effective computability which admits a robust mathematical definition, randomness does not translate mathematically to a single notion, but to a hierarchy of concepts. Nonetheless, randomness admits its own form of robustness, by having many different characterizations based on multiple paradigms. See Downey and Hirschfeldt [33] or Nies [34] for an introduction on algorithmic randomness.

Among the notions of randomness, *Martin-Löf randomness* is widely considered as capturing the intuitive idea of a random sequence.[16] It can be equivalently defined using multiple paradigms:

16: This is known as the Martin-Löf-Chaitin thesis, and plays the same role as the Church-Turing thesis for computability.

- *Incompressibility*: There should be no recognizable pattern in the sequence, which would yield a possibility to compress the sequence. This approach due to Chaitin is based on Kolmogorov complexity.
- *Unpredicability*: One should not be able to predict the the bits of the sequence. This approach is formalized using martingales.
- *Measure*: Random sequences should not satisfy any "rare" properties which can be effectively described.

Kolmogorov complexity is probably the shortest way to define Martin-Löf randomness. A *prefix-free machine* is a partial computable function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ whose domain is prefix-free, that is, if $\sigma, \tau \in \operatorname{dom} M$ with $\sigma \neq \tau$, then they are incomparable. A prefix-free machine M is *universal*[17] if for every prefix-free machine N, there is some $\rho \in 2^{<\mathbb{N}}$ such that $(\forall \sigma \in 2^{<\mathbb{N}}) M(\rho\sigma) = N(\sigma)$.

17: The proof of the existence of a universal prefix-free machine goes as follows: Prove the existence of a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$, $\Phi_f(e)$ is prefix-free and if Φ_e is prefix-free, then $\Phi_{f(e)} = \Phi_e$. Then, let

$$M(1^e 0\sigma) = \Phi_{f(e)}(\sigma)$$

Definition 5.5.1. Fix a universal prefix-free machine M. The *Kolmogorov complexity* $K_M(\sigma)$ of a string $\sigma \in 2^{<\mathbb{N}}$ is the length of the shortest string $\tau \in 2^{<\mathbb{N}}$ such that $M(\tau) = \sigma$. ◊

The Kolmogorov complexity of a string depends on the choice of a universal prefix-free machine. Given another universal prefix-free machine N, $(\forall \sigma \in 2^{<\mathbb{N}}) K_N(\sigma) = K_M(\sigma) + \mathcal{O}(1)$. Kolmogorov complexity is therefore an asymptotic notion of complexity. From now on, we omit the subscript M and work with inequalities to additive constant, noted $\leq^+$.

Exercise 5.5.2. Show that for every $\sigma \in 2^{<\mathbb{N}}$, $K(\sigma) \leq^+ |\sigma| + 2\log_2(|\sigma|)$. ★

Definition 5.5.3 (Chaitin [35] and Levin [36]). A set $X \in 2^{\mathbb{N}}$ is *Martin-Löf random*[18] if for every $n \in \mathbb{N}$, $K(X\restriction_n) \geq^+ n$. ◊

18: This definition is independently due to Chaitin and Levin, but coincides with the notion of Martin-Löf randomness based of measure.

The *Lebesgue measure* on Cantor space $2^{\mathbb{N}}$ is the measure μ induced by letting $\mu([\sigma]) = 2^{-|\sigma|}$ for every $\sigma \in 2^{<\mathbb{N}}$. In particular, every open class $\mathcal{U} \subseteq 2^{\mathbb{N}}$ being of the form $\bigcup_{\sigma \in W}[\sigma]$ for some prefix-free set $W \subseteq 2^{<\mathbb{N}}$, $\mu(\mathcal{U}) = \sum_{\sigma \in W}[\sigma]$. It follows that the Lebesgue measure of a closed class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is $1 - \mu(2^{\mathbb{N}} \setminus \mathcal{P})$. In the case of closed classes, one can give a more direct definition in terms of trees:

Exercise 5.5.4. The *measure* of a tree $T \subseteq 2^{<\mathbb{N}}$ is defined as

$$\mu(T) = \lim_n \frac{\operatorname{card}\{\sigma \in T : |\sigma| = n\}}{2^n}$$

Show that $\mu(T) = \mu([T])$. ★

The following exercise shows the existence of a Π^0_1 class of positive measure containing only (but not all) Martin-Löf random sets.

Exercise 5.5.5. Fix a universal prefix-free machine M. For every $c \geq 1$, let $\mathcal{U}_c$ be the Σ^0_1 class $\{X : \exists n K_M(X\restriction_n) < n - c\}$ and let $V_c \subseteq 2^{<\mathbb{N}}$ be a prefix-free set of strings such that $[\![V_c]\!] = \mathcal{U}_c$ and such that for every $\sigma \in V_c$, $K_M(\sigma) < |\sigma| - c$.

1. Show that $\sum_{\sigma \in V_c} 2^{-|\sigma|+c} \leq \sum_{\sigma \in V_c} 2^{-K_M(\sigma)} \leq 1$. [19]
2. Deduce that $\mu(\mathcal{U}_c) \leq 2^{-c}$, hence that the Π^0_1 class $2^{\mathbb{N}} \setminus \mathcal{U}_c$ has positive measure.[20] ★

19: For every prefix-free machine M and every set of strings $S \subseteq 2^{<\mathbb{N}}$,

$$\sum_{\sigma \in S} 2^{-K_M(\sigma)} \leq 1$$

20: If $V \subseteq 2^{<\mathbb{N}}$ is prefix-free, then

$$\mu([\![V]\!]) = \sum_{\sigma \in V} 2^{-|\sigma|}$$

Given a measurable class $\mathscr{C}$ and a cylinder $[\sigma]$, we write $\mu(\mathscr{C}|[\sigma]) = \frac{\mu(\mathscr{C} \cap [\sigma])}{\mu([\sigma])}$ for the measure of $\mathscr{C}$ relative to $[\sigma]$. The Lebesgue measure satisfies the following theorem which happens to be a very powerful tool for the computability-theoretic study of measure:

Theorem 5.5.6 (Lebesgue density)
Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a measurable class of positive measure. For almost every $X \in \mathscr{C}$, $\lim_n \mu(\mathscr{C}|[X{\upharpoonright}_n]) = 1$.

It follows from Lebesgue density theorem that for every $\epsilon > 0$, there is a cylinder $[\sigma]$ such that $\mu(\mathscr{C}|[\sigma]) > 1 - \epsilon$.

Weak weak König's lemma is the restriction of weak König's lemma to trees of positive measure, that is, the statement "Every infinite binary tree of positive measure admits an infinite path." WWKL_0 is RCA_0 augmented with weak weak König's lemma. By Exercise 5.5.5, there exists a Π^0_1 class of positive measure containing only Martin-Löf random sequences. Conversely, for every Π^0_1 class $\mathscr{P} \subseteq 2^{\mathbb{N}}$ of positive measure and every Martin-Löf random sequence Z, there exists a string $\sigma \in 2^{<\mathbb{N}}$ such that $\sigma \cdot Z \in \mathscr{P}$. Thus, WWKL_0 is equivalent to the statement "For every set X, there exists a Martin-Löf random sequence relative to X". For these reasons, WWKL_0 is considered as capturing probabilistic arguments.

Seeing WWKL_0 as a restriction of WKL_0 which itself captures compactness arguments, WWKL_0 can be seen as a weaker notion of compactness. We now prove that weak weak König's lemma admits PA avoidance using a forcing with closed classes of positive measure.[21]

21: Note that we prove a much stronger statement since the closed class is not assumed to be effectively closed. This actually corresponds to a proof that weak weak König's lemma admits strong PA avoidance.

Theorem 5.5.7
Every closed class $\mathscr{P} \subseteq 2^{\mathbb{N}}$ of positive measure admits a member of non-PA degree.

Proof. Consider the notion of forcing $\mathbb{P}$ whose conditions are closed classes $\mathbb{Q} \subseteq 2^{\mathbb{N}}$ of positive measure, partially ordered by inclusion. A condition is its self interpretation.

Lemma 5.5.8. For every condition $\mathbb{Q} \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\mathscr{R} \leq \mathbb{Q}$ forcing Φ^G_e not to be a DNC_2 function. ★

Proof. By Lebesgue density theorem (Theorem 5.5.6), there is some $\sigma \in 2^{<\mathbb{N}}$ such that $\mu(\mathbb{Q}|[\sigma]) > 0.9$. For every $x \in \mathbb{N}$ and $v < 2$, let $\mathcal{U}_{x,v} = \{X : \Phi^{\sigma \cdot X}_e(x){\downarrow} = v\}$. Consider the following set

$$U = \{(x, v) \in \mathbb{N} \times 2 : \mu(\mathcal{U}_{x,v}) > 0.2\}$$

Note that the classes $\mathcal{U}_{x,v}$ are uniformly Σ^0_1, so the set U is Σ^0_1. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x){\downarrow}$. By assumption, $\mu(\mathcal{U}_{x,\Phi_x(x)}) > 0.2$. Let $\mathscr{C} \subseteq \mathcal{U}_{x,\Phi_x(x)}$ be a clopen[22] subclass such that $\mu(\mathscr{C}) > 0.2$. Let $\mathbb{Q}_\sigma = \{X \in 2^{\mathbb{N}} : \sigma \cdot X \in \mathbb{Q}\}$. By choice of σ,

22: A class is *clopen* if it is both closed and open. Here, we use the fact that if $\bigcup_{\sigma \in W}[\sigma]$ is an open class, for every $\epsilon > 0$, there is a finite subset $F \subseteq W$ such that

$$\mu(\bigcup_{\sigma \in F}[\sigma]) > \mu(\bigcup_{\sigma \in W}[\sigma]) - \epsilon$$

$\mu(\mathbb{Q}_\sigma) > 0.9$, so $\mu(\mathbb{Q}_\sigma \cap \mathcal{C}) > 0.1$. Finally, let $\mathcal{R} = \{\sigma \cdot X : X \in \mathbb{Q}_\sigma \cap \mathcal{C}\}$. The class $\mathcal{R}$ is a closed subclass of $\mathbb{Q}$ such that $\mu(\mathcal{R}|[\sigma]) > 0.1$, thus $\mathcal{R}$ is a valid extension. Furthermore, $\mathcal{R}$ forces $\Phi_e^G(x)\downarrow = \Phi_x(x)$.

- Case 2: $(x,0),(x,1) \notin U$ for some $x \in \mathbb{N}$. By assumption, $\mu(\mathbb{Q}_{x,0}) \leq 0.2$ and $\mu(\mathbb{Q}_{x,1}) \leq 0.2$, so $\mu(\mathbb{Q}_{x,0} \cup \mathbb{Q}_{x,1}) \leq 0.4$. Let $\mathcal{R} = \{\sigma \cdot X \in \mathbb{Q} : X \notin \mathbb{Q}_{x,0} \cup \mathbb{Q}_{x,1}\}$. Since $\mu(\mathbb{Q}|[\sigma]) > 0.9$, then $\mu(\mathcal{R}|[\sigma]) > 0.5$). So $\mathcal{R}$ is a valid extension of $\mathbb{Q}$ forcing $\neg(\Phi_e^G(x)\downarrow = 0) \wedge \neg(\Phi_e^G(x)\downarrow = 1)$, hence forcing Φ_e^G not to be a DNC_2 function.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_1 graph of a $\{0,1\}$-valued DNC function. This contradicts the fact that $\mathbf{0}$ is not PA. ■

We are now ready to prove Theorem 5.5.7. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing Φ_e^G not to be a DNC_2 function. It follows from Lemma 5.5.8 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_\mathcal{F}$ is not of PA degree. This completes the proof of Theorem 5.5.7. ■

Exercise 5.5.9. Consider the notion of forcing of Theorem 5.5.7. Given a condition $\mathcal{P} \subseteq 2^\mathbb{N}$, a string $\sigma \in 2^{<\mathbb{N}}$ such that $\mu(\mathbb{Q}|[\sigma]) > 0.9$, and a Σ^0_1 formula $\varphi(G)$, let $\mathcal{P}\ ?\!\vdash \varphi(G)$ iff $\mu\{X : \varphi(\sigma \cdot X)\} > 0.2$.

1. Show that $\mathcal{C}\ ?\!\vdash \varphi(G)$ is a Σ^0_1-preserving, Π^0_1-merging forcing question.
2. Deduce that if C is a non-computable set and $\mathcal{P} \subseteq 2^\mathbb{N}$ is a closed class of positive measure, there is a member $G \in \mathcal{P}$ such that $C \not\leq_T G$. ★

5.6 Avoiding closed classes

The notion of PA avoidance is an avoidance of a particular closed class: the Π^0_1 class $\mathcal{P} \subseteq 2^\mathbb{N}$ of DNC_2 functions. This class has two particularities: First, it is effectively closed, hence can be represented by a computable tree. Second, it is *homogeneous*, that is, if one considers the pruned[23] tree $T \subseteq 2^{<\mathbb{N}}$ corresponding to $\mathcal{P}$, for every $\sigma, \tau \in T$ at the same level, the sub-trees below σ and τ coincide.

23: A tree is *pruned* it it has no leaves, in other words if every node is extendible.

In this section, we generalize PA avoidance to avoid a larger collection of closed classes, with no effectiveness or homogeneity constraint. Many natural closed classes in $2^\mathbb{N}$ with no computable member cannot even be computably approximated by giving arbitrarily large initial segments of members.

Given a closed class $\mathcal{C} \subseteq 2^\mathbb{N}$, a *trace* is a collection of finite coded sets of strings $F_0, F_1, \dots$ such that for each $n \in \mathbb{N}$, F_n contains only strings of length exactly n, and $\mathcal{C} \cap \bigcup_{\sigma \in F_n}[\sigma] \neq \emptyset$.[24] In other words, for every $n \in \mathbb{N}$, there is a string $\sigma \in F_n$ and some $P \in \mathcal{C}$ such that $\sigma \prec P$. A k*-trace* is a trace such that $\operatorname{card} F_n = k$ for every $n \in \mathbb{N}$. A *constant-bound trace* (c.b-trace) of $\mathcal{C}$ is a k-trace for some $k \in \mathbb{N}$.

24: One usually writes $[\![F_n]\!]$ for the clopen class generated by F_n. Indeed, using $[F_n]$ would be confusing with the collection of paths through a tree.

Definition 5.6.1. A problem P admits *constant-bound trace avoidance*[25] if for every set Z and every closed class $\mathcal{C} \subseteq 2^\mathbb{N}$ with no Z-computable c.b-trace, every Z-computable instance X of P admits a solution Y such that $\mathcal{C}$ has no $Z \oplus Y$-computable c.b-trace. ◇

25: We defined the notion of closed classes in Cantor space $2^\mathbb{N}$, but all the theorems work equally for effectively compact classes in Baire space $\mathbb{N}^\mathbb{N}$. More precisely, it works for every closed class $\mathcal{C} \subseteq h^\mathbb{N}$ for some total computable function $h : \mathbb{N} \to \mathbb{N}$.

Before proving that some problems admit constant-bound trace avoidance, we shall start with a few exercises to get familiar with this seemingly artificial notion. The two following exercises show that for a homogeneous Π^0_1 class, every constant-bound trace computes a member. Hence, c.b-trace avoidance generalizes PA avoidance.

Exercise 5.6.2. Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a Π^0_1 class. Show that every k-trace of $\mathscr{C}$ computes a 1-trace of $\mathscr{C}$. ⋆

Exercise 5.6.3. Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a homogeneous closed class. Show that every 1-trace of $\mathscr{C}$ computes a member of $\mathscr{C}$. ⋆

The following exercise shows that c.b-trace avoidance generalizes cone avoidance.

Exercise 5.6.4. Let C be a non-computable set. Show that $\{C\}$ does not admit any computable c.b-trace. ⋆

As usual, the core lemma involved in proofs of constant-bound trace avoidance is based on a 3-case analysis. As in PA avoidance for weakly merging forcing questions, the case analysis for preservation of c.b-traces is non-trivial and based on a combinatorial lemma. Let us introduce some piece of terminology which will be helpful in working with constant-bound traces.

A *block* is a finite set of strings all of which have the same length. We write $\mathscr{B}_n$ for the set of all blocks $F \subseteq 2^n$ and $\mathscr{B} = \bigcup_n \mathscr{B}_n$. Given a closed class $\mathscr{C} \subseteq 2^{\mathbb{N}}$, a block $F \in \mathscr{B}_n$ is $\mathscr{C}$*-correct* if $F = \{\mu \in 2^n : \mathscr{C} \cap [\mu] \neq \emptyset\}$. In other words, F is $\mathscr{C}$-correct if it is some level in the pruned tree representing $\mathscr{C}$. Given $n, k \in \mathbb{N}$, a finite collection of blocks $V \subseteq \mathscr{B}_n$ is k*-disperse* if for every k-partition $(P_s : s < k)$ of V, there is some $s < k$ such that $\bigcap_{F \in P_s} F = \emptyset$. The following exercise emphasises a core property of k-disperse sequences:

Exercise 5.6.5. Fix $n, k \in \mathbb{N}$, and let $V \subseteq \mathscr{B}_n$ be a k-disperse sequence. If $E \in \mathscr{B}_n$ is a block which intersects[26] every element of V, then $\operatorname{card} E > k$.⋆

26: By *intersects*, we mean that $F \cap E \neq \emptyset$ for every $F \in V$.

We now prove the core combinatorial lemma which frames the 3-case analysis.

Lemma 5.6.6 (Liu [32]). Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable c.b-trace. Let $U \subseteq \mathscr{B}$ be a c.e. set of blocks. Either U contains a $\mathscr{C}$-correct block, or for every $k \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that the set $\mathscr{B}_n \setminus U$ is k-disperse. ⋆

Proof. Suppose that U does not contain any $\mathscr{C}$-correct block.[27] For every $n \in \mathbb{N}$, let $V_n = \mathscr{B}_n \setminus U$. Fix some $k \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$, V_n is not k-disperse, otherwise we are done. Since V_n is co-c.e. uniformly in n, there exists a co-c.e. enumeration $(V_{n,t})_{t \in \mathbb{N}}$ of V_n. Since V_n is not k-disperse, there exists some $t \in \mathbb{N}$ and a k-partition $(P_{n,s} : s < k)$ of $V_{n,t}$ such that for each $s < k$, $\bigcap_{F \in P_{n,s}} F \neq \emptyset$. Such k-partition can be computed uniformly in n. Moreover, since V_n contains a $\mathscr{C}$-correct block, then there is some $s < k$ such that $P_{n,s}$ contains a $\mathscr{C}$-correct block, hence for every $\sigma \in \bigcap_{F \in P_{n,s}} F$, $\mathscr{C} \cap [\sigma] \neq \emptyset$. For each n, let E_n be obtain by picking a string in each set $\bigcap_{F \in P_{n,s}} F$ for each $s < k$. The sequence $(E_n)_{n \in \mathbb{N}}$ is a computable k-trace of $\mathscr{C}$, contradicting the hypothesis. ■

27: The proof actually shows that if $\mathscr{U}$ is a c.e. set of blocks with no $\mathscr{C}$-correct block and if there is no k-disperse sequence of blocks outside of U, then there is a computable k-trace of $\mathscr{C}$.

Let us illustrate preservation of constant-bound traces using the simplest notion of forcing, namely, Cohen forcing.

Theorem 5.6.7
Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable c.b-trace. For every sufficiently Cohen generic set G, $\mathscr{C}$ admits no G-computable c.b-trace.

Proof. It suffices to prove the following lemma.

Lemma 5.6.8. For every condition $\sigma \in 2^{<\mathbb{N}}$, every Turing index $e \in \mathbb{N}$ and every $k \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing Φ_e^G not to be a k-trace of $\mathscr{C}$.★

Proof. We can assume without loss of generality that Φ_e is a k-trace functional, that is, whenever $\Phi_e^X(n)\downarrow$, then the output is a block of size k, whose strings have length n. Fix a condition σ. Consider the following set:

$$U = \{F \in \mathscr{B}_n : n \in \mathbb{N},\ \exists \tau \geq \sigma\ \Phi_e^\tau(n)\downarrow \cap F = \emptyset\}$$

Note that the set U is Σ_1^0. There are three cases:

- Case 1: there is some $n \in \mathbb{N}$ such that $U \cap \mathscr{B}_n$ contains some $\mathscr{C}$-correct block F. Let $\tau \geq \sigma$ witness $F \in U$, that is, let $\tau \geq \sigma$ be such that $\Phi_e^\tau(n)\downarrow \cap F = \emptyset$. Then τ forces Φ_e^G not to be a k-trace of $\mathscr{C}$.
- Case 2: there is some $n \in \mathbb{N}$ such that $\mathscr{B}_n \setminus U$ is k-disperse. We claim that for every $F \in \mathscr{B}_n \setminus U$, σ forces $\Phi_e^G(n)\uparrow \vee \Phi_e^G(n)\downarrow \cap F \neq \emptyset$. Indeed, if for some $Z \in [\sigma]$, $\Phi_e^Z(n)\downarrow \cap F = \emptyset$, then by the use property, there is some $\tau \preceq Z$ such that $\Phi_e^\tau(x)\downarrow \cap F = \emptyset$, contradicting the fact that $F \in \mathscr{B}_n \setminus U$. Thus σ forces

$$\Phi_e^G(n)\uparrow \vee (\forall F \in \mathscr{B}_n \setminus U)\ \Phi_e^G(n)\downarrow \cap F \neq \emptyset$$

  Since Φ_e is a k-trace functional, and $\mathscr{B}_n \setminus U$ is k-disperse, then by Exercise 5.6.5, σ forces $\Phi_e^G(n)\uparrow$.
- Case 3: None of Case 1 and Case 2 holds. This cannot happen by Lemma 5.6.6. ■

We are now ready to prove Theorem 5.6.7. Given $e, k \in \mathbb{N}$, let $\mathscr{D}_{e,k}$ be the set of all conditions τ forcing Φ_e^G not to be a k-trace of $\mathscr{C}$. It follows from Lemma 5.6.8 that every $\mathscr{D}_{e,k}$ is dense, hence for every $\{\mathscr{D}_{e,k} : e, k \in \mathbb{N}\}$-generic set G, $\mathscr{C}$ admits no G-computable c.b-trace. ■

Looking more closely at the previous proof, the key feature of the forcing we exploited was the existence of a Σ_1^0-preserving forcing question such that, if it does not hold for a finite number of Σ_1^0 formulas, then there exists an extension forcing all negations simultaneously. This motivates the following definition, which is a strong form of Γ-merging.

Definition 5.6.9. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *finitely Γ-merging* if for every $p \in \mathbb{P}$ and every finite sequence of Γ-formulas $\varphi_0(G), \dots, \varphi_{\ell-1}(G)$, if $p\ ?\!\nvdash \varphi_s(G)$ holds for every $s < \ell$, then there is an extension $q \leq p$ forcing $\bigwedge_{s<\ell} \varphi_s(G)$. ◇

As for Γ-merging forcing questions, we say that a forcing question for Σ_n^0 formulas is finitely Π_n^0-merging if negation of the forcing question is finitely Π_n^0-merging. At this point, it should be clear how to prove the abstract theorem for constant-bound trace avoidance. We leave it as an exercise:

Exercise 5.6.10. Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving, finitely Π^0_1-merging forcing question. Prove that for every sufficiently generic filter $\mathcal{F}$, $\mathscr{C}$ admits no $G_{\mathcal{F}}$-computable constant-bound trace. ⋆

Exercise 5.6.11. Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Adapt the proof of Theorem 3.2.4 to show that for any set A, there exists a set G such that $G' \geq_T A$ and $\mathscr{C}$ admits no G-computable constant-bound trace. ⋆

Exercise 5.6.12. Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Use computable Mathias forcing to prove that for every uniformly computable sequence of sets $\vec{R} = R_0, R_1, \dots$, there is an infinite $\vec{R}$-cohesive set G such that $\mathscr{C}$ admits no G-computable constant-bound trace. ⋆

Recall that some disjunctive or tree-like forcing questions are not even Π^0_1-merging. One can generalize Exercise 5.6.10 to such notions as we did in Section 5.2.

Definition 5.6.13. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *weakly finitely Γ-merging* if for every $p \in \mathbb{P}$, there is a $d \in \mathbb{N}$ such that for every finite sequence of Γ-formulas $\varphi_0(G), \dots, \varphi_{\ell-1}(G)$, if $p \operatorname{?\vdash} \varphi_s(G)$ holds for every $s < \ell$, there is a d-partition $(P_t : t < d)$ of $\{0, \dots, \ell - 1\}$ such that for every $t < d$, there is an extension $q \leq p$ forcing $\bigwedge_{s \in P_t} \varphi_s(G)$. ◊

The previous definition is quite technical, but contains exactly the hypothesis necessary to prove the following abstract theorem.

Theorem 5.6.14
Let $\mathscr{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable c.b-trace. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving weakly finitely Π^0_1-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $\mathscr{C}$ admits no $G_{\mathcal{F}}$-computable c.b-trace.

Proof. It suffices to prove the following diagonalization lemma.

Lemma 5.6.15. For every condition $p \in \mathbb{P}$, every Turing index $e \in \mathbb{N}$ and every $k \in \mathbb{N}$, there is an extension $q \leq p$ forcing Φ^G_e not to be a k-trace of $\mathscr{C}$.⋆

Proof. Let $d \in \mathbb{N}$ witness that the forcing question is weakly finitely Π^0_1-merging for p. Consider the following set

$$U = \{F \in \mathscr{B}_n : n \in \mathbb{N}, p \operatorname{?\vdash} \Phi^G_e(n)\downarrow \cap F = \emptyset\}$$

Since the forcing question is Σ^0_1-preserving, the set U is Σ^0_1. There are three cases:

- Case 1: there is some $n \in \mathbb{N}$ such that $\mathcal{U} \cap \mathscr{B}_n$ contains some $\mathscr{C}$-correct block F. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi^G_e(n) \cap F = \emptyset$. In particular, q forces Φ^G_e not to be a k-trace of $\mathscr{C}$.
- Case 2: there is some $n \in \mathbb{N}$ such that $\mathscr{B}_n \setminus \mathcal{U}$ is $k \cdot d$-disperse. Since the forcing question is weakly finitely Π^0_1-merging with witness d, there

is a d-partition $(P_t : t < d)$ of $\mathcal{B}_n \setminus \mathcal{U}$ such that for every $t < d$, there is an extension $q_t \leq p$ forcing

$$\bigwedge_{F \in P_t} \left(\Phi_e^G(n)\uparrow \vee \Phi_e^G(n) \cap F \neq \emptyset\right)$$

Let $t < d$ be such that P_t is k-disperse.[28] Since Φ_e is a k-trace functional, by Exercise 5.6.5, the extension $q_t \leq p$ forces $\Phi_e^G(n)\uparrow$.

- Case 3: None of Case 1 and Case 2 holds. This case cannot happen by Lemma 5.6.6. ■

28: For any d-partition of a $k \cdot d$-disperse family, one of the parts is k-disperse. Indeed, otherwise, for each part $t < d$, there is a k-partition witnessing the failure. Putting all these k-partitions together, we obtain a failure of $k \cdot d$-dispersity of the family.

We are now ready to prove Theorem 5.6.14. Given $e, k \in \mathbb{N}$, let $\mathcal{D}_{e,k}$ be the set of all conditions $q \in \mathbb{P}$ forcing Φ_e^G not to be a k-trace of $\mathcal{C}$. It follows from Lemma 5.2.5 that every $\mathcal{D}_{e,k}$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_{e,k} : e, k \in \mathbb{N}\}$-generic, so $\mathcal{C}$ admits no $G_{\mathcal{F}}$-computable c.b-trace. This completes the proof of Theorem 5.6.14. ■

Liu [32] proved that Ramsey's theorem for pairs admits constant-bound trace avoidance, following the same structure as his proof of PA avoidance, *mutatis mutandis*. We leave the steps as exercises.

Exercise 5.6.16 (Liu [32]). Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Adapt the proof of Theorem 5.3.3 to show that for any non-empty Π_1^0 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists an infinite set H homogeneous for $\mathcal{P}$ such that $\mathcal{C}$ admits no H-computable constant-bound trace. ⋆

Exercise 5.6.17 (Liu [32]). Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Adapt the proof of Theorem 5.4.3 using Exercise 5.6.16 to show that for any set A, there exists an infinite subset H of A or $\overline{A}$ such that $\mathcal{C}$ admits no H-computable constant-bound trace. ⋆

Exercise 5.6.18 (Liu [32]). Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be a closed class with no computable constant-bound trace. Combine Exercise 5.6.12 and Exercise 5.6.17 to show that for any computable coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite f-homogeneous set $H \subseteq \mathbb{N}$ such that $\mathcal{C}$ admits no H-computable constant-bound trace. ⋆

The notion of constant-bound trace avoidance is the right invariant property strongly preserved by the pigeonhole principle to prevent it from computing a 1-trace of a closed class $\mathcal{C} \subseteq 2^{\mathbb{N}}$. Indeed, if $\mathcal{C}$ admits a computable k-trace $F_0, F_1, \dots$ for some $k \in \mathbb{N}$, one application of the pigeonhole principle for k colors yields an infinite 1-trace of $\mathcal{C}$. This however leaves open the case of closed classes with no computable member, but admitting a computable 1-trace.

Question 5.6.19. Is there a natural characterization of the closed classes strongly avoided by the pigeonhole principle? ⋆

5.7 DNC and compactness

Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally non-computable* (DNC) if $\forall e\ f(e) \neq \Phi_e(e)$. PA degrees are those computing a $\{0, 1\}$-valued DNC function. In this section, we consider the computational power of $\mathbb{N}$-valued DNC functions. We shall see that the existence of DNC functions is equivalent to a Ramsey-type form of compactness, called the Ramsey-type weak weak König's lemma. A Turing degree is *DNC* if it computes a DNC function. It is often useful to think of DNC degrees as those computing a function which can escape finite c.e. sets when a bound to their size is known.

Proposition 5.7.1 (Bienvenu, Patey and Shafer [37]). Let X be a set. The following are equivalent:

1. X computes a DNC function ;
2. X computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, b \in \mathbb{N}$, if $\operatorname{card} W_e \leq b$, then $g(e,b) \notin W_e$. ⋆

Proof. $(1) \to (2)$[29]: Let $f : \mathbb{N} \to \mathbb{N}$ be a DNC function. For every $e, b \in \mathbb{N}$ and $i < b$, let $h(e,b,i)$ be the index of the partial computable function $\Phi_{h(e,b,i)}$ which on any input x, waits for the ith element y_i of W_e to appear, in order of apparition. It $\operatorname{card} W_e \leq i$, then the program will never terminate, and $\Phi_{h(e,b,i)}$ will be the nowhere-defined function. If $\operatorname{card} W_e > i$, then y_i is eventually found. Then, interpret y_i as a b-tuple $\langle y_i^0, \dots, y_i^{b-1} \rangle$ and output y_i^i. In this case, $\Phi_{h(e,b,i)}(h(e,b,i)) \downarrow = y_i^i$, and $f(h(e,b,i)) \neq y_i^i$. Let $g(e,b) = \langle f(h(e,b,0)), \dots f(h(e,b,b-1)) \rangle$. Suppose for the contradiction that $\operatorname{card} W_e \leq b$ and $g(e,b) \in W_e$. Say $g(e,b) = y_i \in W_e$. Then $f(h(e,b,i)) = y_i^i = \Phi_{h(e,b,i)}(h(e,b,i))$, contradicting the fact that f is a DNC function.

$(2) \to (1)$: Let $g : \mathbb{N}^2 \to \mathbb{N}$ be such that for every $e, b \in \mathbb{N}$, if $\operatorname{card} W_e < b$, then $g(e,b) \notin W_e$. For every $e \in \mathbb{N}$, let $h(e)$ be an index of the partial computable function $\Phi_{h(e)}$ which, on input x, waits until $\Phi_e(e)\downarrow$. If $x = \Phi_e(e)\downarrow$, then the program halts, otherwise it loops forever. In other words, $W_{h(e)} = \{\Phi_e(e)\}$ if $\Phi_e(e)\downarrow$, and $W_{h(e)} = \emptyset$ otherwise. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(e) = g(h(e), 1)$ is diagonally non-computable. ■

29: The idea is the following: Given a list $y_0, \dots, y_{b-1}$ of b integers, interpret each integer as a b-tuple of integers, based on a computable bijection.

$$\begin{array}{c|cccc} y_0 & \boxed{y_0^0} & y_0^1 & \cdots & y_0^{b-1} \\ y_1 & y_1^0 & \boxed{y_1^1} & \cdots & y_1^{b-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{b-1} & y_{b-1}^0 & y_{b-1}^1 & \cdots & \boxed{y_{b-1}^{b-1}} \end{array}$$

Then, given b-many b-tuples of elements, by a diagonal argument, one can create a b-tuple of integers which is different from each element of this list, and re-interpret it as an integer.
The difficulty comes from the fact that the list $y_0, \dots, y_{b-1}$ is c.e., so one uses a DNC function to create this diagonal b-tuple.

DNC degrees can be expressed as a form of compactness as follows: The *Ramsey-type weak weak König lemma* (RWWKL) is the problem whose instances are binary trees of positive measure, and whose solutions are infinite homogeneous sets for the tree. It is a problem at the intersection between weak weak König's lemma – corresponding to the existence of random sequences – and the Ramsey-type König's lemma, – the compactness part of Ramsey's theorem for pairs.

Proposition 5.7.2. Let X be a set. The following are equivalent:

1. X computes a DNC function;
2. Every Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ of positive measure admits an infinite X-computable homogeneous set. ⋆

Proof. $(1) \to (2)$: Fix a Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ with $\mu(\mathcal{P}) \geq 2^{-c}$ for some $c \geq 3$. Given a set $H \subseteq \mathbb{N}$, let $\mathbb{Q}_H = \{X \in 2^{\mathbb{N}} : H \subseteq X\}$, and let $\mathbb{Q}_n = \mathbb{Q}_{\{n\}}$. A finite set $F \subseteq \mathbb{N}$ is *valid* if $\mu(\mathcal{P} \cap \mathbb{Q}_F) \geq 2^{-c \cdot 2^{\operatorname{card} F}}$. Note that $\emptyset$ is valid, and that if F is valid, then it is homogeneous for $\mathcal{P}$. For every finite set $F \subseteq \mathbb{N}$, let $W_{h(F)}$ be the c.e. set of all $n \in \mathbb{N}$ such that $F \cup \{n\}$ is not valid. Let $g : \mathbb{N}^2 \to \mathbb{N}$ be the function given by Proposition 5.7.1. By a measure-theoretic argument[30], for any valid set F, $\operatorname{card} W_{h(F)} < 2 \cdot c \cdot 2^{\operatorname{card} F}$, so $g(h(F), 2 \cdot c \cdot 2^{\operatorname{card} F}) \notin W_{h(F)}$. We can define an infinite set $H \subseteq \mathbb{N}$ such that every initial segment is valid. In particular, H is homogeneous for $\mathcal{P}$.

30: If $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is a closed class with $\mu(\mathcal{C}) \geq 2^{-c}$ for some $c \geq 3$, then

$$\operatorname{card}\{n \in \mathbb{N} : \mu(\mathcal{C} \cap \mathbb{Q}_n) < 2^{-2c}\} < 2c.$$

Indeed, let F be a subset of it of size $2c$ and let $\mathcal{R}_F = \{X \in 2^{\mathbb{N}} : F \cap X = \emptyset\}$. Note that

$$2^{\mathbb{N}} = \mathcal{R}_F \cup \bigcup_{n \in F} \mathbb{Q}_n$$

We have $\mu(\mathcal{C} \cap \mathcal{R}_F) \leq 2^{-2c}$, and $\mu(\mathcal{C} \cap \bigcup_{n \in F} \mathbb{Q}_n) < 2c \cdot 2^{-2c}$, so

$$2^{-c} \leq \mu(\mathcal{C}) \leq 2^{-2c} + 2c \cdot 2^{-2c}$$

which yields a contradiction when $c \geq 3$.

$(2) \to (1)$: For every $e \in \mathbb{N}$, let $\mathcal{P}_e$ be the Π^0_1 class of all elements X such that if $\Phi_e(e)\downarrow$, then interpreting the output as a $(e+3)$-tuple $\langle x^0_e, \dots, x^{e+2}_e \rangle$, there is some $s < t < e+3$ such that $X(x^s_e) \neq X(x^t_e)$. Let $\mathcal{P} = \bigcap_e \mathcal{P}_e$. First, notice that for every infinite homogeneous set $H = \{y_0 < y_1 < \dots\}$ for $\mathcal{P}$, the H-computable function defined by $f(e) = \langle y_0, \dots, y_{e+1} \rangle$ is diagonally non-computable. Second, for every e, $\mu(2^{\mathbb{N}} \setminus \mathcal{P}_e) \leq 2 \cdot 2^{-e-3} = 2^{-e-2}$, so $\mu(\mathcal{P}) \geq 1 - \sum_e 2^{-e-2} = 1/2$. Thus, $\mathcal{P}$ has positive measure. ∎

The Ramsey-type weak weak König lemma is a particular case of RWKL, hence follows from Ramsey's theorem for pairs. Thus, the existence of DNC functions does not imply the existence of random sequences, and a fortiori of PA degrees.

5.8 DNC avoidance

We now develop the techniques to prove that a problem does not imply the existence of this weak notion of compactness. The framework of closed classes avoidance of Section 5.6 admits a straightforward generalization to effectively compacts in the Baire space $\mathbb{N}^{\mathbb{N}}$. The class of $\mathbb{N}$-valued DNC functions is Π^0_1 in the Baire space, but not compact, thus it does not fall within the scope of this framework.

Definition 5.8.1. A problem P admits *DNC avoidance*[31] if for every pair of sets Z and $D \leq_T Z$ such that Z is not of DNC degree over D, every Z-computable instance X of P admits a solution Y such that $Y \oplus Z$ is not of DNC degree over D. ◊

31: Note the similarity between PA and DNC avoidance.

Due to the similar nature of $\{0,1\}$-valued and $\mathbb{N}$-valued DNC functions, proofs of DNC avoidance are very similar to those of PA avoidance.

Exercise 5.8.2. Adapt the proof of Theorem 5.1.3 to show that for every sufficiently Cohen generic set G, G is not of DNC degree. ★

In the proof of PA avoidance, the Π^0_1-merging property of the forcing question is used in the second case, for forcing partiality. Since the functionals are $\{0,1\}$-valued, it suffices to merge two Π^0_1 properties simultaneously to force partiality. In the case of $\mathbb{N}$-valued functionals, infinitely many Π^0_1 properties need to be forced simultaneously.

Definition 5.8.3. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *countably Γ-merging* if for every $p \in \mathbb{P}$ and every countable sequence of Γ-formulas $(\varphi_s(G))_{s \in \mathbb{N}}$, if $p \mathrel{?\!\vdash} \varphi_s(G)$ for each $s \in \mathbb{N}$, then there is an extension $q \leq p$ forcing $\forall s \varphi_s(G)$. ◊

Being countably Π^0_1-merging is a very strong properties, satisfied by very few notions of forcing in practice. Indeed, DNC degrees being computationally very weak, many natural problems imply their existence.

Theorem 5.8.4
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving, countably Π^0_1-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is not of DNC

degree.

PROOF. It suffices to prove the following lemma:

Lemma 5.8.5. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing Φ_e^G not to be a DNC function. ⋆

32: Note that contrary to PA avoidance, this set ranges over $\mathbb{N} \times \mathbb{N}$ instead of $\mathbb{N} \times 2$. This difference is important in Case 2, where one needs to force countably many Π_1^0 formulas simultaneously.

PROOF. Consider the following set[32]

$$U = \{(x, v) \in \mathbb{N}^2 : p \mathrel{?\vdash} \Phi_e^G(x)\downarrow = v\}$$

Since the forcing question is Σ_1^0-preserving, the set U is Σ_1^0. There are three cases:

- Case 1: $(x, \Phi_x(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x(x)\downarrow$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^G(x)\downarrow = \Phi_x(x)$.
- Case 2: there is some $x \in \mathbb{N}$ such that for every $y \in \mathbb{N}$, $(x, y) \notin U$. Since the forcing question is countably Π_1^0-merging, there is an extension $q \leq p$ forcing $\forall y \neg(\Phi_e^G(x)\downarrow = y)$, hence forcing Φ_e^G not to be a DNC function.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_1^0 graph of a DNC function. This contradicts the fact that **0** is not DNC. ■

We are now ready to prove Theorem 5.8.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing Φ_e^G not to be a DNC function. It follows from Lemma 5.8.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G_{\mathcal{F}}$ is not of DNC degree. This completes the proof of Theorem 5.8.4. ■

Exercise 5.8.6. Adapt the proof of Theorem 3.2.4 to show that for any set A, there exists a set G such that $G' \geq_T A$ and G is not of DNC degree. ⋆

5.9 Comparing avoidances

We have seen in Sections 3.5 and 3.6 that cone avoidance coincides with other preservation notions, such as preservation of 1 non-Σ_1^0 definition and of 1 hyperimmunity. Cone avoidance does not imply PA avoidance, as WKL satisfies the former, but not the latter. On the other hand, one can prove that PA avoidance implies cone avoidance. For this, we need the following theorem, which informally says that the computational distance between a set and its Turing jump can be any non-zero Turing degree.

Theorem 5.9.1 (Posner and Robinson [38])
Let A be a non-computable set. There exists a set G such that $A \oplus G \geq_T G'$.

33: One can modify the construction to encode any set Z instead of $\emptyset'$. The construction is then $A \oplus Z \oplus \emptyset'$-computable. This generalization is due to Jockusch and Shore [39].

PROOF. The idea is to build a 1-generic set G, which will encode $\emptyset'$[33], so that G and A allow to find the construction sequence. The construction itself will be computable in $A \oplus \emptyset'$. We can assume without loss of generality that A is not a c.e. set (otherwise, one replaces A by its complement). Let $(W_e)_{e \in \mathbb{N}}$ be an enumeration of the Σ_1^0 subsets of $2^{<\mathbb{N}}$.

Let $\sigma_0 = \epsilon$, the empty word. Suppose σ_e defined. Consider the set

$$D_e = \{m : \exists \tau \text{ such that } \sigma_e \emptyset'(e)0^m 1\tau \in W_e\}.$$

Note that D_e is a c.e. set. In particular as A is not c.e. there is some $m \in D_e$ with $m \notin A$ or some $m \notin D_e$ with $m \in A$. Consider the smallest m such that we are in one case or the other. Note that $\emptyset' \oplus A$ allows to find uniformly this integer m.

In the first case, let $\sigma_{e+1} = \sigma_e \emptyset'(e)0^m 1\tau$ for the first string τ such that $\sigma_e \emptyset'(e)0^m 1\tau$ is listed in W_e. In the second case, let $\sigma_{e+1} = \sigma_e \emptyset'(e)0^m 1$. Note that in this case no string of W_e can extend σ_{e+1}. We define G as being $\sigma_0 \preceq \sigma_1 \preceq \sigma_2 \preceq \dots$. This completes the construction.

It is clear that G is 1-generic and computable in $A \oplus \emptyset'$. How do you now compute $\emptyset'$ from $G \oplus A$? Suppose we know the string σ_e. We then necessarily know the e-th bit of $\emptyset'$: it is the bit i such that $\sigma_e i \prec G$. We can then find σ_{e+1} as follows: we look at the number m of 0 which follows $\sigma_e i$ in G. If $m \in A$, this means that $\sigma_{e+1} = \sigma_e i 0^m 1$. If $m \notin A$, this means that $\sigma_{e+1} = \sigma_e i 0^m 1\tau$ for the first string τ found in W_e. Finding this string τ is then a computable process. We can therefore in all cases find σ_{e+1}, and by repeating the process, compute $\emptyset'$ from $A \oplus G$. Thus, $G \oplus \emptyset' \leq_T G \oplus A$. Since every 1-generic set is generalized low, then $G' \leq_T G \oplus A$. ■

Corollary 5.9.2
If a problem P *admits PA avoidance, then it admits cone avoidance.*

Proof. Fix a set Z, a non-Z-computable set C and a P-instance $X \leq Z$. By Theorem 5.9.1 relativized to Z, there is a set G such that $C \oplus Z \oplus G \geq_T (Z \oplus G)'$. Since P admits PA avoidance, there is a solution Y to X such that $Y \oplus Z \oplus G$ is not of PA degree over $Z \oplus G$. In particular, $Y \oplus Z \not\geq_T C$, otherwise $Y \oplus Z \oplus G \geq_T C \oplus Z \oplus G \geq_T (Z \oplus G)'$, but $(Z \oplus G)'$ is of PA degree over $Z \oplus G$. ■

Constant-bound trace avoidance generalizes PA avoidance, since the Π^0_1 class of $\{0, 1\}$-valued DNC functions does not admit any computable constant-bound trace. On the other hand, some problems such as WWKL admit PA avoidance, but not constant-bound trace avoidance. Indeed, there is a Π^0_1 class of positive measure with no computable constant-bound trance.

An infinite set $X \subseteq \mathbb{N}$ is *immune* iff it has no computable infinite subset, or equivalently no c.e. infinite subset. We have already seen a strong form of immunity, namely, hyperimmunity, for which one cannot even approximate an infinite subset by pairwise disjoint blocks of finite sets.

Definition 5.9.3. A problem P admits *preservation of 1 immunity* if for every set Z and every Z-immune set I, every Z-computable instance X of P admits a solution Y such that I is $Z \oplus Y$-immune. ◊

As for DNC avoidance, the existence of a Σ^0_1-preserving, countably Π^0_1-merging forcing question is sufficient to prove preservation of 1 immunity.

Theorem 5.9.4
Fix an infinite immune set I. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-

preserving, countably Π^0_1-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, I is $G_{\mathcal{F}}$-immune.

PROOF. It suffices to prove the following lemma:

Lemma 5.9.5. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing W_e^G not to be an infinite subset of I. ★

PROOF. Consider the following set

$$U = \{x \in \mathbb{N} : p \mathrel{?\vdash} x \in W_e^G\}$$

Since the forcing question is Σ^0_1-preserving, the set U is Σ^0_1. There are three cases:

- Case 1: $x \in U \setminus I$ for some $x \in \mathbb{N}$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $x \in W_e^G$, hence forcing $W_e^G \not\subseteq I$.
- Case 2: U is finite. Since the forcing question is countably Π^0_1-merging, there is an extension $q \leq p$ forcing $\forall x \notin U\ x \notin W_e^G$, hence forcing W_e^G to be finite.
- Case 3: U is an infinite c.e. subset of I. This contradicts the immunity of I. ■

We are now ready to prove Theorem 5.9.4. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing W_e^G not to be an infinite subset of I. It follows from Lemma 5.9.5 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so I is $G_{\mathcal{F}}$-immune. This completes the proof of Theorem 5.9.4. ■

There exists some problems, such as the Ascending Descending sequence principle (ADS) which admits DNC avoidance, but not preservation of 1 immunity. This naturally raises the following question:

Question 5.9.6. Does preservation of 1 immunity imply DNC avoidance? ★

Custom properties | 6

The classical study of computability theory puts the emphasis on some concepts such as hyperimmunity, PA degrees, or the arithmetic hierarchy. These notions induce invariant properties like preservation of hyperimmunity, PA avoidance, or low$_n$ness, enabling to separate second-order statements in reverse mathematics. However, the diversity of second-order statements makes it impossible to always separate them with classical notions.

In this chapter, we explain how to design custom computability-theoretic properties to separate two mathematical problems. As it turns out, their design is once again driven by the definability and combinatorial properties of their corresponding forcing questions. The main ideas are presented in this chapter through the study of three important statements: the Erdős-Moser theorem (EM), the Ascending Descending Sequence principle (ADS) and the Chain-AntiChain principle (CAC).

Prerequisites: Chapters 2, 3 and 5

6.1 Separation framework

Consider two Π^1_2 problems P and Q. In order to separate P from Q over RCA_0, one needs to build a model $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{P}$ containing an instance X_Q, but such that $\mathcal{M}$ contains no Q-solution to X_Q. The model $\mathcal{M}$ is usually built as a limit of a countable increasing sequence $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots$ of Turing ideals as follows. First, construct a Q-instance X_Q with no X_Q-computable solution, and let $\mathcal{M}_0 = \{Y \in 2^\mathbb{N} : Y \leq_T X_\mathsf{Q}\}$. Then, assuming $\mathcal{M}_n$ is a Turing ideal of the form $\{Y \in 2^\mathbb{N} : Y \leq_T Z_n\}$[1] for some set Z_n, pick a P-instance X_P in $\mathcal{M}_n$ with no solution in $\mathcal{M}_n$, construct a solution Y_P to X_P, and let $\mathcal{M}_{n+1} = \{Y \in 2^\mathbb{N} : Y \leq_T Z_n \oplus Y_\mathsf{P}\}$. One furthermore wants to maintain the invariant that X_Q has no Q-solution in $\mathcal{M}_n$, so the difficulty is to build a solution Y_P to X_P such that X_Q has no $Z_n \oplus Y_\mathsf{P}$-computable solution, assuming it has no Z_n-computable solution. Usually, one needs to find a stronger invariant than just having no Z_n-computable solution. A class $\mathcal{W} \subseteq 2^\mathbb{N}$ is a *weakness property* if it is downward-closed under the Turing reduction.

1: Turing ideal of this form are called topped. A model of RCA_0 is *topped* if its corresponding Turing ideal is topped.

Definition 6.1.1. A problem P *preserves* a weakness property $\mathcal{W}$ if for every $Z \in \mathcal{W}$ and every Z-computable instance X, there is a solution Y to X such that $Z \oplus Y \in \mathcal{W}$. ◊

This previous definition generalizes many properties defined in the previous chapters. For instance, a problem P admits cone avoidance iff it preserves $\mathcal{W}_C = \{X \in 2^\mathbb{N} : C \not\leq_T X\}$ for every set C.[2]

2: Note that if C is computable, then $\mathcal{W}_C = \emptyset$, and then P vacuously preserves $\mathcal{W}_C$.

Exercise 6.1.2. Formulate PA avoidance (Definition 5.1.1) as a preservation of a family of weakness properties. ⋆

The following theorem gives the general construction underlying almost all the separation proofs over ω-models.

Theorem 6.1.3
Let P *be a* Π^1_2 *problem preserving a weakness property* $\mathcal{W}$*. Then for every set* $Z \in \mathcal{W}$*, there is an* ω*-model* $\mathcal{M}$ *of* $\mathsf{RCA}_0 + \mathsf{P}$ *such that* $\mathcal{M} \subseteq \mathcal{W}$ *and* $Z \in \mathcal{M}$*.*

Proof. We are going to define a countable sequence of Turing ideals $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots$, where $\mathcal{M}_n = \{Y \in 2^{\mathbb{N}} : Y \leq_T Z_n\}$, such that for all $n \in \mathbb{N}$,

(1) if $n = \langle a, b \rangle$ and X is the a-th P-instance of $\mathcal{M}_b$, then Z_{n+1} computes a P-solution to X;
(2) $Z_{n+1} \in \mathcal{W}$, or equivalently $\mathcal{M}_n \subseteq \mathcal{W}$.

First $Z_0 = Z$. Suppose we have defined $Z_n \in \mathcal{W}$ and say $n = \langle a, b \rangle$. Let X be the a-th P-instance of $\mathcal{M}_b$, Since P preserves $\mathcal{W}$, there is a solution Y to X such that $Y \oplus Z_n \in \mathcal{W}$. Let $Z_{n+1} = Z_n \oplus Y$. This completes the construction.

Let $\mathcal{M} = \bigcup_n \mathcal{M}_n = \{Y \in 2^{\mathbb{N}} : \exists n\ Y \leq_T Z_n\}$. By construction, the class $\mathcal{M}$ is a Turing ideal, thus $\mathcal{M} \models \mathsf{RCA}_0$. Moreover, by (1), every P-instance $X \in \mathcal{M}$ admits a solution in $\mathcal{M}$. By (2), $\mathcal{M} \subseteq \mathcal{W}$ and by construction, $Z \in \mathcal{M}$. ∎

Corollary 6.1.4
Fix a weakness property $\mathcal{W}$*. Let* P *and* Q *be two* Π^1_2 *problems such that* P *preserves* $\mathcal{W}$ *but* Q *does not. Then* $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{Q}$*.*

Proof. Since Q does not preserve $\mathcal{W}$, there is some $Z \in \mathcal{W}$ and some Z-computable instance X_{Q} of Q such that for every solution Y to X_{Q}, $Y \oplus X_{\mathsf{Q}} \notin \mathcal{W}$. Since P preserves $\mathcal{W}$, by Theorem 6.1.3, there is an ω-model $\mathcal{M}$ of $\mathsf{RCA}_0 + \mathsf{P}$ such that $\mathcal{M} \subseteq \mathcal{W}$ and $Z \in \mathcal{M}$. In particular, $X_{\mathsf{Q}} \in \mathcal{M}$, but $\mathcal{M}$ does not contain any Q-solution to X_{Q}, so $\mathcal{M} \not\models \mathsf{Q}$. ∎

The purpose of this chapter is to emphasize the relation between the combinatorial features of the forcing question of a problem P and the invariant properties it preserves, and to learn through examples how to design a custom invariant property to separate two problems.

6.2 Immunity and variants

The early study of reverse mathematics has shown the emergence of an empirical structural phenomenon: the vast majority of ordinary theorems of mathematics, once formulated as second-order statements, are either provable over RCA_0, or provably equivalent over RCA_0 to one among four main systems of axioms, namely, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0.[3] These systems can be separated over ω-models using standard notions from computability theory or higher recursion theory. Thus, when considering two second-order statements, they are likely to be either equivalent over RCA_0, or to belong to two of the above-mentioned systems, and therefore separable using standard notions.

Some exceptions exist to this structural phenomenon, mostly coming from Ramsey theory.[4] Overall, Ramsey's theory seeks to understand the inherent structure and order that can arise within large sets by investigating the existence of specific patterns, colorings, or configurations. In the setting of second-order arithmetic, statements from Ramsey theory assert the existence of infinite

3: These systems are known as the "Big Five" (see Montalbán [40]).

4: One can often define "Ramsey-type" versions of standard problems, where a solution is an infinite number of bits of information of the original solution. For instance, the Ramsey-type weak König's lemma (RWKL) is a Ramsey-type version of weak König's lemma, stating the existence of an infinite set homogeneous for one of the path.

sets satisfying some property which is closed under subset. For instance, Ramsey's theorem states the existence, for every coloring $f : [\mathbb{N}]^n \to k$, of an infinite f-homogeneous set H, and every infinite subset $G \subseteq H$ is also f-homogeneous, hence also a solution. We shall therefore give a particular attention to statements such that the collection of solutions is closed under infinite subsets.

It follows that if Q is a statement from Ramsey theory and X is an instance with no computable solution, then every solution Y is immune.[5] Thus, when separating a Π^1_2 problem P from a Q over ω-models, one usually considers preservations of strong notions of immunity. Some of the invariant properties studied in previous chapters can already be formulated in terms of preservation of strong immunity.

5: Recall that an infinite set A is *immune* if it has no infinite computable subset, or equivalently if it has no infinite c.e. subset.

Hyperimmunity. As explained in Section 3.6, cone avoidance is equivalent to preservation of 1 hyperimmunity. In Chapter 2, hyperimmunity is defined in terms of domination of functions, but the original definition over sets is formulated as a strong variant of immunity.

Definition 6.2.1. Let $D_0, D_1, \dots$ be a canonical enumeration of all non-empty finite sets.[6] A *c.e. array*[7] is a collection of finite sets for the form $\{D_{f(n)} : n \in \mathbb{N}\}$ for some computable function $f : \mathbb{N} \to \mathbb{N}$, such that $\min D_{f(n)} > n$ for every $n \in \mathbb{N}$. An infinite set A is *hyperimmune* if for every c.e. array $\{D_{f(n)} : n \in \mathbb{N}\}$, there is some $n \in \mathbb{N}$ such that $A \cap D_{f(n)} = \emptyset$.

6: One can let D_n be such that $\sum_{x \in D_n} 2^x = n + 1$, in other words, the binary representation of $n+1$ is seen as the characteristic function of D_n.

7: One usually requires a c.e. array to be made of pairwise disjoint sets rather than requiring that $\min D_{f(n)} > n$. Both definitions yield the same notion of hyperimmunity, but our formulation will be more convenient for merging c.e. arrays.

Intuitively, an infinite set A is hyperimmune if not only one cannot find an infinite subset of it, but one cannot even approximate an infinite subset by giving blocks of elements, each of them capturing an element of A. It is clear from the definition that if A is hyperimmune, then A is immune.

Exercise 6.2.2 (Kuznecov, Medvedev, Uspenskii). Recall that the *principal function* of an infinite set $A = \{x_0 < x_1 < \dots\}$ is the function $p_A : \mathbb{N} \to \mathbb{N}$ defined by $p_A(n) = x_n$. Show that an infinite set A is hyperimmune iff its principal function p_A is hyperimmune, that is, is not dominated by any computable function. ★

Diagonal non-computability. Recall that a total function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally non-computable* (DNC) if $f(e) \neq \Phi_e(e)$ for every $e \in \mathbb{N}$. The degrees computing DNC function admit many characterizations, and thus are arguably natural. By Proposition 5.7.2, a set X computes a DNC function iff every Π^0_1 class of positive measure admits an infinite X-computable homogeneous set. Such degrees can also be formulated in terms of strong immunity.

Definition 6.2.3. Given a function $h : \mathbb{N} \to \mathbb{N}$, an infinite set A is *h-immune* if for every c.e. set W_e such that $W_e \subseteq A$, then $\operatorname{card} W_e \leq h(e)$. An infinite set is *effectively immune* if it is h-immune for some computable function $h : \mathbb{N} \to \mathbb{N}$. ◇

Theorem 6.2.4 (Jockusch [41])
Let X be a set. The following are equivalent.

1. *X computes a DNC function;*

2. X computes an effectively immune set;
3. X computes a fixpoint-free function.

Proof. $(1) \to (2)$: By Proposition 5.7.1, X computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, b \in \mathbb{N}$, if $\operatorname{card} W_e \leq b$, then $g(e,b) \notin W_e$. Let $D_0, D_1, \dots$ be a canonical enumeration of all non-empty finite sets. Let $h : \mathbb{N} \to \mathbb{N}$ be a partial computable function such that for every $e \in \mathbb{N}$, if $\operatorname{card} W_e > e$, then $D_{h(e)} \subseteq W_e$ and $\operatorname{card} D_{h(e)} = e + 1$. We shall construct an infinite increasing, X-computable sequence of integers $x_0 < x_1 < \dots$ such that for every $s \in \mathbb{N}$,

$$\forall e \leq s,\ (\operatorname{card} W_e > e \to D_{h(e)} \subsetneq \{x_i : i \leq s\}). \qquad (\star)$$

Then, $A = \{x_n : n \in \mathbb{N}\}$ is effectively immune, as witnessed by the identity function. Indeed, if $W_e \subseteq A$, then $\operatorname{card} W_e \leq e$. Assume $x_0 < \dots < x_s$ is already constructed, satisfying $(\star)$. Let [8]

$$W_{v(s)} = \{y : y \leq x_s\} \cup \bigcup_{e \leq s+1 \,\wedge\, h(e)\downarrow} D_{h(e)}$$

Note that the function $v : \mathbb{N} \to \mathbb{N}$ is X-computable, and $\operatorname{card} W_{v(s)} \leq x_s + 1 + \sum_{n \leq s+2} n$, so, letting $x_{s+1} = g(v(s), x_s + 1 + \sum_{n \leq s+2} n)$, we have $x_{s+1} \notin W_{v(s)}$. In particular, $x_{s+1} > x_s$ and $x_0, \dots, x_{s+1}$ satisfies $(\star)$. This completes the construction.

8: The left part $\{y : y \leq x_s\}$ of the union is to ensure that $x_{s+1} > x_s$, hence the set A is X-computable.

$(2) \to (3)$: Let $A \leq_T X$ be an h-effectively immune set, for some computable function $h : \mathbb{N} \to \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be an X-computable function such that $W_{f(e)}$ is the set of the $h(e) + 1$ first elements of A. We claim that f is a fixpoint-free function. Suppose for the contradiction that $W_{f(e)} = W_e$ for some $e \in \mathbb{N}$. Then $W_e \subseteq A$, but $\operatorname{card} W_e > h(e)$, contradiction.

$(3) \to (1)$: Let $f \leq_T X$ be a fixpoint-free function. Let $g : \mathbb{N} \to \mathbb{N}$ be the X-computable function such that for every n, $g(n)$ creates the code e_n of the function $m \mapsto \Phi_{\Phi_n(n)}(m)$[9], and outputs $f(e_n)$. We claim that g is DNC. Suppose for the contradiction that $g(n) = \Phi_n(n)$ for some $n \in \mathbb{N}$. Then by definition of g, $f(e_n) = \Phi_n(n)$. In particular, $\Phi_{f(e_n)} = \Phi_{\Phi_n(n)} = \Phi_{e_n}$. This contradicts the fact that f is fixpoint-free. ■

9: Here, $m \mapsto \Phi_{\Phi_n(n)}(m)$ is an abuse of notation for the program which, on input m, first executes $\Phi_n(n)$, and if it halts and outputs some e, executes $\Phi_e(m)$. In other words, the computation of $\Phi_n(n)$ is not part of the computation of g, hence g is total even if $\Phi_n(n)\uparrow$.

6.3 Hyperimmunity and WKL

Immunity and its variants form a unifying language to express custom invariant enabling to separate statements from Ramsey theory. The difficulty to separate to statements P and Q is to find a notion of immunity which is strong enough to be preserved by P, but weak enough not to be preserved by Q. This strengthening can often be obtained by studying the combinatorial features of the forcing question for P.

Let us consider the case of weak König's lemma, which captures the notion of compactness. Suppose one wants to prove that WKL preserves 1 immunity. This proof will fail, but one will exploit this failure to design a custom invariant. Fix an infinite immune set A, and let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be a non-empty Π^0_1 class. The natural notion of forcing to build members of Π^0_1 classes is Jockusch-Soare forcing $(\mathbb{P}, \leq)$, that is, the set of all infinite computable binary trees partially ordered by inclusion. Given a Turing index $e \in \mathbb{N}$, one wants to force the following requirement:

$\mathcal{R}_e$: W_e^G is not an infinite subset of A.

Recall that Jockusch-Soare forcing admits the following natural forcing question for Σ^0_1 formulas: Given a Σ^0_1-formula $\varphi(G)$, let $T \mathrel{?\vdash} \varphi(G)$ hold if there is some level $\ell \in \mathbb{N}$ such that for every $\sigma \in T \cap 2^\ell$, $\varphi(\sigma)$ holds. This forcing question is Σ^0_1-preserving and Σ^0_1-compact. The proof of $\mathcal{R}_e$ usually goes as follows: Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Turing index e, if T does not force W_e^G to be an infinite subset of A, then there is an extension $S \subseteq T$ forcing $\mathcal{R}_e$. If, on the other hand, T already forces W_e^G to be an infinite subset of A, then exploit the forcing question to compute an infinite subset of A, contradicting immunity of A.

Suppose we are in the second case. Given some $n \in \mathbb{N}$, one wants to find computably an element $x > n$ in A. The problem comes from the difference between the following two statements:

$$T \mathrel{?\vdash} \exists x(x > n \wedge x \in W_e^G) \quad \text{and} \quad \exists x\ (T \mathrel{?\vdash} x > n \wedge x \in W_e^G)$$

Assuming T forces W_e^G to be an infinite subset of A, the left statement holds, as otherwise, one would find an extension forcing W_e^G to be bounded by n, hence to be finite. On the other hand, the right statement does not hold in general. It might be that for each individual $x > n$, $T \mathrel{?\nvdash} x \in W_e^G$, but $T \mathrel{?\vdash}$ "W_e^G is infinite ". Thankfully, by Σ^0_1-compactness of the forcing question, one has the following implication

$$T \mathrel{?\vdash} \exists x(x > n \wedge x \in W_e^G) \ \rightarrow\ \exists F \text{ finite } (T \mathrel{?\vdash} \min F > n \wedge F \cap W_e^G \neq \emptyset)$$

Moreover, for any such F, we claim that $A \cap F \neq \emptyset$. Indeed, by definition of the forcing question, there is an extension $S \subseteq T$ forcing $F \cap W_e^G \neq \emptyset$, but S also forces $W_e^G \subseteq A$. Last, since the forcing question is Σ^0_1-preserving, for every n, one can computably find some F_n such that $F_n \cap A \neq \emptyset$ and $\min F_n > n$. In order to obtain a contradiction, one therefore must assume that no infinite subset of A can be approximated by finite sets, hence that A is hyperimmune. It happens that this is a sufficient invariant. Indeed, a finite union of finite sets is again a finite set.[10]

Statements from Ramsey theory do not usually imply weak König's lemma, and therefore might preserve a weaker form of immunity. For instance, the "compactness part" of Ramsey's theorem for pairs is the Ramsey-type weak König's lemma (RWKL).[11] However, it is often not necessary to consider the optimal invariant, and in many cases, on works with variants of hyperimmunity as soon as the statement contains some amount of compactness.

10: The computably dominated basis theorem for Π^0_1 classes is a much stronger form of preservation of 1 hyperimmunity, in the sense that every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ has a member G such that every hyperimmune function is G-hyperimmune.

11: This sentence has to be taken in an informal sense. On one hand, $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \rightarrow \mathsf{RWKL}$, so the compactness part of RT^2_2 is at least RWKL. For the converse, the usual notion of forcing for Ramsey's theorem for pairs with a good first-jump control can be done with reservoirs restricted to any ω-model of $\mathsf{RCA}_0 + \mathsf{RWKL}$.

6.4 Erdős-Moser theorem

Let us step up and separate two statements from Ramsey's theory with very similar combinatorics: the Erdős-Moser theorem and Ramsey's theorem for pairs. The *Erdős-Moser theorem* is a statement about tournaments at the intersection of graph theory and Ramsey theory. A *tournament*[12] over an infinite domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation $T \subseteq D^2$ such that for every $a, b \in D$ with $a \neq b$, $T(a, b)$ iff $\neg T(b, a)$. The tournament T is *transitive* if for all $a, b, c \in D$, if $T(a, b)$ and $T(b, c)$ hold, then $T(a, c)$ also holds.[13] A

12: This formalizes real-world tournaments: Intuitively, $T(a, b)$ if Player a beats Player b in a tournament. In general, a tournament is not transitive, that is, it might be that Player a beats Player b, who beats Player c, who himself beats Player a.

13: It is important to note that transitivity is a property over $[D]^3$. Thus, if a tournament is not transitive, then it is witnessed by a 3-tuple of elements of D.

sub-tournament of T is the restriction of T to a subdomain $D_1 \subseteq D$. Thus, given T, a sub-tournament is fully specified by the sub-domain D_1, and is therefore identified with it, and we say that D_1 is T*-transitive* if T is transitive on D_1.

The Erdős-Moser theorem states that every infinite tournament admits an infinite transitive subtournament. It can be seen as a Π^1_2 problem EM whose instances are tournaments on $\mathbb{N}$, and whose solutions are infinite domains on which the tournament is transitive. It follows from Ramsey's theorem for pairs and two colors by defining, given a tournament T on $\mathbb{N}$, a coloring $f : [\mathbb{N}]^2 \to 2$ such that for every $a < b$, $f(a,b) = 1$ iff $T(a,b)$. Then any infinite f-homogeneous set is T-transitive.[14]

14: The Erdős-Moser theorem was first studied in reverse mathematics by Bovykin and Weiermann [42]. Lerman, Solomon and Towsner [43] proved that EM is strictly weaker than RT^2_2 over RCA_0, later simplified by Patey [44].

Recall from Section 5.3 that RWKL is the Π^1_2 problem whose instances are infinite binary trees, and whose solutions are infinite homogeneous sets.[15] The following lemma shows that EM has the same amount of compactness as RT^2_2.

15: By Definition 5.3.1, given an infinite tree $T \subseteq 2^{<\mathbb{N}}$, a finite set $F \subseteq \mathbb{N}$ is T-*homogeneous* for color $i < 2$ if $\{\sigma \in T : (\forall x \in F)\sigma(x) = i\}$ is infinite. An infinite set H is T*-homogeneous* if every finite subset of H is T-homogeneous.

Exercise 6.4.1 (Bienvenu, Patey and Shafer [37]). Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. For each $s \in \mathbb{N}$, let σ_s be the left-most element of T of length s. Define a tournament T as follows: For $x < s$, if $\sigma_s(x) = 1$, then $R(x,s)$ holds and $R(s,x)$ fails. Otherwise, if $\sigma_s(x) = 0$, then $R(x,s)$ fails and $R(s,x)$ holds. Show that every infinite transitive subtournament computes an infinite T-homogeneous set. ★

Looking at the standard notion of forcing for Ramsey's theorem for pairs and for the Erdős-Moser theorem, the combinatorics are very similar, except that Ramsey's theorem for pairs is a disjunctive statement. Forcing multiple requirements is not an issue for the Erdős-Moser theorem. On the other hand, the situation for disjunctive statements is more delicate: if one forces requirements of the form $\mathscr{R} \vee \mathscr{R}$ and $\mathscr{S} \vee \mathscr{S}$, it might be that the $\mathscr{R}$-requirements and the $\mathscr{S}$-requirements are not satisfied on the same side.[16] This motivates the following definition:

16: It is sometimes possible to satisfy multiple requirements using a pairing argument, by forcing all the possible disjunctive pairs: $\mathscr{R} \vee \mathscr{R}$, $\mathscr{S} \vee \mathscr{S}$, $\mathscr{R} \vee \mathscr{S}$ and $\mathscr{S} \vee \mathscr{R}$.

Definition 6.4.2. A problem P admits *preservation of* k *hyperimmunities* if for every set Z and every k-tuple of Z-hyperimmune functions $f_0, \dots, f_{k-1}$, every Z-computable instance X of P admits a solution Y such that each f_i is $Z \oplus Y$-hyperimmune. ◊

We now prove that the Erdős-Moser theorem admits preservation of ω hyperimmunities.

Theorem 6.4.3 (Patey [44])
Let $h_0, h_1, \dots$ be a countable collection of hyperimmune functions, and let $T \subseteq \mathbb{N}^2$ be a computable tournament. There is an infinite T-transitive subtournament $G \subseteq T$ such that every h_i is G-hyperimmune.

Proof. Given two sets $E, F \subseteq \mathbb{N}$, we write $E \to_T F$ if for every $x \in E$ and every $y \in F$, $T(x,y)$. A set X is in a *minimal* T*-interval* of F if for every $a \in F$, either $\{a\} \to_T X$, or $X \to_T \{a\}$.[17]

17: One can actually define the notion of T-interval $(a,b)_T$ to be the set of all $x \in \mathbb{N}$ such that $T(a,x)$ and $T(x,b)$ (see [43]), but for our purpose, it is sufficient to work with a coarser definition.

Consider the notion of forcing whose *conditions*[18] are Mathias conditions (σ, X) such that

18: One would naturally be tempted to define a condition as a pair satisfying Items 1 and 3. Actually, Item 2 is already sufficient to ensure extendibility of the stem, but it requires some extra work. With the actual definition, one can simply apply the Erdős-Moser theorem to $T{\restriction}[X]^2$ to obtain an infinite T-transitive subset $Y \subseteq X$, and thanks to Item 1 and Item 2, $\sigma \cup Y$ is T-transitive.

1. $\sigma \cup \{x\}$ is T-transitive for every $x \in X$;
2. X is in a minimal T-interval of σ;[19]

19: Note that this property can be obtained for free by considering the map $g : X \to 2^{|\sigma|}$ which to x associates the string ρ of length $|\sigma|$ such that for every $y < |\sigma|$, $\rho(y) = 1$ iff $T(y,x)$ holds. By the pigeonhole principle, there is an infinite X-computable g-homogeneous subset $Y \subseteq X$. Any such Y is in a minimal T-interval of σ.

3. h_i is X-hyperimmune for every $i \in \mathbb{N}$.

The notion of extension is exactly Mathias extension. Every filter $\mathcal{F}$ induces a set $G_{\mathcal{F}}$ defined by $\bigcup\{\sigma : (\sigma, X) \in \mathcal{F}\}$. The following lemma shows that $G_{\mathcal{F}}$ is infinite for every sufficiently generic filter $G_{\mathcal{F}}$.

Lemma 6.4.4. Let $p = (\sigma, X)$ be a condition. There is an extension (τ, Y) of p and some $n > |\sigma|$ such that $n \in \tau$. ★

Proof. Pick any $n \in X$. Let $\tau = \sigma \cup \{n\}$, and Y be either $\{x \in X : T(n, x)\}$ or $\{x \in X : T(x, n)\}$, depending on which one is infinite. Then, $(\tau, Y \setminus \{0, \dots, n-1\})$ is an extension of p such that $n \in \tau$. ■

This notion of forcing admits a non-disjunctive, Σ^0_1-preserving, Σ^0_1-compact forcing question.

Definition 6.4.5. Let $p = (\sigma, X)$ be a condition, and let $\varphi(G)$ be a Σ^0_1-formula. Let $p \mathrel{?\vdash} \varphi(G)$ hold if for every 2-partition $Z_0 \sqcup Z_1 = X$, there is some $i < 2$ and some finite T-transitive set $\rho \subseteq Z_i$ such that $\varphi(\sigma \cup \rho)$ holds.[20] ◊

20: Note the similarity of this forcing question with the one from Exercise 3.4.12.

Note that by compactness, the forcing question is $\Sigma^0_1(X)$. The following lemma states that the forcing question meets its specification.

Lemma 6.4.6. Let $p = (\sigma, X)$ be a condition, and let $\varphi(G)$ be a Σ^0_1-formula.

1. If $p \mathrel{?\vdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\varphi(G)$;
2. If $p \mathrel{?\nvdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\neg\varphi(G)$. ★

Proof. Suppose first $p \mathrel{?\vdash} \varphi(G)$. Then, by compactness, there is some threshold $\ell \in \mathbb{N}$ such that for every 2-partition $Z_0 \sqcup Z_1 = X{\upharpoonright}\ell$, there is some $i < 2$ and some finite T-transitive set $\rho \subseteq Z_i$ such that $\varphi(\sigma \cup \rho)$ holds. For every $x \in X \setminus \{0, \dots, \ell\}$, let σ_x be the binary string of length ℓ such that for every $y < \ell$, $T(y, x) = \sigma_x(y)$. By the pigeonhole principle, there is some string σ of length ℓ and an infinite X-computable subset $Y \subseteq X \setminus \{0, \dots, \ell\}$ such that for $\sigma = \sigma_x$ for every $x \in Y$. Let $Z_i = X \cap \{y : \sigma(y) = i\}$ for each $i < 2$. By assumption, there is some $i < 2$ and some finite T-transitive set $\rho \subseteq Z_i$ such that $\varphi(\sigma \cup \rho)$ holds. We claim that $(\sigma \cup \rho, Y)$ is an extension of p forcing $\varphi(G)$.

Suppose now $p \mathrel{?\nvdash} \varphi(G)$. Let $\mathscr{C}$ be the $\Pi^0_1(X)$ class of all $Z_0 \oplus Z_1$ such that, $Z_0 \sqcup Z_1 = X$ and for every $i < 2$ and every finite T-transitive set $\rho \subseteq Z_i$, $\varphi(\sigma \cup \rho)$ does not hold. By the computably dominated basis theorem (see Jockusch and Soare [9]), there is some 2-partition $Z_0 \sqcup Z_1 = X$ such that $Z_0 \oplus Z_1 \oplus X$ is computably X-dominated. In particular, each h_i is $Z_0 \oplus Z_1 \oplus X$-hyperimmune. Let $i < 2$ be such that Z_i is infinite. Then (σ, Z_i) is an extension of p forcing $\neg\varphi(G)$. ■

The following lemma is an adaptation of Theorem 3.6.4.

Lemma 6.4.7. Let $p = (\sigma, X)$ be a condition. For every Turing index e and every $i \in \mathbb{N}$, there is an extension $q \leq p$ forcing Φ^G_e not to dominate h_i.[21] ★

21: By this, we mean forcing either Φ^G_e to be partial, or $\Phi^G_e(x) < h_i(x)$ for some $x \in \mathbb{N}$.

Proof. Let $?\vdash$ be the forcing question of Definition 6.4.5. Suppose first that $p \mathrel{?\nvdash} \exists v \Phi_e^G(x)\downarrow = v$ for some $x \in \mathbb{N}$. Then by Lemma 6.4.6(2), there is an extension $q \leq p$ forcing $\Phi_e^G(x)\uparrow$, and we are done. Suppose now that for every $x \in \mathbb{N}$, $p \mathrel{?\vdash} \exists v \Phi_e^G(x)\downarrow = v$. By Σ_1^0-compactness of the forcing question, for every $x \in \mathbb{N}$, there is a finite set $F_x \subseteq \mathbb{N}$ such that $p \mathrel{?\vdash} \exists v \in F_x\, \Phi_e^G(x)\downarrow = v$. Let $g : \mathbb{N} \to \mathbb{N}$ be the function which on input x, looks for some finite set F_x such that $p \mathrel{?\vdash} \exists v \in F_x\, \Phi_e^G(x)\downarrow = v$ and outputs $\max F_x$. Such a function is total by hypothesis, and X-computable since the forcing question is $\Sigma_1^0(X)$. Since h_i is X-hyperimmune, $g(x) < h_i(x)$ for some $x \in \mathbb{N}$. By Lemma 6.4.6(1), there is an extension $q \leq p$ forcing $\exists v \in F_x \Phi_e^G(x)\downarrow = v$. Since $h_i(x) > \max F_x$, q forces $\Phi_e^G(x)\downarrow < h_i(x)$. ■

We are now ready to prove Theorem 6.4.3. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing,. By Lemma 6.4.4, $G_{\mathcal{F}}$ is infinite. Moreover, by Lemma 6.4.7, h_i is $G_{\mathcal{F}}$-hyperimmune for every $i \in \mathbb{N}$. This completes the proof of Theorem 6.4.3. ■

The following proposition shows that RT^2_2 does not admit preservation of 2 hyperimmunities.

Proposition 6.4.8. There exists two hyperimmune functions $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ and a computable coloring $f : [\mathbb{N}]^2 \to 2$ such that for every infinite f-homogeneous set H for color i, g_i is not H-hyperimmune. ⋆

Proof. Let $A_0 \sqcup A_1$ be a Δ_2^0 2-partition such that A_0 and A_1 are hyperimmune, and let $g_i = p_{A_i}$ be the principal function of A_i for each $i < 2$. By Shoenfield's limit lemma, there is a computable function $f : [\mathbb{N}]^2 \to 2$ such that for every x, $\lim_y f(x, y)$ exists, and equals i iff $x \in A_i$. For every infinite f-homogeneous set H for color i, $H \subseteq A_i$. In particular, p_H dominates g_i, so g_i is not H-hyperimmune. ■

Corollary 6.4.9 (Lerman, Solomon and Towsner [43])
EM *does not imply* RT^2_2 *over* RCA_0.

Proof. Immediate by Proposition 6.4.8, Theorem 6.4.3 and Corollary 6.1.4. ■

Consider three kinds of requirement $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$. Suppose one can construct solutions to Ramsey's theorem for pairs and two colors by satisfying requirements of type $\mathcal{R} \vee \mathcal{R}$, $\mathcal{S} \vee \mathcal{S}$ and $\mathcal{T} \vee \mathcal{T}$. By the pigeonhole principle, there must be a side preserving two kinds of requirements simultaneously. In the case of preservation of hyperimmunities, it yields that, given 3 hyperimmune functions, one can always construct solutions to computable instances of RT^2_2 while preserving two among the three hyperimmunities simultaneously. We leave the proofs as an exercise.

Exercise 6.4.10 (Patey [45]). A problem P admits *preservation of ℓ among k hyperimmunities* if for every set Z and every k-tuple of Z-hyperimmune functions $f_0, \dots, f_{k-1}$, every Z-computable instance X of P admits a solution Y and some finite set $F \in [k]^\ell$ such that for each $i \in F$, f_i is $Z \oplus Y$-hyperimmune.

1. Show that RT^2_3 does not admit preservation of 3 among 3 hyperimmuni-

ties.[22]

2. Show that RT^2_2 admits preservation of 2 among 3 hyperimmunities.[23] ★

22: Hint: Adapt the proof of Proposition 6.4.8).

23: Hint: Adapt the proof of Theorem 6.4.3, but with the notion of forcing of Exercise 3.4.12.

6.5 Partial orders

Partial orders also provide a good family of Ramsey-type theorems requiring custom preservations properties. A *partial order* is a pair $\mathcal{P} = (D, <_{\mathcal{P}})$, where $D \subseteq \mathbb{N}$ and $<_{\mathcal{L}}$ is an irreflexive transitive binary relation over D. A set $X \subseteq D$ is an *chain (antichain)* if every two elements of X are comparable (incomparable) over $<_{\mathcal{P}}$. A set $X \subseteq D$ is an *ascending (descending) sequence* if for every $x, y \in X$, $x < y$ iff $x <_{\mathcal{P}} y$ ($x >_{\mathcal{P}} y$). The *Chain AntiChain* principle[24] (CAC) is the Π^1_2-problem whose instances are partial orders over $\mathbb{N}$ and whose solutions are infinite chains or infinite antichains.

24: This principle was studied by Hermann [21] and Hirschfeldt and Shore [23] in reverse mathematics.

Exercise 6.5.1 (Hirschfeldt and Shore [23]). Show that $\mathsf{RCA}_0+\mathsf{CAC}$ proves that every partial order on $\mathbb{N}$ admits either an infinite ascending or descending sequence, or an infinite antichain. ★

Exercise 6.5.2 (Hirschfeldt and Shore [23]). A coloring $f : [\mathbb{N}]^2 \to k$ is *transitive for color $i < k$* if for every $x < y < z$ such that $f(x, y) = f(y, z) = i$, then $f(x, z) = i$. Show that CAC is equivalent over RCA_0 to the statement "For every transitive coloring $f : [\mathbb{N}]^2 \to 2$ for some color, there is an infinite f-homogeneous set." ★

Exercise 6.5.3 (Herrmann [21]). Construct a computable partial order on $\mathbb{N}$ with no infinite computable chain or antichain. ★

As it happens, building either an ascending or a descending sequence has better combinatorial properties than building a chain. We shall therefore build a strong solution to CAC, in the sense of Exercise 6.5.1. The corresponding notion of forcing admits a forcing question for Σ^0_1 formulas which is strongly Σ^0_1-compact, in that if $p \mathrel{?\vdash} \exists x \varphi(G, x)$, then there is a set F of size 3 such that $p \mathrel{?\vdash} (\exists x \in F)\varphi(G, x)$. Following the process of Section 6.3, this yields the following notion of immunity:

Definition 6.5.4. A *c.e. k-array* is a c.e. array $\{D_{f(n)} : n \in \mathbb{N}\}$ such that $\operatorname{card} D_{f(n)} \leq k$ for each n. An infinite set $A \subseteq \mathbb{N}$ is *k-immune* if for every c.e. k-array $\{D_{f(n)} : n \in \mathbb{N}\}$, there is some n such that $A \cap D_{f(n)} = \emptyset$. A set A is *constant-bound immune (c.b-immune)* if it is k-immune for every $k \in \mathbb{N}$.◇

Constant-bound immunity is a strong form of immunity. The following exercise shows that two notions coincide on co-c.e. sets.

Exercise 6.5.5. Let A be a co-c.e. set. Show that A is immune iff A is c.b-immune. ★

As usual, every notion of immunity induces a preservation property.

Definition 6.5.6. A problem P admits *preservation of 1 c.b-immuniy* if for every set Z and every c.b-Z-immune set A, every Z-computable instance X of P admits a solution Y such that A is c.b-$Z \oplus Y$-immune. ◇

We now prove that CAC admits preservation of 1 c.b-immuniy.

Theorem 6.5.7 (Patey [46])
Let A be a c.b-immune set, and $\mathcal{P} = (\mathbb{N}, <_{\mathcal{P}})$ be a computable partial order. Then there is either an infinite ascending or descending sequence G, or an infinite antichain G such that A is c.b-G-immune.

Proof. Consider the notion of forcing whose *conditions* are 4-tuples $(\sigma_0, \sigma_1, \sigma_2, X)$, where

1. (σ_i, X) is a Mathias condition for each $i < 3$;
2. $\sigma_0 \cup \{x\}$, $\sigma_1 \cup \{x\}$ and $\sigma_2 \cup \{x\}$ form respectively an ascending sequence, a descending sequence and an antichain, for each $x \in X$;
3. X is computable.[25]

25: Having a notion of forcing with a good first-jump control while keeping the reservoir computable is a good indicator that the statement does not imply any form of compactness.

A condition $(\tau_0, \tau_1, \tau_2, Y)$ *extends* $(\sigma_0, \sigma_1, \sigma_2, X)$ if (τ_i, Y) Mathias extends (σ_i, X) for every $i < 3$. One can therefore see a condition as three simultaneous Mathias conditions sharing a same reservoir. Every filter $\mathcal{F}$ induces three sets: $G_{0,\mathcal{F}}$, $G_{1,\mathcal{F}}$ and $G_{2,\mathcal{F}}$, defined by $G_{i,\mathcal{F}} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, \sigma_2, X) \in \mathcal{F}\}$.

As in the proof of Theorem 3.4.6, if $\mathcal{F}$ is a sufficiently generic filter, then $G_{i,\mathcal{F}}$ is not necessarily infinite. We shall therefore make the following hypothesis:

> (H1): For every infinite computable set X, there is some $x_0, x_1, x_2 \in X$ such that $\{y \in X : x_0 <_{\mathcal{P}} y\}$, $\{y \in X : x_1 >_{\mathcal{P}} y\}$ and $\{y \in X : x_2 \mid_{\mathcal{P}} y\}$ are all infinite.

If the (H1) hypothesis fails for some set X, one can computably thin it out to obtain an infinite subset $Y \subseteq X$ which avoids one of the three behaviors. One then restarts the construction with conditions whose reservoirs are subsets of Y. The conditions will then have less stems, and the forcing questions must be adapted accordingly.

Lemma 6.5.8. Suppose (H1) holds. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $i < 3$. There is an extension $(\tau_0, \tau_1, \tau_2, Y)$ of p and some $x > |\sigma_i|$ such that $x \in \tau_i$. ★

Proof. Say $i = 0$. Then two other cases are similar. By (H1), there is some $x_0 \in X$ such that $Y = \{y \in X : x_0 <_{\mathcal{P}} y\}$ is infinite. Let $\tau_0 = \sigma_0 \cup \{x_0\}$, and $\tau_i = \sigma_i$ otherwise. Then, $(\tau_0, \tau_1, \tau_2, Y)$ is an extension of p such that $x_0 \in \tau_0$. ■

We now define a disjunctive forcing question for Σ^0_1-formulas. Given a condition $p = (\sigma_0, \sigma_1, \sigma_2, X)$, a *split triple* is a 3-tuple (ρ_0, ρ_1, ρ_2) such that $\rho_i \subseteq X$ for each $i < 3$, ρ_0 is ascending, ρ_1 is descending, ρ_2 is an antichain, and for every $x \in \rho_2$, $\max_{\mathcal{P}}(\rho_0) <_{\mathcal{P}} x <_{\mathcal{P}} \min_{\mathcal{P}}(\rho_1)$.[26]

26: In other words, every element of the ascending sequence ρ_0 is below (with respect to $<_{\mathcal{P}}$) every element of the antichain ρ_2, and every element of ρ_2 is below every element of the descending sequence ρ_1.

Definition 6.5.9. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $\varphi_0(G)$, $\varphi_1(G)$ and $\varphi_2(G)$ be three Σ^0_1-formulas. Let $p \mathrel{?{\vdash}} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$ hold if there is a split triple (ρ_0, ρ_1, ρ_2) such that for each $i < 3$, $\varphi_i(\sigma_i \cup \rho_i)$ holds.◇

Note that being a split triple is a decidable predicate, hence the forcing question is Σ^0_1-preserving. The following lemma shows that the forcing question meets its specification.

Lemma 6.5.10. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $\varphi_0(G)$, $\varphi_1(G)$ and $\varphi_2(G)$ be three Σ^0_1-formulas.

1. If $p \mathrel{?{\vdash}} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$, then there is some $i < 3$ and some

extension $q \leq p$ forcing $\varphi_i(G_i)$.

2. If $p \mathrel{?}\nvdash \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$, then there is some $i < 3$ and some extension $q \leq p$ forcing $\neg\varphi_i(G_i)$. ⋆

Proof. Suppose first $p \mathrel{?}\vdash \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$ holds, as witnessed by some split triple (ρ_0, ρ_1, ρ_2). By the pigeonhole principle, there is some infinite X-computable subset $Y \subseteq X$ such that for every $x \in \rho_0 \cup \rho_1 \cup \rho_2$, either for every $y \in Y$, $x <_{\mathcal{P}} y$, or for every $y \in Y$, $x >_{\mathcal{P}} y$, or for every $y \in Y$, $x |_{\mathcal{P}} y$. We say that x is *small* if it is on the first case, *large* if it is on the second case, and *isolated* if it is on the third case. If every $x \in \rho_2$ is isolated, then the condition $(\sigma_0, \sigma_1, \sigma_2 \cup \rho_2, Y)$ is an extension of p forcing $\varphi_2(G_2)$. If some $x \in \rho_2$ is small, then every element in ρ_0 is small, so $(\sigma_0 \cup \rho_0, \sigma_1, \sigma_2, Y)$ is an extension of p forcing $\varphi_0(G_0)$. Last, if some $x \in \rho_2$ is large, then every element in ρ_1 is large, thus $(\sigma_0, \sigma_1 \cup \rho_1, \sigma_2, Y)$ is an extension of p forcing $\varphi_1(G_1)$.

Suppose now $p \mathrel{?}\nvdash \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$. We have two cases. Case 1: there are two sets $\rho_0, \rho_1 \subseteq X$ such that ρ_0 is ascending, ρ_1 is descending, and the set $Y = \{x \in X : \max_{\mathcal{P}} \rho_0 <_{\mathcal{P}} x <_{\mathcal{P}} \min_{\mathcal{P}} \rho_1\}$ is infinite. Then the condition $q = (\sigma_0 \cup \rho_0, \sigma_1 \cup \rho_1, \sigma_2, Y)$ is an extension forcing $\neg\varphi_2(G_2)$. Indeed, if there is an extension $r = (\tau_0, \tau_1, \tau_2, Z)$ of q such that $\varphi_2(\tau_2)$ holds, then, letting $\rho_2 = \tau_2 \setminus \sigma_2$, the tuple (ρ_0, ρ_1, ρ_2) forms a split triple contradicting our hypothesis. Case 2: there are no such two sets. Then we claim that p already forces $\neg\varphi(G_0) \vee \neg\varphi(G_1)$. Indeed, if there is some extension $q = (\tau_0, \tau_1, \tau_2, Y)$ of p such that $\varphi_0(\tau_0)$ and $\varphi_1(\tau_1)$ both hold, then, letting $\rho_i = \tau_i \setminus \sigma_i$, the sets ρ_0, ρ_1 witness Case 1. Thus there is an extension of p forcing either $\neg\varphi(G_0)$, or $\neg\varphi(G_1)$. ∎

By definition of the forcing question, if

$$p \mathrel{?}\vdash \exists x \varphi_0(G_0, x) \vee \exists x \varphi_1(G_1, x) \vee \exists x \varphi_2(G_2, x)$$

then there are three elements $n_0, n_1, n_2 \in \mathbb{N}$ such that

$$p \mathrel{?}\vdash \varphi_0(G_0, n_0) \vee \varphi_1(G_1, n_1) \vee \varphi_2(G_2, n_2)$$

This can be seen as some strong form of Σ^0_1-compactness, where the finite set is of size at most 3.

Lemma 6.5.11. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $\Phi_{e_0}, \Phi_{e_1}, \Phi_{e_2}$ be three c.e. k-array functionals.[27] There is an extension q of p forcing $\Phi^{G_i}_{e_i}$ to be partial, or $\Phi^{G_i}_{e_i}(n)\downarrow \cap A = \emptyset$ for some $n \in \mathbb{N}$. ⋆

27: By this, we mean that for every oracle Z, if $\Phi^Z_{e_i}(n)\downarrow$, then its output is a finite set F of size at most k with $\min F > n$.

Proof. Suppose first that $p \mathrel{?}\nvdash \Phi^{G_0}_{e_0}(n)\downarrow \vee \Phi^{G_1}_{e_1}(n)\downarrow \vee \Phi^{G_2}_{e_2}(n)\downarrow$ for some n. Then by Lemma 6.5.10(2), there is an extension q of p forcing $\Phi^{G_i}_{e_i}(n)\uparrow$ for some $i < 3$.

Suppose now that for every $n \in \mathbb{N}$, $p \mathrel{?}\vdash \Phi^{G_0}_{e_0}(n)\downarrow \vee \Phi^{G_1}_{e_1}(n)\downarrow \vee \Phi^{G_2}_{e_2}(n)\downarrow$. Then for each $n \in \mathbb{N}$, there is some finite set E_n of size at most $3k$ such $p \mathrel{?}\vdash \Phi^{G_0}_{e_0}(n)\downarrow\subseteq E_n \vee \Phi^{G_1}_{e_1}(n)\downarrow\subseteq E_n \vee \Phi^{G_2}_{e_2}(n)\downarrow\subseteq E_n$. Moreover, since the forcing question is Σ^0_1-preserving, then the map $n \mapsto E_n$ is computable, so $(E_n : n \in \mathbb{N})$ forms a c.e. $3k$-array. By c.b-immunity of A, there is some $n \in \mathbb{N}$ such that $E_n \cap A = \emptyset$. By Lemma 6.5.10(1), there is an extension q of p forcing $\Phi^{G_i}_{e_i}(n)\downarrow\subseteq E_n$ for some $i < 3$. In particular, q forces $\Phi^{G_i}_{e_i}(n)\downarrow \cap A = \emptyset$. ∎

We are now ready to prove Theorem 6.5.7 in the case (H1) holds. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. For each $i < 3$, let $G_i = G_{\mathcal{F},i}$. By Lemma 6.5.8, G_i is infinite for every $i < 3$. By Lemma 6.5.11, there is some $i < 3$ such that A is c.b-G_i-immune. The case where (H1) does not hold is left to the reader, and consists in a degenerate forcing construction. This completes the proof of Theorem 6.5.7. ■

Looking at the proof of Theorem 6.5.7, the core of the combinatorics lies in the existence of a Σ^0_1-preserving forcing question which admits the following strong form of Σ^0_1-compactness.

Definition 6.5.12. Given a notion of forcing $(\mathbb{P}, \leq)$, a forcing question is *constant-bound Σ^0_n-compact* if for every $p \in \mathbb{P}$, there is some $k \in \mathbb{N}$ such that for every Σ^0_n formula $\varphi(G, x)$, if $p \mathrel{?\vdash} \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ of size k such that $p \mathrel{?\vdash} \exists x \in F\ \varphi(G, x)$. ◊

We leave the following abstract theorem of preservation of 1 c.b-immunity as an exercise.

Exercise 6.5.13. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a constant-bound Σ^0_1-compact, Σ^0_1-preserving forcing question. Show that for every c.b-immune set A and every sufficiently generic filter $\mathcal{F}$, A is c.b-immmune relative to $G_{\mathcal{F}}$.★

Let DNC be the Π^1_2-problem whose instances are any sets, and, given a set X, a solution is a DNC function relative to X. Recall that by Section 5.7, DNC can be seen as a form of compactness statement, in that it is equivalent to the Ramsey-type weak weak König's lemma (see Proposition 5.7.2). The following theorem therefore shows, as expected, that DNC not to admit preservation of constant-bound immunity.

Theorem 6.5.14 (Patey [46])
There is a Δ^0_2, c.b-immune set $A \subseteq \mathbb{N}$ such that every DNC function computes an infinite subset.

Proof. Let $\mu_{\emptyset'}$ be the modulus of $\emptyset'$, that is, such that $\mu_{\emptyset'}(x)$ is the minimum stage s at which $\emptyset'_s \upharpoonright x = \emptyset' \upharpoonright x$.[28]

28: Note that this modulus is *left-c.e.*, that is, there is a uniformly computable sequence of functions $g_0, g_1, \dots$ such that for every $s, x \in \mathbb{N}$, $g_s(x) \leq g_{s+1}(x) \leq \mu_{\emptyset'}(x)$. In other words, the set $\{(x, y) : y < \mu_{\emptyset'}(x)\}$ is c.e.

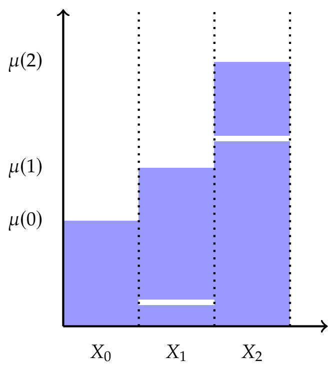

Figure 6.1: The set A (in blue) is a countable union of some finite initial segments $F_0, F_1, \dots$ of the columns $X_0, X_1, \dots$, from which finitely many elements have been removed in a c.e. way. The holes in the columns are the elements of W.

Computably split $\mathbb{N}$ into countably many columns $X_0, X_1, \dots$ of infinite size. For example, set $X_i = \{\langle i, n\rangle : n \in \mathbb{N}\}$ where $\langle \cdot, \cdot \rangle$ is the Cantor bijection from $\mathbb{N}^2$ to $\mathbb{N}$. For each i, let F_i be the set of the $\mu_{\emptyset'}(i)$ first elements of X_i. The sequence $F_0, F_1, \dots$ is $\emptyset'$-computable. Assume for now that we have defined a c.e. set W such that the Δ^0_2 set $A = \bigcup_i F_i \setminus W$ is c.b-immune, and such that $|X_i \cap W| \leq i$. We claim that every DNC function computes an infinite subset of A.

Let f be any DNC function. By Proposition 5.7.1, f computes a function $g(\cdot, \cdot, \cdot)$ such that whenever $|W_e| \leq n$, then $g(e, n, i) \in X_i \setminus W_e$.[29] For each i, let e_i be the index of the c.e. set $W_{e_i} = W \cap X_i$, and let $n_i = g(e_i, i, i)$. Since $|X_i \cap W| \leq i$, then $|W_{e_i}| \leq i$, so $n_i = g(e_i, i, i) \in X_i \setminus W_{e_i}$, which implies $n_i \in X_i \setminus W$. We then have two cases.

29: The function g can be obtained from Proposition 5.7.1 by "renaming" the elements of X_i using the bijection between X_i and $\mathbb{N}$.

- Case 1: $n_i \in F_i$ for infinitely many i's. One can f-computably find infinitely many of them since $\mu_{\emptyset'}$ is left-c.e. and the sequence of the n's is f-computable. Therefore, one can f-computably find an infinite subset of $\bigcup_i F_i \setminus W = A$.

- Case 2: $n_i \in F_i$ for only finitely many i's. Then the sequence of the n_i's eventually dominates the modulus function $\mu_{\emptyset'}$, and therefore computes the halting set. Since the set A is Δ^0_2, f computes an infinite subset of A.

We now detail the construction of the c.e. set W. In what follows, interpret Φ_e as a partial computable sequence of finite sets such that if $\Phi_e(x)$ halts, then $\min(\Phi_e(x)) > x$. We need to satisfy the following requirements for each $e, k \in \mathbb{N}$:

$$\mathcal{R}_{e,k}: \quad \begin{array}{l}[\Phi_e \text{ total } \wedge (\forall i)(\forall^\infty x)(\Phi_e(x) \cap X_i = \emptyset)] \\ \rightarrow (\exists x)\,[|\Phi_e(x)| > k \vee \Phi_e(x) \subseteq W]\end{array}$$

We furthermore want to ensure that $|X_i \cap W| \leq i$ for each i. We can prove by induction over k that if $\mathcal{R}_{e,\ell}$ is satisfied for each $\ell \leq k$, then the set $A = \bigcup_i F_i \backslash W$ is k-immune. The case $k = 1$ is trivial, since if Φ_e is a total c.e. 1-array and $\exists^\infty x \Phi_e(x) \cap X_i \neq \emptyset$, then $\exists^\infty x \Phi_e(x) \subseteq X_i$, so $\exists x \Phi_e(x) \subseteq (X_i \backslash F_i) \subseteq \overline{A}$. For the case $k \geq 2$, assume that Φ_e is a total c.e. k-array. If the right-hand side of the implication $\mathcal{R}_{e,k}$ holds, then we are done, so suppose it does not hold. In particular, the set $Y_i = \{x : \Phi_e(x) \cap X_i \neq \emptyset\}$ is infinite for some $i \in \mathbb{N}$. Let $Z_i \subseteq Y_i$ be a computable infinite subset such that $\min Z_i > \max F_i$. Say $Z_i = \{x_0 < x_1 < \dots\}$. Since $x < \min(\Phi_e(x))$, then for every $n \in \mathbb{N}$, $F_i < \Phi_e(x_n)$, hence $\Phi_e(x_n) \cap X_i \subseteq \overline{A}$. Let $E_0 < E_1 < \dots$ be defined by $E_n = \Phi_e(x_n) \setminus X_i$. Then $|E_n| < k$ for every n, so by induction hypothesis, there is some n such that $E_n \cap A = \emptyset$. In particular, $\Phi_e(x_n) \cap A = \emptyset$.

We now explain how to satisfy $\mathcal{R}_{e,k}$ for each $e, k \in \mathbb{N}$. For each pair of indices $e, k \in \mathbb{N}$, let $i_{e,k} = \sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k'$. A strategy for $\mathcal{R}_{e,k}$ *requires attention* at stage $s > \langle e, k\rangle$ if there is an $x < s$ such that $\Phi_{e,s}(x)\downarrow$, $|\Phi_{e,s}(x)| \leq k$, and $\Phi_{e,s}(x) \subseteq \bigcup_{j \geq i_{e,k}} X_j$. Then, the strategy enumerates all the elements of $\Phi_{e,s}$ in W, and is declared *satisfied*, and will never require attention again. First, notice that if Φ_e is total, outputs k-sets, and meets finitely many times each X_i, then it will require attention at some stage s and will be declared satisfied. Therefore each requirement $\mathcal{R}_{e,k}$ is satisfied. Second, suppose for the sake of contradiction that $|X_i \cap W| > i$ for some i. Let s be the stage at which it happens, and let $\langle e, k\rangle < s$ be the maximal pair such that $\mathcal{R}_{e,k}$ has enumerated some element of X_i in W. In particular, $i_{e,k} \leq i$. Since the strategy for $\mathcal{R}_{e',k'}$ enumerates at most k' elements in W,

$$\sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k' \geq |X_i \cap W| > i \geq i_{e,k} = \sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k'$$

Contradiction. ■

Corollary 6.5.15 (Hirschfeldt and Shore [23])
CAC *implies neither* DNC *nor* RT^2_2 *over* RCA_0.[30]

30: Actually, this separation was originally proven using DNC avoidance. However, the design c.b-immunity is more straightforward from an analysis for the combinatorial properties of the forcing question for CAC.

Proof. By Theorem 6.5.7, Theorem 6.5.14 and Corollary 6.1.4, CAC does not imply DNC over RCA_0. By Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [47], $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \rightarrow$ DNC, so CAC does not imply RT^2_2 over RCA_0. ■

6.6 Linear orders

A *linear order* is a pair $\mathscr{L} = (D, <_{\mathscr{L}})$ where $D \subseteq \mathbb{N}$ and $<_{\mathscr{L}}$ is an irreflexive and transitive total binary relation over D. A set $X \subseteq D$ is an *ascending (descending) sequence* if for every $x, y \in X$, $x < y$ iff $x <_{\mathscr{L}} y$ ($x >_{\mathscr{L}} y$). Let ADS be the Π^1_2 problem whose instances are infinite linear orders over $\mathbb{N}$ and whose solutions are infinite ascending or descending sequences.

Exercise 6.6.1 (Hirschfeldt and Shore [23]). Show that $\mathsf{RCA}_0 \vdash \mathsf{CAC} \to \mathsf{ADS}$. ★

Exercise 6.6.2 (Hirschfeldt and Shore [23]). Let $\vec{R} = R_0, R_1, \dots$ be a countable sequence of sets. Let $\mathscr{L} = (\mathbb{N}, <_{\mathscr{L}})$ be the linear order defined by setting $x <_{\mathscr{L}} y$ iff $\langle R_i(x) : i \leq x \rangle <_{\mathtt{lex}} \langle R_i(y) : i \leq y \rangle$, where $<_{\mathtt{lex}}$ is the lexicographic order on $2^{<\mathbb{N}}$. Show that every infinite ascending or descending sequence of $\mathscr{L}$ is $\vec{R}$-cohesive. ★

The Ascending Descending Sequence plays a dual role with the Erdős-Moser theorem with respect to RT^2_2 in the following sense: Any coloring $f : [\mathbb{N}]^2 \to 2$ can be interpreted as a tournament $T \subseteq \mathbb{N}^2$ by letting $T(x, y)$ hold if $x < y$ and $f(\{x, y\}) = 1$, or if $x > y$ and $f(\{y, x\}) = 0$. Every infinite T-transitive sub-tournament $U \subseteq \mathbb{N}$ induces a linear order $(U, <_{\mathscr{U}})$ defined by $x <_{\mathscr{U}} y$ iff $T(x, y)$ holds. Then, every infinite ascending and descending sequence is f-homogeneous for colors 1 and 0, respectively.

Exercise 6.6.3 (Montálban, see [42]). Show that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{EM} \wedge \mathsf{ADS}$. ★

One can naturally ask whether a reversal exists, that is, whether ADS implies CAC over RCA_0. The goal of this section is to separate the two statements. The natural notion of forcing for ADS is a degenerate version of the notion of forcing for CAC used in Theorem 6.5.7. The combinatorics are therefore very similar, with one notable exception:

> **Definition 6.6.4.** Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*extremal* if for every formula $\varphi \in \Gamma$ and every condition $p \in \mathbb{P}$, if $p \operatorname{?}\!\vdash \varphi(G)$ then p forces $\varphi(G)$. ◊

By extension, we say that a forcing question for Σ^0_n-formulas is Π^0_n-*extremal* if for every Σ^0_n-formula φ and every condition $p \in \mathbb{P}$, if $p \operatorname{?}\!\nvdash \varphi(G)$, then p forces $\neg\varphi(G)$.

Contrary to CAC, the notion of forcing for ADS admits a disjunctive forcing question which satisfies some form of Π^0_1-extremality. This extremality can be exploited to force countably many Π^0_1 facts simultaneously, yielding the following notion of immunity.

> **Definition 6.6.5.** A formula $\varphi(U, V)$ is *essential*[31] if for every $x \in \mathbb{N}$, there is a finite set $R > x$ such that for every $y \in \mathbb{N}$, there is a finite set $S > y$ such that $\varphi(R, S)$ holds. A pair of sets $A_0, A_1 \subseteq \mathbb{N}$ is *dependently X-hyperimmune*[32] if for every essential $\Sigma^{0,X}_1$ formula $\varphi(U, V)$, $\varphi(R, S)$ holds for some $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$. ◊

The following exercise shows that dependent hyperimmunity can be seen as a strong form of hyperimmunity. The two notions coincide on co-c.e. sets.

31: The terminology comes from Lerman, Solomon and Towsner [43] who first proved that ADS does not imply CAC over RCA_0. The proof was then simplified by Patey [46].

32: One could as well have defined the notion of dependently constant-bound X-immune by fixing the cardinality of the sets R and S. This would also yield a notion separating ADS from CAC over RCA_0.

Exercise 6.6.6 (Patey [46]). Show that

1. If A_0, A_1 are dependently hyperimmune, then A_0 and A_1 are both hyperimmune.
2. If A_0, A_1 are both hyperimmune and A_0 is co-c.e., then A_0, A_1 are dependently hyperimmune. ★

As usual, one can define the corresponding notion of preservation.

Definition 6.6.7. A problem P admits preservation of 1 dependent hyperimmunity if for every set Z and every pair A_0, A_1 of dependently Z-hyperimmune sets, every Z-computable instance X of P admits a solution Y such that A_0, A_1 are dependently $Z \oplus Y$-hyperimmune. ◊

We now prove that ADS admits preservation of 1 dependent hyperimmunity, while we shall see later that CAC does not.

Theorem 6.6.8 (Patey [46])
Let A_0, A_1 be dependently hyperimmune, and $\mathscr{L} = (\mathbb{N}, <_{\mathscr{L}})$ be a computable linear order. Then there is an infinite ascending or descending sequence G such that A_0, A_1 is dependently G-hyperimmune.

PROOF. Consider the notion of forcing whose *conditions*[33] are 3-tuples (σ_0, σ_1, X), where

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_0 \cup \{x\}$ and $\sigma_1 \cup \{x\}$ form respectively an ascending and a descending sequence, for each $x \in X$;
3. X is computable.

33: Note that this notion of forcing for building solutions to ADS is a particular case of the one in Theorem 6.5.7, since any linear order is a degenerate partial order.

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X) for every $i < 2$. One can therefore see a condition as two simultaneous Mathias conditions sharing a same reservoir. Every filter $\mathscr{F}$ induces two sets: $G_{0,\mathscr{F}}$ and $G_{1,\mathscr{F}}$, defined by $G_{i,\mathscr{F}} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathscr{F}\}$.

We make the following hypothesis:

> (H1): For every infinite computable set X, there is some $x_0, x_1 \in X$ such that $\{y \in X : x_0 <_{\mathscr{L}} y\}$ and $\{y \in X : x_1 >_{\mathscr{L}} y\}$ are both infinite.

If the (H1) hypothesis fails for some set X, then one can computably thin it out to obtain a computable infinite ascending or descending sequence $Y \subseteq X$. In particular, A_0, A_1 are dependently Y-hyperimmune, so we are done. We can therefore from now on assume that (H1) holds.

Lemma 6.6.9. Suppose (H1) holds. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$. There is an extension (τ_0, τ_1, Y) of p and some $x > |\sigma_i|$ such that $x \in \tau_i$.★

PROOF. Say $i = 0$ as the other case is symmetric. By (H1), there is some $x_0 \in X$ such that $Y = \{y \in X : x_0 <_{\mathscr{L}} y\}$ is infinite. Let $\tau_0 = \sigma_0 \cup \{x_0\}$, and $\tau_1 = \sigma_1$. Then, (τ_0, τ_1, Y) is an extension of p such that $x_0 \in \tau_0$. ■

We now define a disjunctive forcing question for Σ^0_1-formulas. Given a condition $p = (\sigma_0, \sigma_1, X)$, a *split pair*[34] is an ordered pair (ρ_0, ρ_1) such that $\rho_i \subseteq X$ for each $i < 2$, ρ_0 is ascending, ρ_1 is descending, and $\max_{\mathscr{L}}(\rho_0) <_{\mathscr{L}} \min_{\mathscr{L}}(\rho_1)$.[35]

34: Note that the notion of split pair is the restriction of split triples from Theorem 6.5.7 to linear orders.

35: In other words, every element of the ascending sequence ρ_0 is below (with respect to $<_{\mathscr{L}}$) every element of the descending sequence ρ_1.

Definition 6.6.10. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_1-formulas. Let $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ hold if there is a split pair (ρ_0, ρ_1) such that for each $i < 2$, $\varphi_i(\sigma_i \cup \rho_i)$ holds. ◇

Note that being a split pair is a decidable predicate, hence the forcing question is Σ^0_1-preserving. The following lemma shows that the forcing question not only meets its specification, but also satisfies some form of Π^0_1-extremality.

Lemma 6.6.11. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_1-formulas.

1. If $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is some $i < 2$ and some extension $q \leq p$ forcing $\varphi_i(G_i)$.
2. If $p \operatorname{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then p forces $\neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$. ★

Proof. Suppose first $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ holds, as witnessed by some split pair (ρ_0, ρ_1). By the pigeonhole principle, there is some infinite X-computable subset $Y \subseteq X$ such that for every $x \in \rho_0 \cup \rho_1$, either for every $y \in Y$, $x <_{\mathscr{L}} y$, or for every $y \in Y$, $x >_{\mathscr{L}} y$. We say that x is *small* if it is on the first case and *large* otherwise. If $\max_{\mathscr{L}}(\rho_0)$ is small, then every element in ρ_0 is small, so the condition $(\sigma_0 \cup \rho_0, \sigma_1, Y)$ is an extension of p forcing $\varphi_0(G_0)$. If $\max_{\mathscr{L}}(\rho_0)$ is large, then every element in ρ_1 is large, so $(\sigma_0, \sigma_1 \cup \rho_1, Y)$ is an extension of p forcing $\varphi_1(G_1)$.

Suppose now $p \operatorname{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$. Suppose for the contradiction that there is an extension $q = (\tau_0, \tau_1, Y)$ of p such that $\varphi_0(\tau_0)$ and $\varphi_1(\tau_1)$ both hold. Then, letting $\rho_0 = \tau_0 \setminus \sigma_0$ and $\rho_1 = \tau_1 \setminus \sigma_1$, the pair (ρ_0, ρ_1) forms a split pair contradicting our hypothesis. Thus, p already forces $\neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$. ■

We now prove that for every sufficiently generic filter $\mathscr{F}$, there is some $i < 2$ such that A_0, A_1 is dependently $G_{i,\mathscr{F}}$-hyperimmune.

Lemma 6.6.12. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $\varphi_0(G, U, V), \varphi_1(G, U, V)$ be two Σ^0_1-formulas. There is some $i < 2$ and an extension q of p forcing $\varphi_i(G_i, U, V)$ not to be essential, or $\varphi_i(G_i, U, V)$ to hold for some sets $U \subseteq \overline{A}_0$ and $V \subseteq \overline{A}_1$. ★

Proof. Let $\psi(U, V)$ be the Σ^0_1-formula which holds if there is some $U_0, U_1 \subseteq U$ and some $V_0, V_1 \subseteq V$ such that $p \operatorname{?\vdash} \varphi_0(G_0, U_0, V_0) \vee \varphi_1(G_1, U_1, V_1)$.

If $\psi(U, V)$ is essential, then by dependent hyperimmunity of A_0, A_1, there are some finite sets $U \subseteq \overline{A}_0$ and $V \subseteq \overline{A}_1$ such that $\psi(U, V)$ holds. Let U_0, U_1, V_0, V_1 witness this. By Lemma 6.6.11(1), there is some $i < 2$ and an extension q of p forcing $\varphi_i(G_i, U_i, V_i)$. Since $U_i \subseteq \overline{A}_0$ and $V_i \subseteq \overline{A}_1$, then q is the desired extension.

Suppose now that $\psi(U, V)$ is not essential. Unfolding the definition, there is some $x \in \mathbb{N}$ such that for every finite set $R > x$, there is some $y_R \in \mathbb{N}$ such that for every finite set $S > y_R$, $\psi(R, S)$ does not hold. Suppose for the contradiction that there is a filter $\mathscr{F}$ containing p such that $\varphi_0(G_{0,\mathscr{F}}, U, V)$ and $\varphi_1(G_{1,\mathscr{F}}, U, V)$ are both essential. For each $i < 2$, since $\varphi_i(G_{i,\mathscr{F}}, U, V)$ is essential, there is some $R_i > x$ such that for every $y \in \mathbb{N}$, there is some $S_i > y$ such that $\varphi_i(G_{i,\mathscr{F}}, R_i, S_i)$ holds. Let $R = R_0 \cup R_1$, and for each $i < 2$, let $S_i > y_R$ be such that $\varphi_i(G_{i,\mathscr{F}}, R_i, S_i)$ holds. Let $S = S_0 \cup S_1$. Then p does not force $\neg\varphi_0(G_0, R_0, S_0) \vee \neg\varphi_1(G_1, R_1, S_1)$, so by Lemma 6.6.11(2), $p \operatorname{?\vdash} \varphi_0(G_0, R_0, S_0) \vee \varphi_1(G_1, R_1, S_1)$. Thus, $\psi(R, S)$ holds, with $R > x$ and $S > y_R$, contradiction. ■

We are now ready to prove Theorem 6.6.8. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. For each $i < 2$, let $G_i = G_{\mathcal{F},i}$. By Lemma 6.6.9, G_i is infinite for every $i < 2$. Moreover, by construction, G_0 is an ascending sequence and G_1 is a descending sequence. Last, by Lemma 6.6.12, there is some $i < 2$ such that A_0, A_1 is dependently G_i-hyperimmune. This completes the proof of Theorem 6.6.8. ■

We leave the abstract preservation theorem as an exercise.

Exercise 6.6.13. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Π^0_1-extremal, Σ^0_1-preserving forcing question. Show that for every pair A_0, A_1 of dependently hyperimmune sets and every sufficiently generic filter $\mathcal{F}$, A_0, A_1 is dependently $G_{\mathcal{F}}$-hyperimmune. ★

We construct a computable partial order witnessing that CAC does not admit preservation of 1 dependent hyperimmunity. This partial order will satisfy some strong structural properties that we now define. Given a partial order $\mathcal{P} = (D, <_{\mathcal{P}})$, we say that $x \in P$ is *small*, *large* or *isolated* if for all but finitely many $y \in D$, $x \leq_P y$, $x \geq_P y$, or $x|_P y$, respectively. We write $S^*(\mathcal{P})$, $L^*(\mathcal{P})$ and $I^*(\mathcal{P})$ for the set of small, large and isolated elements of $\mathcal{P}$, respectively. A partial order is *weakly stable*[36] if every element is either small, large, or isolated, that is, $D = S^*(\mathcal{P}) \cup L^*(\mathcal{P}) \cup I^*(\mathcal{P})$. A partial order is *stable* if every element is small or isolated, or if every element is large or isolated, that is, $D = S^*(\mathcal{P}) \cup I^*(\mathcal{P})$ or $D = L^*(\mathcal{P}) \cup I^*(\mathcal{P})$.

36: Weak stability is arguably the natural notion of stability for CAC, in that a partial order over $\mathbb{N}$ can be seen as a 3-coloring of $[\mathbb{N}]^2$, and this partial order is weakly stable if the corresponding 3-coloring is stable. The stronger notion of stability was first introduced by Hirschfeldt and Shore [23], who proved that ADS is equivalent to the statement "Every infinite partial order admits an infinite sub-domain over which it is weakly stable."

Theorem 6.6.14 (Patey [46])
There exists a computable, stable partial order $\mathcal{P} = (\mathbb{N}, <_{\mathcal{P}})$ such that the pair $I^*(\mathcal{P}), L^*(\mathcal{P})$ is dependently hyperimmune.

Proof. Fix an enumeration $\varphi_0(U, V), \varphi_1(U, V), \dots$ of all Σ^0_1 formulas. The construction of the partial order $<_{\mathcal{P}}$ is done by a finite injury priority argument with a movable marker procedure. We want to satisfy the following scheme of requirements for each e, where $L^* = L^*(\mathcal{P})$ and $I^* = I^*(\mathcal{P})$.[37]

37: Note that by stability of $\mathcal{P}$, we will have $L^* \sqcup I^* = \mathbb{N}$, thus in the requirement, one must think of I^* as $\overline{L^*}$ and L^* as $\overline{I^*}$.

$$\mathcal{R}_e : \varphi_e(U, V) \text{ essential} \rightarrow (\exists R \subseteq_{\mathtt{fin}} L^*)(\exists S \subseteq_{\mathtt{fin}} I^*)\varphi_e(R, S)$$

The requirements are given the usual priority ordering. We proceed by stages, maintaining two sets I^*, L^* which represent the limit of the partial order $<_{\mathcal{P}}$. At stage 0, $I^*_0 = L^*_0 = \emptyset$ and $<_{\mathcal{P}}$ is nowhere defined. Moreover, each requirement $\mathcal{R}_e$ is given a movable marker m_e initialized to 0.

A strategy for $\mathcal{R}_e$ *requires attention at stage* $s+1$ if $\varphi_e(R, S)$ holds for some $R < S \subseteq (m_e, s]$. The strategy sets $I^*_{s+1} = (I^*_s \setminus (m_e, min(S)) \cup [min(S), s]$ and $L^*_{s+1} = (L^*_s \setminus [min(S), s]) \cup (m_e, min(S))$. Note that $R \subseteq (m_e, min(S))$ since $R < S$. Then it is declared *satisfied* and does not act until some strategy of higher priority changes its marker. Each marker $m_{e'}$ of strategies of lower priorities is assigned the value $s + 1$.

At stage $s + 1$, assume that $I^*_s \cup L^*_s = [0, s)$ and that $<_{\mathcal{P}}$ is defined for each pair over $[0, s)$.[38] For each $x \in [0, s)$, set $x <_{\mathcal{P}} s$ if $x \in L^*_s$ and $x|_{\mathcal{P}} s$ if $x \in I^*_s$. If some strategy requires attention at stage $s + 1$, take the least one and satisfy it. If no such requirement is found, set $L^*_{s+1} = L^*_s$ and $I^*_{s+1} = I^*_s \cup \{s\}$.[39] Then go to the next stage. This ends the construction.

38: By "$<_{\mathcal{P}}$ is defined over $[0, s)$", we don't mean that it is a linear order on $[0, s)$, but that the status "below/above/incomparable" is defined for every pair over $[0, s)$.

39: This choice is arbitrary. One could have defined $L^*_{s+1} = L^*_s \cup \{s\}$ and $I^*_{s+1} = I^*_s$.

Each time a strategy acts, it changes the markers of strategies of lower priority, and is declared satisfied. Once a strategy is satisfied, only a strategy of higher priority can injure it. Therefore, each strategy acts finitely often and the markers stabilize. It follows that $\lim_s I^*_s$ and $\lim_s L^*_s$ both exist, and that $(\mathbb{N}, <_{\mathcal{P}})$ is stable.

Claim. For every $x < y < z$, if $x <_{\mathcal{P}} y$ and $y <_{\mathcal{P}} z$, then $x <_{\mathcal{P}} z$.

Proof. Suppose that $x <_{\mathcal{P}} y$ and $y <_{\mathcal{P}} z$ but $x|_{\mathcal{P}} z$. By construction of $<_{\mathcal{P}}$, $x \in I^*_z$, $x \in L^*_y$ and $y \in L^*_z$. Let $s \leq z$ be the last stage such that $x \in L^*_s$. Then at stage $s+1$, some strategy $\mathcal{R}_e$ receives attention and moves x to I^*_{s+1} and therefore moves $[x, s]$ to I^*_{s+1}. In particular $y \in I^*_{s+1}$ since $y \in [x, s]$. Moreover, the strategies of lower priority have had their marker moved to $s+1$ and therefore will never move any element below s. Since $y <_{\mathcal{P}} z$, then $y \in L^*_z$. In particular, some strategy $\mathcal{R}_i$ of higher priority moved y to L^*_{t+1} at stage $t+1$ for some $t \in (s, z)$. Since $\mathcal{R}_i$ has a higher priority, $m_i \leq m_e$, and since y is moved to L^*_{t+1}, then so is $[m_i, y]$, and in particular $x \in L^*_{t+1}$ since $m_i \leq m_e \leq x \leq y$. This contradicts the maximality of s. ■

Claim. For every $e \in \omega$, $\mathcal{R}_e$ is satisfied.

Proof. By induction over the priority order. Let s_0 be a stage after which no strategy of higher priority will ever act. By construction, m_e will not change after stage s_0. If $\varphi_e(U, V)$ is essential, then $\varphi_e(R, S)$ holds for two sets $m_e < R < S$. Let $s = 1 + max(s_0, S)$. The strategy $\mathcal{R}_e$ will require attention at some stage before s, will receive attention, be satisfied and never be injured. ■

This last claim finishes the proof of Theorem 6.6.14. ■

Corollary 6.6.15 (Lerman, Solomon and Towsner [43])
ADS *does not imply* CAC *over* RCA_0.

Proof. Let $\mathcal{P} = (\mathbb{N}, <_{\mathcal{P}})$ be the partial order of Theorem 6.6.14, and let $A_0 = I^*(\mathcal{P})$ and $A_1 = L^*(\mathcal{P})$. Let H be either infinite chain, or an infinite antichain, and let $\varphi(U, V)$ be the essential $\Sigma^0_1(H)$-formula "$U \cup V \subseteq H$". If H is a chain, then by stability of $\mathcal{P}$, it is an ascending sequence, hence $H \subseteq A_1$. If H is an antichain, then $H \subseteq A_0$. In both cases, φ witnesses the fact that A_0, A_1 is not dependently H-hyperimmune. Thus CAC does not admit preservation of 1 dependent hyperimmunity. On the other hand, by Theorem 6.6.8, ADS admits preservation of 1 dependent hyperimmunity. Thus, by Corollary 6.1.4, ADS does not imply CAC over RCA_0. ■

Conservation theorems 7

Prerequisites: Chapters 2 to 4

The importance of the combinatorial features of the forcing question extends to the proof-theoretic realm, especially for proving conservation theorems. In this setting, one usually starts with a model of a weak theory, and extends it to satisfy a stronger theory, while preserving some features of the original model. When working with models of weak arithmetic, the stake is to add new sets to the model while preserving induction. We shall see that Σ^0_n-induction can be preserved thanks to the existence of a Σ^0_n-preserving forcing question which is able to find a common extension witnessing a positive and a negative answer simultaneously.

In this chapter, we shall consider conservation theorems over RCA_0, a weak theory capturing computable mathematics. Thanks to the correspondence between computability and definability, we shall benefit from the framework of first-jump control to prove our main conservations theorems. However, the translation of computability-theoretic constructions to proof-theoretic ones requires a careful formalization, as many intuitive features of the integers are not necessarily true in models of weak arithmetic.

7.1 Context and motivation

At the end of the 19th century, the various paradoxes arising in the development of set theory led to a foundational crisis of mathematics. Mathematicians started to question the use of infinity in mathematics, partially due to the lack of ground to reality: with the discovery of the atom, and of the finiteness of the universe, infinity seemed to be a purely intellectual construction in which intuition failed. In the early 1920s, David Hilbert proposed a program as a solution to the foundational crisis, called *finitistic reductionism*. The goal was to show that every finitary statement proven by infinitary means, could also be proven finitarily. Thus, infinity would be a convenience language not affecting the truth value of finitary statements.[1]

1: There is an excellent article from Simpson [1] on the subject, presenting reverse mathematics as a partial realization of Hilbert's program.

Sadly, Gödel's incompleteness theorems showed the unrealizability of Hilbert's program in its full generality, as the consistency of Peano arithmetic is a finitary statement which is not provable by finitary means, but provable in set theory. Reverse mathematics can be considered as a partial realization of Hilbert's program, as it showed that many theorems of ordinary mathematics are provable over WKL_0, which is Π_2-conservative over primitive recursive arithmetic (PRA).[2] PRA is considered as capturing finitary mathematics (see Tait [48]), so any Π_2 theorem of WKL_0 can be proved by finitary means.

2: PRA is a system in the language of functions, capturing primitive recursive functions. Technically, the languages being different, saying that WKL_0 is Π_2-conservative over PRA requires some work in translating sentences from one language to the other. See Simpson [5, p. IX.3] for a formal development of the subject.

More generally, it is of foundational importance to understand the *first-order part* of a second-order theory, that is, the set of its first-order theorems. There exist two main methods to characterize the first-order part of a second-order theory T: either directly identify a first-order theory capturing the first-order part of T, or reduce the theory T to a weaker second-order theory for which the first-order part is already known. We shall mostly adopt the second approach, through Π^1_1-conservation.

Definition 7.1.1. Let T_0, T_1 be two theories of second-order arithmetic. A theory T_1 is Π^1_1-*conservative over* T_0 if every Π^1_1 sentence provable in T_1 is also provable in T_0. ◊

If furthermore T_1 implies T_0, then we say that T_1 is a Π^1_1-*conservative extension* of T_0. Proving that a theory T_1 is a Π^1_1-conservative extension of T_0 is a strong way of proving that T_1 and T_0 have the same first-order part. Indeed, the class of Π^1_1 sentences not only contains all the first-order sentences, but also every arithmetic sentence with second-order parameters.

Recall that a model of second-order arithmetic is of the form $\mathcal{M} = (M, S, +, \times, <, 0, 1)$ where $S \subseteq \mathcal{P}(M)$. A model $\mathcal{M}$ is *topped*[3] by a set $Y \in S$ if every $X \in S$ is $\Delta^0_1(Y)$-definable with parameters in M.[4]

Definition 7.1.2. A model $\mathcal{N} = (N, T, +^{\mathcal{N}}, \times^{\mathcal{N}}, <^{\mathcal{N}}, 0^{\mathcal{N}}, 1^{\mathcal{N}})$ is an ω-*extension*[5] of a model $\mathcal{M} = (M, S, +^{\mathcal{M}}, \times^{\mathcal{M}}, <^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$ if $\mathcal{N}$ and $\mathcal{M}$ differ only by their second-order part and $T \supseteq S$. In other words, $M = N$, and the basic operations coincide. ◊

We shall often omit the signature, and simply write $\mathcal{M} = (M, S)$ when there is no ambiguity. Proofs of Π^1_1-conservation are usually done through ω-extensions of countable models.

Proposition 7.1.3. Let T_0 and T_1 be two theories of second-order arithmetic. Suppose that every countable model $\mathcal{M} \models T_0$ can be ω-extended into a model $\mathcal{N} \models T_1$. Then T_1 is Π^1_1-conservative over T_0. ★

Proof. Let $\varphi \equiv \forall X \theta(X)$ be a Π^1_1 sentence, where θ is an arithmetic formula. Suppose that $T_0 \nvdash \varphi$. Then by Gödel's completeness theorem[6], there is a model of $T_0 \cup \{\neg\varphi\}$. By the downward Löwenheim–Skolem theorem[7], there is a countable such model $\mathcal{M} = (M, S) \models T_0 \cup \{\neg\varphi\}$. Let $X \in S$ be such that $\mathcal{M} \models \neg\theta(X)$. By assumption, there is an ω-extension $\mathcal{N} = (M, S_1) \models T_1$ of $\mathcal{M}$. Since $S_1 \supseteq S$, then $X \in S_1$. Moreover, since $\mathcal{N}$ is an ω-extension of $\mathcal{M}$, then $\mathcal{N} \models \neg\theta(X)$, so $\mathcal{N} \models \neg\varphi$. ■

In this chapter, we shall consider two base theories for T_0: RCA_0 and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. The techniques to prove Π^1_1-conservation over these two theories are pretty different, but both use a formalization of first-jump control.

7.2 Induction and collection

Before turning to the actual proofs of conservation, it is important to get familiar with some fundamental concepts of weak arithmetic. Classical mathematicians being used to work with full induction, it can be challenging to get an intuition on what constructions and theorems of mathematics remain valid over weak arithmetic. See Hájek and Pudlák [50] for a development of the basics of mathematics over increasingly strong axiomatic systems. The base system, RCA_0, is a restriction of the full second-order arithmetic on two axis:

- The *comprehension scheme* is restricted to Δ^0_1 predicates with parameters. By Post's theorem, this restrictions allows only the construction of sets computably from existing sets in the model. In ω-models, this ensures that the second-order part is a Turing ideal. The computability-theorist should already be familiar with this restriction.

3: Topped models should not be confused with top models, although there is a lot of beauty in models of weak arithmetic.

4: One can define a notion of Turing functional in weak models of arithmetic, and therefore define the Turing reduction. However, if the theory is too weak, the Turing reduction is not transitive. In order to have a Turing reduction $Y \leq_T X$ with a good behavior, one needs $(M, \{X\}) \models \mathsf{B}\Sigma^0_1$. See Groszek and Slaman [49].

5: The terminology might be confusing, as being an ω-extension has nothing to do with ω-models.

6: Recall that second-order arithmetic is a two-sorted first-order theory. A *Henkin structure* is a structure of second-order arithmetic in which the ownership relation $\in$ has its standard interpretation. Henkin proved that Gödel's completeness theorem also applies to Henkin tructures, that is, a second-order theory is *consistent* iff it admits a Henkin model.

7: The *downward Löwenheim-Skolem theorem* is a classical theorem from model theory, stating that for every structure $\mathcal{M}$ over a signature σ, and every infinite cardinal κ between $\operatorname{card} \mathcal{M}$ and $\operatorname{card} \sigma$, there is an elementary substructure of $\mathcal{M}$ of cardinal κ. In particular, the language of second-order arithmetic is countable, so consistency of a theory T implies the existence of a countable model of T.

- The *induction scheme* is restricted to Σ^0_1 formulas with parameters. This might be the less intuitive part, both in terms of consequences over the theory, and in terms of design choice. Indeed, why restrict induction to capture computable mathematics?

This section therefore focuses on the second restriction, and gives a brief overview on the impact of induction over the models of weak arithmetic. One can define a hierarchy of systems based on the complexity of formulas satisfying induction.

Definition 7.2.1. Given a class of formulas Γ, the Γ-*induction scheme* (written IΓ) states, for every formula $\varphi(x) \in \Gamma$,

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\, \varphi(x)$$

We shall in particular be interested in the theories IΣ^0_n and IΠ^0_n.[8] Recall that Q denotes Robinson arithmetic (see Section 2.2). Most of our equivalences will be stated either over Q, Q + IΔ^0_0 or Q + IΔ^0_0 + $\exp$, where $\exp$ is the statement of the totality of the exponential.[9]

Proposition 7.2.2 (Paris and Kirby [51]). Fix $n \geq 1$. Then Q $\vdash$ I$\Sigma^0_n \leftrightarrow$ IΠ^0_n. ⋆

Proof. We first prove Q $\vdash$ I$\Sigma^0_n \to$ IΠ^0_n. Suppose that IΣ^0_n holds but IΠ^0_n fails. Let $F(x)$ be a Π^0_n formula such that $F(0)$ and $\forall x(F(x) \to F(x+1))$, but $\neg F(a)$ for an integer $a > 0$. Let $G(y)$ be the formula $\exists x\ (a = x + y \wedge \neg F(x))$. Note that $G(y)$ is equivalent to a Σ^0_n formula. Moreover, $G(0)$ is true and $G(a)$ is false. Let y be such that $G(y)$ is true. In particular, there is an x such that $a = x + y$ and $\neg F(x)$. Since $F(0)$ holds, then $x > 0$ and $y < a$. Thus $a = (x-1)+(y+1)$ and by hypothesis, $\neg F(x) \to \neg F(x-1)$, therefore $G(y+1)$ is true. As $G(0)$ and $\forall y\ (G(y) \to G(y+1))$ and $\neg G(a)$, then IΣ^0_n fails.

We now prove Q $\vdash$ I$\Pi^0_n \to$ IΣ^0_n. Suppose IΠ^0_n holds but IΣ^0_n fails. Let $F(x)$ be a Σ^0_n formula such that $F(0)$ and $\forall x(F(x) \to F(x+1))$, but $\neg F(a)$ for an integer $a > 0$. Let $H(y)$ be the formula $\forall x\ (a = x + y \to \neg F(x))$. As before, $H(y)$ is equivalent to a Π^0_n formula. Additionally $H(0)$ is true and $H(a)$ is false. We also show $H(y) \to H(y+1)$. Then, $H(0)$ and $\forall y\ (H(y) \to H(y+1))$ and $\neg H(a)$, so IΠ^0_n fails.[10] ■

Exercise 7.2.3 (Hájek and Pudlák [50]). Given a class of formulas Γ, the Γ-*least principle* (written LΓ) states, for every formula $\varphi(x) \in \Gamma$,

$$\exists x \varphi(x) \to \exists x(\varphi(x) \wedge \forall y < x \neg\varphi(y))$$

Show that Q $\vdash$ I$\Sigma^0_n \leftrightarrow$ LΠ^0_n and Q $\vdash$ I$\Pi^0_n \leftrightarrow$ LΣ^0_n. ⋆

From a computability-theoretic viewpoint, bounded sets are finite and therefore trivially computable. In weak arithmetic on the other hand, not all bounded sets exist in the model, and their existence is closely related to the hierarchy of induction. A set $F \subseteq M$ is M-*coded* if it has a canonical code in M, that is, there is some $s \in M$ such that $s = \sum_{x \in F} 2^x$. Given $s \in M$, we write $\mathrm{Ack}(s)$ for the set coded by s.

8: One should not confuse the arithmetic hierarchy on sets and on formulas. The former is a semantic notion, starting a the first level with computable predicates. The latter is a syntactic hierarchy, starting at the first level with *bounded arithmetic formulas*, that is, formulas with only quantifiers of the form $\forall x < t$ and $\exists x < t$ where t is a term. By a theorem of Gödel, the Σ^0_n sets are exactly the ones definable by a Σ^0_n formula, for $n \geq 1$, so the hierarchies coincide starting from level 1. On the other hand, some computable sets and even some primitive recursive sets are not definable by bounded arithmetic formulas.

Note that the hierarchies of Σ^0_n and Π^0_n formulas allow integer and set parameters, which is equivalent to quantify universally all free variables.

9: Note that Q + IΣ^0_1, and *a fortiori* RCA$_0$, proves $\exp$, so all the implications of this section hold over RCA$_0$, and even over RCA$^*_0$, a weaker system that will be introduced in Section 7.4.

10: Note that in both directions, we used a formula with parameter a to witness failure of the other induction scheme. This is necessary, as the parameter-free versions of IΣ^0_n and IΠ^0_n are not equivalent for $n \geq 1$. [52]

Definition 7.2.4. Let $\mathcal{M} = (M, S)$ be a model. A set $A \subseteq M$ is *M-regular*[11] if every initial segment of A is M-coded. ◇

11: These sets are also called *amenable* or *piecewise coded*. If $\mathcal{M} \models \mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$ then every set in S is M-regular.

The following proposition states that the induction scheme is equivalent to a bounded version of the comprehension scheme. Therefore, restricting the induction corresponds to restricting the complexity of the finite sets in the model.

Proposition 7.2.5 (Hájek and Pudlák [50]). Fix $n \geq 1$. Then the following are equivalent over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$:

1. $\mathsf{I}\Sigma^0_n$;
2. Every Σ^0_n-definable set is regular. ★

Proof. Suppose first that every Σ^0_n-definable set is regular. Let φ be a Σ^0_n formula such that $\varphi(0)$ holds and $\forall x(\varphi(x) \rightarrow \varphi(x+1))$. Fix any $a \in \mathbb{N}$ and let $\sigma \in 2^{a+1}$ be the string defined by $\sigma(x) = 1$ iff $\varphi(x)$ holds. By regularity, σ exists. Let $\psi(x)$ be the Δ^0_0 formula defined by $\psi(x) \equiv (x \leq a \rightarrow \sigma(x) = 1)$. By $\mathsf{I}\Delta^0_0$, $\psi(x)$ holds for every x, so $\varphi(a)$ holds.

Suppose now $\mathsf{I}\Sigma^0_n$. Let φ be a Σ^0_n formula and $a \in \mathbb{N}$. Let $\psi(q)$ be the Π^0_n formula $(\forall x < a)(\varphi(x) \rightarrow x \in q)$, where $x \in q$ means that x belongs to the set canonically coded by q. Note that $2^a - 1$ is a canonical code for $\{x \in \mathbb{N} : x < a\}$, so $\psi(2^a - 1)$ holds. By $\mathsf{L}\Pi^0_n$ (which is equivalent to $\mathsf{I}\Sigma^0_n$ by Exercise 7.2.3), there is a least $q \in \mathbb{N}$ such that $\psi(q)$ holds. Then q is a canonical code of $\{x < a : \varphi(x)\}$. ■

The collection scheme is a principle equivalent to induction, but whose induced hierarchy is interleaved with the induction hierarchy. It plays a very important role in proving closure properties of levels of the arithmetic hierarchy.

Definition 7.2.6. Given a class of formulas Γ, the *Γ-collection scheme* (written $\mathsf{B}\Gamma$) states, for every formula $\varphi(x, y) \in \Gamma$,

$$\forall a[(\forall x < a \exists y \varphi(x, y)) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y)]$$

In other words, the collection scheme states that every bounded family of existential formulas admits a uniform existential bound. By contraction of quantifiers, $\mathsf{B}\Sigma^0_{n+1}$ is equivalent to $\mathsf{B}\Pi^0_n$.

Exercise 7.2.7 (Hájek and Pudlák [50]). Prove that $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_{n+1} \leftrightarrow \mathsf{B}\Pi^0_n$. ★

The following proposition is very useful for formulas manipulation:

Proposition 7.2.8 (Parsons [53]). Fix $n \geq 1$. Let $\varphi_0(x), \varphi_1(x), \varphi(x)$ be Σ^0_n (resp. Π^0_n) formulas. Then the following formulas are provably equivalent to a Σ^0_n (resp. Π^0_n) formula over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{B}\Sigma^0_n$:

(1) $\varphi_0(x) \wedge \varphi_1(x), \varphi_0(x) \vee \varphi_1(x)$;
(2) $\exists x < a\varphi(x), \forall x < a\varphi(x)$;
(3) $\exists x\varphi(x)$ (resp. $\forall x\varphi(x)$). ★

Proof. Say $\varphi_0(x) \equiv \exists y \theta_0(x, y)$, $\varphi_1(x) \equiv \exists y \theta_1(x, y)$ and $\varphi(x) \equiv \exists y \theta(x, y)$. The proof goes by induction, using the following equivalences:

$$\begin{array}{rcll} \varphi_0(x) \wedge \varphi_1(x) & \leftrightarrow & \exists y \exists y_0, y_1 < y(\theta_0(x, y_0) \wedge \theta_1(x, y_1)) & (a) \\ \varphi_0(x) \vee \varphi_1(x) & \leftrightarrow & \exists y(\theta_0(x, y) \vee \theta_1(x, y)) & (b) \\ \exists x < a\varphi(x) & \leftrightarrow & \exists y \exists x < a\theta(x, y) & (c) \\ \forall x < a\varphi(x) & \leftrightarrow & \exists a \forall x < a \exists y < z\theta(x, y) & (d) \\ \exists x \theta(x) & \leftrightarrow & \exists z \exists x, y < z\theta(x, y) & (e) \end{array}$$

Note that (a)(b)(c) and (e) are provable over $\mathsf{Q} + \mathsf{I}\Delta^0_0$, while (d) uses $\mathsf{B}\Sigma^0_n$. ■

The following theorem shows that the hierarchies of induction and collection are interleaved. Paris and Kirby [51] proved the following implications, which are both strict:

Theorem 7.2.9 (Paris and Kirby [51])
Fix $n \geq 1$.

1. $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_n \longrightarrow \mathsf{B}\Sigma^0_n$
2. $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_{n+1} \longrightarrow \mathsf{I}\Sigma^0_n$.

Actually, the levels of the collection hierarchy can be understood in terms of induction, using Δ^0_n predicates. Recall that for $n \geq 1$, Δ^0_n predicates do not form a syntactic class for formulas. Thankfully, one can extend the various schemes to Δ^0_n predicates using a syntactical trick.

Definition 7.2.10. Fix $n \geq 1$. The Δ^0_n-induction scheme (written $\mathsf{I}\Delta^0_n$) states, for every Σ^0_n formula $\varphi(x)$ and every Π^0_n formula $\psi(x)$:

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow [(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))) \rightarrow \forall x \varphi(x)]$$

The Δ^0_n-least principle ($\mathsf{L}\Delta^0_n$) is defined accordingly. By Gandy (see Slaman [54]), $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_n \leftrightarrow \mathsf{L}\Delta^0_n$. The proof of following theorem goes far beyond the scope of this book.

Theorem 7.2.11 (Slaman [54])
Fix $n \geq 1$.

- $\mathsf{Q} + \mathsf{I}\Delta^0_0 \vdash \mathsf{B}\Sigma^0_n \longrightarrow \mathsf{I}\Delta^0_n$;
- $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp \vdash \mathsf{I}\Delta^0_n \longrightarrow \mathsf{B}\Sigma^0_n$.

Exercise 7.2.12 (Hájek and Pudlák [50]). Fix $n \geq 1$. Show that the following are equivalent over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$:

1. $\mathsf{I}\Delta^0_n$;
2. Every Δ^0_n-definable set is regular. ★

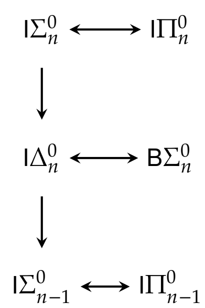

Figure 7.1: Induction hierarchy. Arrows stand for implications in $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$.

7.3 Conservation over RCA_0

The proof-theoretic strength of RCA_0 is relatively well understood. Its first-order part is $\mathsf{Q} + \mathsf{I}\Sigma_1$[12], and it is a Π_2-conservative extension of PRA. In

12: We distinguish the class of Σ^0_n formulas in the language of second-order arithmetic from the class of Σ_n formulas in first-order arithmetic. In particular, in the former case, second-order parameters are allowed.

particular, every primitive recursive function is provably total over RCA_0, and every theorem of RCA_0 is finitistically reducible in the sense of Hilbert's program. Proving that a theory T is Π^1_1 conservative over RCA_0 is therefore a good way to show that T is finitistically reducible.

Given a model $\mathcal{M} = (M, S)$ and a set $G \subseteq M$, we denote by $\mathcal{M} \cup \{G\}$ and $\mathcal{M}[G]$ the ω-extensions whose second-order parts are $S \cup \{G\}$ and the $\Delta^0_1(\mathcal{M}, G)$-definable sets[13], respectively. The following exercise reflects the fact that every Σ^0_1-formula over $\mathcal{M}[G]$ is equivalent to a Σ^0_1-formula over $\mathcal{M} \cup \{G\}$.

13: Given a class of formulas Γ and a structure $\mathcal{M}$, we write $\Gamma(\mathcal{M})$ for the class of formulas with parameters in $\mathcal{M}$.

Exercise 7.3.1 (Friedman [55]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ and $G \subseteq M$ be such that $\mathcal{M} \cup \{G\} \models \mathsf{I}\Sigma^0_1$. Show that $\mathcal{M}[G] \models \mathsf{RCA}_0$. ⋆

Proposition 7.1.3 gives a general proof scheme to obtain conservation theorems between two second-order theories. One can prove a refined proposition in the particular case of conservation of Π^1_2 problems over RCA_0. Recall that a problem P is Π^1_2 if the relations $X \in \operatorname{dom} \mathsf{P}$ and $Y \in \mathsf{P}(X)$ are both arithmetically definable. The sentence $\forall X \in \operatorname{dom} \mathsf{P}\ \exists Y \in \mathsf{P}(X)$ is then Π^1_2.

Proposition 7.3.2. Let P be a Π^1_2 problem. Suppose that for every countable topped model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$, and every $X \subseteq M$ such that $\mathcal{M} \models X \in \operatorname{dom} \mathsf{P}$, there is a set $Y \subseteq M$ such that $\mathcal{M}[Y] \models \mathsf{RCA}_0 + (Y \in \mathsf{P}(X))$. Then $\mathsf{RCA}_0 + \mathsf{P}$ is Π^1_1-conservative over RCA_0.[14] ⋆

14: By Exercise 7.3.1, it is actually sufficient to require that

$$\mathcal{M} \cup \{Y\} \models \mathsf{I}\Sigma^0_1 + (Y \in \mathsf{P}(X))$$

Proof. Let $\varphi \equiv \forall Z \theta(Z)$ be a Π^1_1-sentence, where θ is an arithmetic formula. Suppose that $\mathsf{RCA}_0 \nvdash \varphi$. Then by Gödel's completeness theorem and the downward Löwenheim-Skolem theorem, there is a countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 \cup \{\neg\varphi\}$. Let $Z_0 \in S$ be such that $\mathcal{M} \models \neg\theta(Z_0)$. Let $\mathcal{M}_0 = (M, S_0)$, where S_0 be the set of Δ^0_1-definable sets over $(M, \{Z_0\})$. By Friedman [56], $\mathcal{M}_0 \models \mathsf{RCA}_0$, and by construction, $\mathcal{M}_0$ is topped by Z_0.

We define by external induction a countable sequence of sets $Z_0, Z_1, \dots$ and models $\mathcal{M}_0, \mathcal{M}_1, \dots$ such that for every $n \in \omega$,

1. $\mathcal{M}_n = (M, S_n) \models \mathsf{RCA}_0$ is topped by $Z_0 \oplus \dots \oplus Z_n$;
2. for every $X \in S_n$ such that $\mathcal{M}_n \models X \in \operatorname{dom} \mathsf{P}$, there is some $p \in \omega$ such that $\mathcal{M}_p \models Z_p \in \mathsf{P}(X)$.

Assuming $\mathcal{M}_n$ is defined and given some $X \in \mathcal{M}_n$ such that $\mathcal{M}_n \models X \in \operatorname{dom} \mathsf{P}$, by assumption, there is a set $Z_{n+1} \subseteq M$ such that $\mathcal{M}[Z_{n+1}] \models \mathsf{RCA}_0 + (Z_{n+1} \in \mathsf{P}(X)\}$. Let $\mathcal{M}_{n+1} = \mathcal{M}_n[Z_{n+1}]$. By construction, $\mathcal{M}_{n+1}$ is topped by $Z_0 \oplus \dots \oplus Z_{n+1}$.

Let $\mathcal{N} = (M, T)$ be defined by $T = \bigcup_n S_n$. Note that $\mathcal{N} \models \mathsf{RCA}_0$ since it is a union of models of RCA_0. By construction, $\mathcal{N}$ is an ω-extension of $\mathcal{M}$ and a model of P. Last, since $Z_0 \in T$ and θ is arithmetic $\mathcal{N} \models \neg\theta(Z_0)$, hence $\mathcal{N} \models \neg\varphi$. ■

The first-conservation theorem, due to Harrington (see Simpson [5]), is the most important one for its implications to Hilbert's program. Indeed, many theorems are provable by compactness arguments.

Theorem 7.3.3 (Harrington)
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ be a countable model and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $G \in [T]$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0$.

PROOF. Consider the Jockusch-Soare forcing whose conditions are infinite trees $T_1 \subseteq T$ in S, partially ordered by inclusion. First of all, some simple facts such as the existence of extendible nodes of arbitrary length are not immediate in weak arithmetic. We prove a lemma stating that it is the case in models of RCA_0. Recall that a node σ is *extendible* in T_1 if the set of nodes in T_1 comparable with σ is infinite.

Lemma 7.3.4 (Fernandes et al. [57]). Let T_1 be a condition and $\ell \in M$. There is an extendible node $\sigma \in T_1$ of length ℓ.[15] ★

15: Note that the proof of this lemma only uses $\mathsf{Q} + \mathsf{B}\Sigma^0_1$.

PROOF. Assume by contradiction that for every $\sigma \in 2^\ell$ the tree $\{\tau \in T_1 : \tau \text{ is comparable with } \sigma\}$ is M-bounded. Then

$$\forall \sigma \in 2^\ell \exists b \forall \tau \in 2^b, \sigma \prec \tau \to \tau \notin T_1$$

The formula $\forall \tau \in 2^b, \sigma \prec \tau \to \tau \notin T_1$ is Δ^0_0, so by $\mathsf{B}\Sigma^0_1$ (which holds in RCA_0 by Theorem 7.2.9), there is some $b \in M$ such that

$$\forall \sigma \in 2^\ell \exists c < b \forall \tau \in 2^c, \sigma \prec \tau \to \tau \notin T_1$$

This yields that T_1 is bounded by b, contradicting our assumption that T_1 is M-infinite.[16] ■

16: In general, the predicate "X is finite" is Σ^0_2, so if T_1 was an arbitrary set of strings, the existence of an extendible node would require $\mathsf{B}\Sigma^0_2$. Thanks to prefix closure, the predicate "T is finite" for a tree T is Σ^0_1 and $\mathsf{B}\Sigma^0_1$ is sufficient.

Thanks to Lemma 7.3.5, for every sufficiently generic filter $\mathcal{F}$, the class $\bigcap_{T_1 \in \mathcal{F}} [T_1]$ is a singleton $G_{\mathcal{F}}$. Indeed, for every condition T_1 and $\ell \in M$, letting σ be an extendible node in T_1 of length ℓ, the condition $T_2 = \{\tau \in T_1 : \tau \preceq \sigma \vee \sigma \prec \tau\}$ exists by Δ^0_0-comprehension and is a valid extension of T_1 forcing $\sigma \prec G$.

Exercise 3.3.7 defined a Σ^0_1-preserving forcing question for Jockusch-Soare forcing in a standard context. We re-define it and prove its properties in the context of weak arithmetic.

Given a condition T_1 and a Σ^0_1-formula (with parameters in $\mathcal{M}$) $\varphi(G) \equiv \exists y \psi(y, G \upharpoonright_y)$, let $T_1 \operatorname{?\vdash} \varphi(G)$ hold if there is some $\ell \in M$ such that for every $\sigma \in T$ such that $|\sigma| = \ell$, there is some $y < \ell$ such that $\psi(y, \sigma \upharpoonright y)$ holds. By Theorem 7.2.9, $\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_1$, so by Proposition 7.2.8, Σ^0_1-formulas are closed under bounded quantification. It follows that this relation is Σ^0_1. The following lemma shows that this is a forcing question in a strong sense, that is, if it holds, then the condition already forces the Σ^0_1 formula.

Lemma 7.3.5. Let T_1 be a condition and $\varphi(G)$ be a Σ^0_1 formula.

1. If $T_1 \operatorname{?\vdash} \varphi(G)$ then T_1 forces $\varphi(G)$;
2. If $T_1 \operatorname{?\nvdash} \varphi(G)$ then there is an extension $T_2 \subseteq T_1$ forcing $\neg\varphi(G)$. ★

PROOF. Say $\varphi(G) \equiv \exists y \psi(y, G \upharpoonright_y)$.

1. Suppose $T_1 \operatorname{?\vdash} \varphi(G)$. Then we claim that for every $P \in [T_1]$, $\varphi(P)$ holds. Indeed, let $\ell \in M$ be such that for every $\sigma \in T$ such that $|\sigma| = \ell$, there is some $y < \ell$ such that $\psi(y, \sigma \upharpoonright y)$ holds. Fix some $P \in [T_1]$. Since $P \upharpoonright_\ell \in T$, there is some $y < \ell$ such that $\psi(y, P \upharpoonright_y)$ holds, so $\varphi(P)$ holds.
2. Suppose $T_1 \operatorname{?\nvdash} \varphi(G)$. Let $T_2 = \{\sigma \in T_1 : \forall y < |\sigma| \neg\psi(y, \sigma \upharpoonright_y\}$. By assumption, T_2 is an infinite subtree of T_1 and by Δ^0_0-comprehension it belongs to S. We claim that for every $P \in [T_2]$, $\neg\varphi(P)$ holds. Suppose for the contradiction that $\varphi(P)$ holds for some $P \in [T_2]$. Let $y \in M$ be

such that $\psi(y, P\upharpoonright_y)$ holds. Then $P\upharpoonright y+1 \notin T_2$, contradiction. So T_2 forces $\neg\varphi(G)$. ■

It follows from Lemma 7.3.5 that if $\varphi(G)$ and $\psi(G)$ are two Σ^0_1-formulas such that $T_1 \mathrel{?{\vdash}} \varphi(G)$ and $T_1 \mathrel{?{\nvdash}} \psi(G)$, then there is an extension $T_2 \subseteq T_1$ forcing $\varphi(G) \wedge \neg\psi(G)$. The following lemma shows that if $\mathcal{F}$ is sufficiently generic, then $\mathcal{M} \cup \{G_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$.

Lemma 7.3.6. Let T_1 be a condition and $\varphi(x, X)$ be a Σ^0_1 formula such that T_1 forces $\neg\varphi(b, G)$ for some $b \in M$. Then there is an extension $T_2 \subseteq T_1$ and some $a \in M$ such that T_2 forces $\neg\varphi(a, G)$, and if $a > 0$, then T_2 forces $\varphi(a-1, G)$.[17] ⋆

17: Note that the proof of Lemma 7.3.6 uses essentially two properties of the forcing question: the fact that it is Σ^0_1-preserving, and its ability to find a simultaneous witness extension to a positive and a negative answer.

Proof. Let $A = \{x \in M : T_1 \mathrel{?{\vdash}} \varphi(x, G)\}$. Since the forcing question is Σ^0_1-preserving, the set A is $\Sigma^0_1(\mathcal{M})$. Moreover, T_1 forces $\neg\varphi(b, G)$, so by Lemma 7.3.5, $T_1 \mathrel{?{\nvdash}} \varphi(b, G)$, hence $b \notin A$. Since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$, and $A \neq M$, there is some $a \in M$ such that $a \notin A$, and if $a > 0$, then $a - 1 \in A$. By Lemma 7.3.5, there is an extension $T_2 \subseteq T_1$ forcing $\neg\varphi(a, G)$. Moreover, if $a > 0$, then since $a - 1 \in A$, by Lemma 7.3.5, T_1 forces $\varphi(a-1, G)$, hence so does T_2. This completes the proof of Lemma 7.3.6. ■

We are now ready to prove Theorem 7.3.3. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. By Lemma 7.3.4, there is a unique set $G \in \bigcap_{T_1 \in \mathcal{F}} [T_1]$. In particular, $G \in [T]$. By Lemma 7.3.6, $\mathcal{M} \cup \{G\} \models \mathsf{I}\Sigma^0_1$, so by Exercise 7.3.1, $\mathcal{M}[G] \models \mathsf{RCA}_0$. This completes the proof of Theorem 7.3.3. ■

Corollary 7.3.7 (Harrington)
WKL_0 *is a* Π^1_1*-conservative extension of* RCA_0.

Proof. Immediate by Theorem 7.3.3 and Proposition 7.3.2. ■

Recall that by Theorem 3.2.4, every set can become Δ^0_2 relative to a cone avoiding degree. This can be interpreted as saying that cone avoidance for Δ^0_2 instances and strong cone avoidance are equivalent. A formalization due to Towsner [58] of the notion of forcing yields a conservation theorem over RCA_0, saying informally that from the viewpoint of RCA_0, Δ^0_2 sets are indistiguishable from arbitrary sets.

Theorem 7.3.8 (Toswner [58])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ *be a countable model and* $A \subseteq M$ *be an arbitrary set. There is a set* $G \subseteq M$ *such that* A *is* $\Delta^0_2(G)$ *and* $\mathcal{M}[G] \models \mathsf{RCA}_0$.

Proof. Based on Shoenfield's limit lemma [8], we will construct a stable function $f : \mathbb{N}^2 \to 2$ such that for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists and equals $A(x)$. We are therefore going to build directly the function f by forcing, and let G be the graph of f.

The idea is to use the notion of forcing from Theorem 3.2.4, however there is a technical difficulty: Assume A is not regular, and fix $a \in M$ such that $A\upharpoonright a$ does not belong to M. Then, the condition $(\emptyset, a)$ has no extension (g, b) in $\mathcal{M}$ with $\{0, \dots, a\} \times \{0\} \subseteq \operatorname{dom} g$. Worse, the set of extensions of $(\emptyset, a)$ is not

Δ^0_1-definable with parameters in $\mathcal{M}$. Thankfully, the model being countable, one can lock non-uniformly a standard number of columns for each condition, and still obtain a stable function.

Consider the notion of forcing whose *conditions* are pairs (g, I), such that

- $g \subseteq M^2 \to \{0,1\}$ is a partial function with two parameters whose domain is M-finite, representing an initial segment of the function f that we are building.
- $I \subseteq M$ is a set of "locked" columns with $\operatorname{card} I \in \omega$, meaning that from now on, when we extend the domain of g with a new pair (x, y), if $x \in I$ then $g(x, y) = A(x)$.

The *interpretation* $[g, I]$ of a condition (g, I) is the class of all partial or total functions $h \subseteq M^2 \to 2$ such that

(1) $g \subseteq h$, i.e. $\operatorname{dom} g \subseteq \operatorname{dom} h$ and for all $(x, y) \in \operatorname{dom} g$, $g(x, y) = h(x, y)$;
(2) for all $(x, y) \in \operatorname{dom} h \setminus \operatorname{dom} g$, if $x \in I$, then $h(x, y) = A(x)$.[18]

A condition (h, J) *extends* (g, I) (denoted $(h, J) \leq (g, I)$) if $J \supseteq I$ and $h \in [g, I]$.

18: Even if A is not regular, the set I being of standard cardinality, the restriction $A \upharpoonright I$ belongs to M. Therefore, the extension relation is Δ^0_1-definable with parameters in $\mathcal{M}$.

For every condition (g, I) and every $x \in M$, $(g, I \cup \{x\})$ is a valid extension. Moreover, for every condition (g, I) and every $(x, y) \in M^2$, there is an extension $(h, I) \leq (g, I)$ such that $(x, y) \in \operatorname{dom} h$. Therefore, if $\mathcal{F}$ is a sufficiently generic filter, then, letting $f_{\mathcal{F}} = \bigcup\{g : (g, I) \in \mathcal{F}\}$, $\operatorname{dom} f_{\mathcal{F}} = M^2$ and every column will eventually be locked, so $f_{\mathcal{F}}$ is stable with limit A.

Given a condition (g, I) and a Σ^0_1-formula (with parameters in $\mathcal{M}$) $\varphi(G) \equiv \exists y \psi(y, G \upharpoonright y)$, let $(g, I) \mathrel{?\vdash} \varphi(G)$ hold if there is a finite $h \in [g, I]$ and some $y \in M$ such that $\psi(y, h \upharpoonright y)$ holds. The formula is Σ^0_1-preserving. We show that it is a forcing question in a strong sense, that is, if it does not hold, then the condition already forces the Π^0_1 formula.

Lemma 7.3.9. Let (g, I) be a condition and $\varphi(G)$ be a Σ^0_1 formula.

- If $(g, I) \mathrel{?\vdash} \varphi(G)$ then there is an extension (h, I) forcing $\varphi(G)$;
- If $(g, I) \mathrel{?\nvdash} \varphi(G)$, then (g, I) forces $\neg\varphi(G)$. ★

Proof. Say $\varphi(G) \equiv \exists y \psi(y, G \upharpoonright_y)$.

1. Suppose $(g, I) \mathrel{?\vdash} \varphi(G)$. Then, letting $h \in [g, I]$ and $y \in M$ witness it, the condition (h, I) is an extension forcing $\varphi(G)$.
2. Suppose $(g, I) \mathrel{?\nvdash} \varphi(G)$. Suppose for the contradiction that there is some $h \in [g, I]$ such that $\varphi(h)$ holds. Unfolding the definition, there is some $y \in M$ such that $\psi(y, h \upharpoonright y)$ holds. Let $h_1 \subseteq h$ be a finite function such that $\operatorname{dom} g \subseteq \operatorname{dom} h_1$ and $h \upharpoonright y = h_1 \upharpoonright y$, then y and h_1 witness the fact that $(g, I) \mathrel{?\vdash} \varphi(G)$. Contradiction. So (g, I) forces $\neg\varphi(G)$. ■

It follows from Lemma 7.3.9 that if $\varphi(G)$ and $\psi(G)$ are two Σ^0_1-formulas such that $(g, I) \mathrel{?\vdash} \varphi(G)$ and $(g, I) \mathrel{?\nvdash} \psi(G)$, then there is an extension $(h, I) \leq (g, I)$ forcing $\varphi(G) \wedge \neg\psi(G)$. The following lemma shows that if $\mathcal{F}$ is sufficiently generic, then $\mathcal{M} \cup \{f_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$.

Lemma 7.3.10. Let (g, I) be a condition and $\varphi(x, X)$ be a Σ^0_1 formula such that (g, I) forces $\neg\varphi(b, G)$ for some $b \in M$. Then there is an extension $(h, I) \leq (g, I)$ and some $a \in M$ such that (h, I) forces $\neg\varphi(a, G)$, and if $a > 0$, (h, I)

forces $\varphi(a-1, G)$.[19] ★

19: Note the similarity of the proof of Lemma 7.3.10 with the proof of Lemma 7.3.6. We again only exploit some abstract properties of the forcing question.

Proof. Let $A = \{x \in M : (g, I) ?\vdash \varphi(x, G)\}$. Since the forcing question is Σ^0_1-preserving, the set A is $\Sigma^0_1(\mathcal{M})$. Moreover, (g, I) forces $\neg\varphi(b, G)$, so by Lemma 7.3.9, $(g, I) ?\nvdash \varphi(b, G)$, hence $b \notin A$. Since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$, and $A \neq M$, there is some $a \in M$ such that $a \notin A$, and if $a > 0$, then $a - 1 \in A$. By Lemma 7.3.9, (g, I) forces $\neg\varphi(a, G)$. Moreover, if $a > 0$, then since $a-1 \in A$, by Lemma 7.3.9, there is an extension (h, I) forcing $\varphi(a-1, G)$. Note that (h, I) forces $\neg\varphi(a, G)$. This completes the proof of Lemma 7.3.10. ■

We are now ready to prove Theorem 7.3.8. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. As mentioned, it induces a stable function $f_{\mathcal{F}} = \bigcup\{g : (g, I) \in \mathcal{F}\}$ whose limit is A. By Lemma 7.3.10, $\mathcal{M} \cup \{f_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$, so by Exercise 7.3.1, $\mathcal{M}[f_{\mathcal{F}}] \models \mathsf{RCA}_0$. This completes the proof of Theorem 7.3.8.■

The careful reader will have recognized some common pattern in the proofs of Theorem 7.3.3 and Theorem 7.3.8. Indeed, in both theorems, the lemma stating the preservation of Σ^0_1-induction used the existence of a Σ^0_1-preserving function which was able to give simultaneously a positive and a negative answer to two independent Σ^0_1 questions. This motivates the following definition.

Definition 7.3.11. Given a notion of forcing $(\mathbb{P}, \leq)$ and some $n \in \mathbb{N}$, a forcing question is (Σ^0_n, Π^0_n)-*merging* if for every $p \in \mathbb{P}$ and every pair of Σ^0_n formulas $\varphi(G), \psi(G)$ such that $p ?\vdash \varphi(G)$ but $p ?\nvdash \psi(G)$, then there is an extension $q \leq p$ forcing $\varphi(G) \wedge \neg\psi(G)$. ◊

Recall that a forcing question can be seen as a dividing line within the slice of conditions which do not already decide a formula (see Figure 7.2).

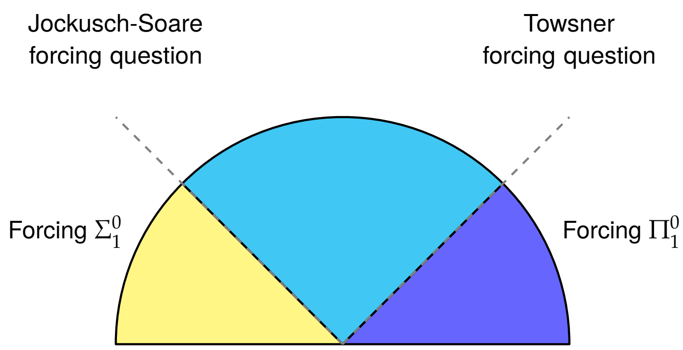

Figure 7.2: The yellow part and the dark blue part represent the conditions forcing a fixed Σ^0_1 and its negation, respectively. The light blue part represent the conditions of the third category. With Jockusch-Soare forcing (Theorem 7.3.3), the dividing line is at the left-most position, while for Towsner forcing (Theorem 7.3.8), the dividing line is at the opposite position.

As shown in the picture, Jockush-Soare forcing and Towsner forcing have extremal values. Any forcing question at one of these extremes is (Σ^0_1, Π^0_1)-merging, as if $p ?\vdash \varphi(G)$ and $p ?\nvdash \psi(G)$ for two Σ^0_1 formulas φ and ψ, then either p forces $\varphi(G)$ or p forces $\neg\psi(G)$, and one simply has to take the extension witnessing the answer to the other question. We now prove the abstract theorem associated to preservation of Σ^0_1-induction.

Theorem 7.3.12
Let $\mathcal{M} = (M, S) \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$ be a countable model and let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving (Σ^0_1, Π^0_1)-merging forcing question. For every sufficiently generic filter $\mathcal{F}$, $\mathcal{M} \cup \{G_{\mathcal{F}}\} \models \mathsf{I}\Sigma^0_1$.

Proof. It suffices to prove the following lemma:

Lemma 7.3.13. For every condition $p \in \mathbb{P}$ and every Σ^0_1-formula such that p forces $\neg\varphi(b, G)$ for some $b \in M$, there is an extension $q \leq p$ and some $a \in M$ such that q forces $\neg\varphi(a, G)$, and if $a > 0$, then q forces $\varphi(a-1, G)$. ★

Proof. Let $A = \{x \in M : p \mathrel{?\vdash} \varphi(x, G)\}$. Since the forcing question is Σ^0_1-preserving, the set A is $\Sigma^0_1(\mathcal{M})$. Moreover, p forces $\neg\varphi(b, G)$, so by definition of the forcing question, $p \mathrel{?\nvdash} \varphi(b, G)$, hence $b \notin A$. Since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$, and $A \neq M$, there is some $a \in M$ such that $a \notin A$, and if $a > 0$, then $a - 1 \in A$. If $a = 0$, then by definition of the forcing question, there is an extension $q \leq p$ forcing $\neg\varphi(0, G)$. If $a > 0$, then since the forcing question is (Σ^0_1, Π^0_1)-merging, there is an extension $q \leq p$ forcing $\neg\varphi(a, G)$ and $\varphi(a-1, G)$. ■

We are now ready to prove Theorem 7.3.12. Given a Σ^0_1 formula φ, let $\mathcal{D}_\varphi$ be the set of all conditions $q \in \mathbb{P}$ forcing either $\forall b \varphi(b, G)$, or $\neg\varphi(0, G)$, or $\varphi(a-1, G) \wedge \neg\varphi(a, G)$ for some $a > 0$. It follows from Lemma 7.3.13 that every $\mathcal{D}_\varphi$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_\varphi : \varphi \in \Sigma^0_1\}$-generic, so $\mathcal{M} \cup \{G_\mathcal{F}\} \models \mathsf{I}\Sigma^0_1$. This completes the proof of Theorem 7.3.12.■

Exercise 7.3.14 (Cholak, Jockusch and Slaman [27]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ be a countable model and $\vec{R} = R_0, R_1, \dots$ be a sequence of sets in $\mathcal{M}$. Use a formalized notion of computable Mathias forcing (see Exercise 3.2.8) to prove the existence of an infinite $\vec{R}$-cohesive set $G \subseteq M$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0$. Deduce that $\mathsf{RCA}_0 + \mathsf{COH}$ is Π^1_1-conservative over RCA_0. ★

7.4 Isomorphism theorem

The choice of RCA_0 as a base theory capturing computable mathematics can be questioned because of Σ^0_1-induction. Indeed, by Proposition 7.2.5, Σ^0_n-induction corresponds to Σ^0_n-regularity, so Σ^0_1-induction will add every bounded c.e. set in the model. By Post's theorem, one would arguably restrict the base theory to Δ^0_1-induction to have Δ^0_1-regularity.[20] Simpson and Smith [59] introduced RCA^*_0, the theory based on Robinson arithmetic (Q), together with the Δ^0_1-comprehension scheme, the Δ^0_0-induction scheme ($\mathsf{I}\Delta^0_0$) and the statement of the totality of the exponential (exp).

Exercise 7.4.1. Show that RCA^*_0 proves $\mathsf{I}\Delta^0_1$ and $\mathsf{B}\Sigma^0_1$. ★

Although RCA_0 remains the mainstream base theory to found reverse mathematics, RCA^*_0 is useful to compare very weak statements of arithmetic [59]. In particular, the notion of infinity is not robust in RCA^*_0, as some unbounded sets may not be in bijection with $\mathbb{N}$. As it turns out, RCA^*_0 became an essential tool in the study of models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, through the notion of jump model.

Definition 7.4.2. Given a model $\mathcal{M} = (M, S)$, its *jump model* is the structure $\mathcal{N} = (M, \Delta^0_2\text{-Def}(\mathcal{M}))$, where $\Delta^0_2\text{-Def}(\mathcal{M})$ denotes the Δ^0_2 definable sets with parameters in $\mathcal{M}$. We then call $\mathcal{M}$ a *ground model* of $\mathcal{N}$. ◇

The following exercise puts a bridge between models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ and models of RCA^*_0.

20: There are mostly two reasons why RCA_0 was chosen as the base theory rather than RCA^*_0: a historical and a pragmatical one.
Historically, Friedman used a language of functions rather than sets, with a Δ^0_0-recursion principle which turned out to be equivalent to Σ^0_1-induction. See Hirschfeldt [7, Chapter 4] for a more thorough discussion on the subject.
Pragmatically, basic features such as the equivalence of the various notions of infinity, are equivalent to Σ^0_1-induction. One expects from a base theory to be able to prove the robustness of the core concepts. In particular, the provably total functions over RCA_0 are the primitive recursive functions, while RCA^*_0 only proves the totality of the elementary recursive functions.

Exercise 7.4.3 (Belanger [60]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$. Show that $\mathcal{M} \models \mathsf{B}\Sigma^0_2$ iff $(M, \Delta^0_2\text{-Def}(\mathcal{M})) \models \mathsf{RCA}^*_0$. ★

Models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ play an important role in the study of Ramsey's theorem for pairs. Let RT^1 be the statement $\forall a \mathsf{RT}^1_a$. This statement easily follows from $\mathsf{RCA}_0 + \mathsf{RT}^2_2$. Indeed, given a coloring $f : \mathbb{N} \to a$ for some $a \in \mathbb{N}$, one can define the coloring $g : [\mathbb{N}]^2 \to 2$ by $g(x, y) = 1$ iff $f(x) = f(y)$. Any infinite g-homogeneous set is f-homogeneous. The following proposition therefore shows that any model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ satisfies $\mathsf{B}\Sigma^0_2$.

Proposition 7.4.4 (Hirst [61]). $\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_2 \leftrightarrow \mathsf{RT}^1$. ★

Proof.

- Assume $\mathsf{B}\Sigma^0_2$. Let $f : \mathbb{N} \to a$ be an instance of RT^1 for some $a \in \mathbb{N}$. Suppose that there is no infinite f-homogeneous set. Then $(\forall x < a)(\exists y)(\forall w)[w > y \to f(w) \neq x]$. Then by $\mathsf{B}\Sigma^0_2$, there is some $b \in \mathbb{N}$ such that $(\forall x < a)(\exists y < b)(\forall w)[w > y \to f(w) \neq x]$. Then $(\forall x < a)[f(b) \neq x]$, contradiction.
- Assume RT^1. Let $\theta(x, y, w)$ be a Δ^0_0-formula. Fix $a \in \mathbb{N}$ and suppose that $(\forall x < a)(\exists y)(\forall z)\theta(x, y, w)$. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(t)$ is the least $b < t$ such that $(\forall x < a)(\exists y < b)(\forall w < t)\theta(x, y, w)$, if such a b exists. Otherwise, let $f(t) = t$. Suppose first that there exists an infinite f-homogeneous set H, for some color b. Then $(\forall x < a)(\exists y < b)\forall w \theta(x, y, w)$ holds by RT^1. Suppose now that there is no infinite f-homogeneous set. Then by RT^1, the range of f is unbounded. Construct a strictly increasing sequence $(t_s)_{s \in \mathbb{N}}$ such that $f(t_s) < f(t_{s+1})$ for every $s \in \mathbb{N}$. Let $g : \mathbb{N} \to a$ be such that $g(s)$ is the least $x < a$ such that $(\forall y < f(t_s) - 1)(\exists w < t_s)\neg\theta(x, y, w)$. By RT^1, there is an infinite g-homogeneous set S for some color x. Fix some $y \in \mathbb{N}$. Since S is infinite, there is some $s \in S$ such that $f(t_s) - 1 > y$. So $(\exists w < t_s)\neg\theta(x, y, w)$ holds. Hence $(\forall y)(\exists w)\neg\theta(x, y, w)$, contradiction. ■

Π^1_1-conservation theorems over RCA^*_0 follow the same structure as over RCA_0, mutatis mutandis.

Exercise 7.4.5 (Simpon and Smith [59]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}^*_0$ and fix a set $G \subseteq M$. Show that

1. If G is M-regular, then $\mathcal{M}[G] \models \mathsf{I}\Delta^0_0$.
2. If moreover $\mathcal{M} \cup \{G\} \models \mathsf{B}\Sigma^0_1$, then $\mathcal{M}[G] \models \mathsf{RCA}^*_0$. ★

Exercise 7.4.6 (Simpon and Smith [59]). Let P be a Π^1_2 problem. Suppose that for every countable topped model $\mathcal{M} = (M, S) \models \mathsf{RCA}^*_0$, and every $X \in S$ such that $\mathcal{M} \models X \in \operatorname{dom} \mathsf{P}$, there is set $Y \subseteq M$ such that $\mathcal{M}[Y] \models \mathsf{RCA}^*_0 + (Y \in \mathsf{P}(X))$. Adapt the proof of Proposition 7.3.2 to show that $\mathsf{RCA}^*_0 + \mathsf{P}$ is Π^1_1-conservative over RCA^*_0. ★

Let WKL^*_0 be the theory RCA^*_0 augmented with the statement "Every infinite binary tree admits an infinite path". Simpson and Smith proved that WKL^*_0 is Π^1_1-conservative over RCA^*_0, and we shall see that this is the best result possible, in the sense that weak König's lemma is the strongest Π^1_2 statement that is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$.

Theorem 7.4.7 (Simpson and Smith [59])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0^*$ be a countable model and $T \subseteq 2^{<M}$ be an infinite tree in S. There is an M-regular path $G \in [T]$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0^*$.[21]

Proof. The proof of Theorem 7.4.7 is very similar to that of Theorem 7.3.3. It also uses Jockusch-Soare forcing whose conditions are infinite trees $T_1 \subseteq T$ in S, partially ordered by inclusion. Lemma 7.3.4 and Lemma 7.3.5 both hold in models of RCA_0^*, so for every sufficiently generic filter $\mathcal{F}$, $\bigcap_{T_1 \in \mathcal{F}}[T_1]$ is a singleton $G_{\mathcal{F}}$, which is M-regular. The main difference lies in the following lemma:

Lemma 7.4.8. Let T_1 be a condition, $a \in M$, and $\varphi(x, y, X)$ be a Σ^0_1 formula forcing $(\forall x < a)(\exists y)\varphi(x, y, G)$. Then there is some $b \in M$ such that T_1 forces $(\forall x < a)(\exists y < b)\varphi(x, y, G)$. ★

Proof. Let $\theta(x, z) \equiv T_1 \mathrel{?\vdash} (\exists y < z)\varphi(x, y, G)$. Since the forcing question is Σ^0_1-preserving, the formula θ is $\Sigma^0_1(\mathcal{M})$. Moreover, T_1 forces $(\forall x < a)(\exists y)\varphi(x, y, G)$, so by Lemma 7.3.5, for every $x < a$, $T_1 \mathrel{?\vdash} \exists y \varphi(x, y, G)$. By Σ^0_1-compactness[22] of the forcing question, for every $x < a$, there is some $z \in M$ such that $T_1 \mathrel{?\vdash} (\exists y < z)\varphi(x, y, G)$. Thus, for every $x < a$, there is some $z \in M$ such that $\theta(x, z)$ holds. By $\mathsf{B}\Sigma^0_1$, there is some $b \in M$ such that $(\forall x < a)(\exists z < b)\theta(x, z)$. Unfolding the definition of θ, $(\forall x < a)(\exists z < b) T_1 \mathrel{?\vdash} (\exists y < z)\varphi(x, y, G)$. By Lemma 7.3.5, for every $x < a$, there is some $z < b$ such that T_1 forces $(\exists y < z)\varphi(x, y, G)$, so T_1 forces $(\exists y < b)\varphi(x, y, G)$. ■

We are now ready to prove Theorem 7.4.7. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. By Lemma 7.3.4, there is a unique M-regular set $G \in \bigcap_{T_1 \in \mathcal{F}}[T_1]$. In particular, $G \in [T]$. By Lemma 7.3.6, $\mathcal{M} \cup \{G\} \models \mathsf{B}\Sigma^0_1$, so by Exercise 7.4.5, $\mathcal{M}[G] \models \mathsf{RCA}_0^*$. This completes the proof of Theorem 7.4.7.■

21: The proof of preservation of $\mathsf{B}\Sigma^0_1$ (Lemma 7.4.8) uses the existence of a Σ^0_1-preserving, Σ^0_1-compact forcing question such that if $p \mathrel{?\vdash} \varphi(G)$ holds for some Σ^0_1 formula φ, then p already forces $\varphi(G)$. Since weak König's lemma is the strongest Π^1_2 theory which is Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$, the Jockusch-Soare forcing is in some sense the strongest notion of forcing with the existence of a forcing question with the above mentioned properties.

22: Recall that a forcing question is Σ^0_n-*compact* if for every $p \in \mathbb{P}$ and every Σ^0_n formula $\varphi(G, x)$, if $p \mathrel{?\vdash} \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ such that $p \mathrel{?\vdash} \exists x \in F\ \varphi(G, x)$.

Corollary 7.4.9 (Simpson and Smith [59])
WKL_0^* *is a Π^1_1-conservative extension of* RCA_0^*.

Proof. Immediate by Theorem 7.4.7 and Exercise 7.4.6. ■

Fiori-Carones, Kołodziejczyk, Wong and Yokoyama [62] proved a beautiful isomorphism theorem for countable models of $\mathsf{WKL}_0^* + \neg\mathsf{I}\Sigma^0_1$ with many consequences, not only for provability over RCA_0^*, but also for conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Theorem 7.4.10 (Fiori-Carones et al [62])
Let (M, S_0) and (M, S_1) be countable models of WKL_0^* such that $(M, S_0 \cap S_1) \models \neg\mathsf{I}\Sigma^0_1$. Let $\vec{c}$ be a tuple of elements of M and $\vec{C}$ be a tuple of elements of $S_0 \cap S_1$. Then there is an isomorphism h between (M, S_0) and (M, S_1) such that $h(\vec{c}) = \vec{c}$ and $h(\vec{C}) = \vec{C}$.

Proof. Let $\mathcal{M} = (M, S_0 \cap S_1)$ and $\mathcal{M}_i = (M, S_i)$ for each $i < 2$. A *cut* is an initial segment of M which is closed under successor. Any model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$ contains a proper Σ^0_1-definable cut. Indeed, since $\varphi(x)$ be a Σ^0_1 formula such that $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))$ holds, but $\neg\varphi(a)$ for some $a \in \mathbb{N}$.

Let $I = \{x \in \mathbb{N} : (\forall x' < x)\varphi(x')\}$. By B$\Sigma^0_1$, I is Σ^0_1-definable, and by construction, I is a proper cut. Such a cut I is not necessarily closed under other operations such as addition, multiplication or exponentiation. With some extra work, one can prove that every model of IΔ^0_0 + exp +¬IΣ^0_1 contains a proper Σ^0_1-definable cut which is closed under exp (see [63, Lemma 9]). Therefore, fix a $\Sigma^0_1(\mathcal{M})$ proper cut I which is closed under exp.

Let $\psi(x,y)$ be a $\Delta^0_0(\mathcal{M})$ formula such that $I = \{x \in M : \mathcal{M} \models \exists y \psi(x,y)\}$. Let $a_0 \in M \setminus I$ and let B be the set of all pairs $\langle i, a_i \rangle \in \mathbb{N}$ such that a_{i+1} is the least element greater than a_i satisfying $(\forall x \leq i)(\exists y \leq a_{i+1})\psi(x,y)$. The set B is $\Delta^0_0(\mathcal{M})$-definable, of cardinality I and the sequence $(a_i)_{i \in I}$ is enumerated in increasing order and cofinal in M. Note that B belongs $S_0 \cap S_1$ by Δ^0_0-comprehension. By adding the set B to the tuple $\vec{C}$, we ensure that the relation $\theta(x,i) \equiv x = a_i$ is $\Delta_0(\vec{C})$.

We build the isomorphism h by a back-and-forth construction. Let $\mathbb{P}$ be the notion of forcing[23] whose conditions are tuples $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ such that

1. $\vec{r}$ and $\vec{s}$ are finite vectors of same standard length, of elements of M ;
2. $\vec{R}$ and $\vec{S}$ are finite vectors of same standard length, of elements of S_0 and S_1, respectively ;
3. $b \in M$ is such that $b > I$;
4. for each $i \in I$ and each Δ^0_0-formula δ with $\ulcorner \delta \urcorner < b$, $\mathcal{M}_0 \models \delta(a_i, \vec{r}, \vec{R})$ iff $\mathcal{M}_1 \models \delta(a_i, \vec{s}, \vec{S})$.[24] [25]

Intuitively, a condition $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ is a partial assignment of h over the domain $\vec{r} \cup \vec{R}$ and with range $\vec{s} \cup \vec{S}$. The initial condition is $(\vec{c}, \vec{c}, \vec{C}, \vec{C}, b)$ for a fixed $b > I$. A condition $(\vec{r}', \vec{s}', \vec{R}', \vec{S}', b')$ *extends* $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ if $b' \leq b$, $\vec{r} \leq \vec{r}'$, $\vec{s} \leq \vec{s}'$, $\vec{R} \leq \vec{R}'$ and $\vec{S} \leq \vec{S}'$.

Before proving our main density lemmas, we need to state a technical coding lemma which generalizes Proposition 7.2.5.

Lemma 7.4.11 (Chong and Mourad [64]). Let $\mathcal{M} = (M, S) \models$ RCA$^*_0$. Then for every pair of bounded disjoint Σ^0_1 sets $X, Y \subseteq M$, there exists some $s \in M$ such that $\mathrm{Ack}(s) \cap (X \cup Y) = X$.[26] ⋆

Proof. Let φ and ψ be two Δ^0_0 formulas such that $X = \{x \in M : \mathcal{M} \models (\exists z)\varphi(x,z)\}$ and $Y = \{x \in M : \mathcal{M} \models (\exists z)\psi(x,z)\}$. Let $a \in M$ be a common bound for X and Y and let $b \in M$ be such that $\mathrm{Ack}(b) = \{0, \dots, a-1\}$. Suppose for the contradiction that for all $s \leq b$, $\mathrm{Ack}(s) \cap (X \cup Y) \neq X$. Then

$$(\forall s < b)(\exists x < a)[(x \in \mathrm{Ack}(s) \wedge x \in Y) \vee (x \notin \mathrm{Ack}(s) \wedge x \in X)]$$

By BΣ^0_1, there is a uniform bound $\hat{z} \in M$ such that

$$(\forall s < b)(\exists x < a)\left[\begin{array}{ll} & (x \in \mathrm{Ack}(s) \wedge (\exists z < \hat{z})\psi(x,z)) \\ \vee & (x \notin \mathrm{Ack}(s) \wedge (\exists z < \hat{z})\varphi(x,z)) \end{array}\right]$$

Let $S = \{x < a : (\forall z < \hat{z})\neg\psi(x,z)\}$. The set S is Δ^0_0, hence is M-coded by some $s \leq b$. Moreover, $S \cap (X \cup Y) = X$, contradiction. ■

The following lemma shows that one can add any first-order element to the domain of h while preserving the invariant. Since the models (M, S_0) and (M, S_1) play a symmetric role, it is also dense to add any first-order element to the range of h.

23: The construction uses the language of forcing for convenience, but it will not use its whole machinery, such as the forcing relation.

24: We write $\ulcorner \delta \urcorner$ for the Gödel number of a formula. One can think of it as the integer whose binary representation is the string of the formula. In particular, the Gödel number of a standard formula is a standard integer. Note that we work with Δ^0_0-formulas with first-order parameters, that is, in a language enriched with symbol constants for each first-order element. The constraint $\ulcorner \delta \urcorner < b$ prevents from using first-order parameters larger than $\log b$.

25: Since we also consider non-standard Δ^0_0-formulas, the satisfaction relation $\models$ is replaced by a Σ^0_1-formula Sat$_0$ expressing the truth definition for Δ^0_0-formulas (see Hájek and Pudlák [50]).

26: Recall that given $s \in M$, we write $\mathrm{Ack}(s)$ for the set $F \subseteq M$ coded by s, that is, such that $s = \sum_{x \in F} 2^x$.

Lemma 7.4.12. Let $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ be a condition and $d \in M$. There is an extension $(\vec{r}d, \vec{s}e, \vec{R}, \vec{S}, b')$ for some $e, b' \in M$. ⋆

Proof. Let $b' > I$ be sufficiently small with respect to b. Let $D \subseteq I \times b'$ be the following set

$$\{(i, \ulcorner \delta \urcorner) \in I \times b' : \delta \text{ is } \Delta^0_0 \text{ and } \mathcal{M}_0 \models \delta(a_i, \vec{r}d, \vec{R})\}$$

Both D and $(I \times b') \setminus D$ are bounded and Σ^0_1-definable, so by Lemma 7.4.11, there is some $t \in M$ such that $\mathrm{Ack}(t) \cap (I \times b') = D$. Moreover, since $D \subseteq I \times b'$ and $I < b'$, we can assume $t < 2^{b' \times b'}$. Let $i' \in I$ be such that $d \leq a_{i'}$. By choice of t, for every $i \in I$, the structure $\mathcal{M}_0$ satisfies

$$(\exists y \leq a_{i'})(\forall j \leq i) \bigwedge_{\ulcorner \delta \urcorner < b'} [\delta(a_j, \vec{r}y, \vec{R}) \leftrightarrow (j, \ulcorner \delta \urcorner) \in \mathrm{Ack}(t)]$$

as witnessed by taking $y = d$. For every $i \in I$ such that $i \geq i'$, $\mathcal{M}_0$ therefore satisfies the Δ^0_0-formula $\gamma(a_i, \vec{r}, \vec{R})$ defined by

$$(\exists x, z \leq a_i)(\exists y \leq x)(x = a_{\mathbf{i'}} \wedge z = \mathbf{t} \wedge (\forall j \leq i)(\forall v \leq a_i)$$
$$(v = a_j \rightarrow \bigwedge_{\ulcorner \delta \urcorner < \mathbf{b'}} [\delta(v, \vec{r}y, \vec{R}) \leftrightarrow (j, \ulcorner \delta \urcorner) \in \mathrm{Ack}(z)]))$$

For each $i \in I$, the formula γ is written in a language enriched with symbol constants for i', b', t.[27] The formula γ written in binary starts with a part of length $\mathcal{O}(\log(i') + \log(b') + \log(t))$. It is then followed by a conjunction composed of b' conjuncts, each of length $\mathcal{O}(b')$. Since $i' < b'$ and $\log(t) < b' \cdot b'$, the formula γ has length $\mathcal{O}(b' \times b')$. Since I is an exponential cut, we can take b' sufficiently small so that $\ulcorner \gamma \urcorner < b$.

27: The relation $\theta(x, i) \equiv x = a_i$ being $\Delta_0(\vec{C})$, the parameter i can be obtained from a_i, and conversely, $a_{i'}$ can be obtained from i'. Thus, i and $a_{i'}$ are not considered as parameters.

The big conjunction is not part of the language, hence is a shorthand for a non-standard conjunction with b' many conjuncts. Because of this and because of the non-standard parameters i', b' and t, the formula has a non-standard length.

The variable z is introduced to move the parameter t outside of the big conjunction. Therefore, t is coded only once, instead of b' many times.

By definition of a condition, $\mathcal{M}_1 \models \gamma(a_i, \vec{s}, \vec{S})$ for each $i \in I$ such that $i \geq i'$. Therefore $\mathcal{M}_1$ satisfies

$$(\exists y \leq a_{i'})(\forall j \leq i) \bigwedge_{\ulcorner \delta \urcorner < b'} [\delta(a_j, \vec{s}y, \vec{S}) \leftrightarrow (j, \ulcorner \delta \urcorner) \in \mathrm{Ack}(t)]$$

Since $\mathcal{M}_1 \models \mathsf{B}\Sigma^0_1$, there is some fixed $e \in M$ that witnesses the first existential above for every $i \in I$ such that $i \geq i'$. Then $(\vec{r}d, \vec{s}e, \vec{R}, \vec{S}, b')$ is our desired extension. ∎

The following lemma shows that one can add any second-order element to the domain of h. Here again, by symmetry, any second-order element can also be added to the range of h.

Lemma 7.4.13. Let $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b)$ be a condition and $X \in S_0$. There is an extension $(\vec{r}, \vec{s}, \vec{R}X, \vec{S}Y, b')$ for some $b' \in M$ and $Y \in S_1$. ⋆

Proof. Let $b' > I$ be sufficiently small with respect to b and $D \subseteq I \times b'$ be the following set

$$\{(i, \ulcorner \delta \urcorner) \in I \times b' : \delta \text{ is } \Delta^0_0 \text{ and } \mathcal{M}_0 \models \delta(a_i, \vec{r}, \vec{R}X)\}$$

Again, D and $(I \times b') \setminus D$ are bounded and Σ^0_1-definable, so by Lemma 7.4.11, there is some $t < 2^{b' \times b'}$ such that $\mathrm{Ack}(t) \cap (I \times b') = D$. By choice of t, there is some $i' \in I$ such that for every $i \in I$ with $i \geq i'$, the structure $\mathcal{M}_0$ satisfies the formula

$$(\exists F \subseteq [0, \log a_i))(\forall j \leq i)(\forall v \leq \log\log a_i)$$
$$(v = a_j \to \bigwedge_{\ulcorner\delta\urcorner < b'}[\delta(a_j, \vec{r}, \vec{R}F) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(t)]$$

as witnessed by taking $F = X \cap [0, \log a_i)$.[28] For every $i \in I$ such that $i \geq i'$, $\mathcal{M}_0$ therefore satisfies the Δ^0_0-formula $\gamma(a_i, \vec{r}, \vec{R})$ defined by

$$(\exists F \subseteq [0, \log a_i))(\exists z \leq a_i)(\forall j \leq i)(\forall v \leq \log\log a_i)$$
$$(z = \mathbf{t} \wedge v = a_j \to \bigwedge_{\ulcorner\delta\urcorner < \mathbf{b}'}[\delta(a_j, \vec{r}, \vec{R}F) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(z)]$$

28: It is not clear at first sight that $\mathcal{M}_0$ satisfies this formula, since δ is witnessed by $F = X \cap [0, \log a_i)$ instead of X. However, since the first-order parameters of δ are smaller than $\max(\log\log a_i, \vec{r})$, then the gödel number the formula δ evaluated on its parameters is smaller than $\log a_i$, hence its evaluation is left unchanged by replacing X with $X \cap [0, \log a_i)$.

For each $i \in I$, the formula γ is written in a language enriched with symbol constants for b' and t. By a similar analysis to Lemma 7.4.12, if b' is sufficiently small with respect to b, then $\ulcorner\gamma\urcorner < b$. Thus by definition of a condition, for every $i \in I$ such that $i \geq i'$, $\mathcal{M}_1$ satisfies

$$(\exists F \subseteq [0, \log a_i))(\forall j \leq i)(\forall v \leq \log\log a_i)$$
$$(v = a_j \to \bigwedge_{\ulcorner\delta\urcorner < b'}[\delta(a_j, \vec{s}, \vec{S}F) \leftrightarrow (j, \ulcorner\delta\urcorner) \in \mathrm{Ack}(t)]$$

Let $T \subseteq 2^{<M}$ be the Π^0_1 tree of all σ such that for every $i \in I$ with $i' \leq i \leq |\sigma|$, the set $F = \{s < \log a_i : \sigma(s) = 1\}$ witnesses the first existential of the previous formula. Since $\mathcal{M}_1 \models \mathsf{WKL}^*_0$, there is an infinite path Y through T in $\mathcal{M}_1$. Then $(\vec{r}, \vec{s}, \vec{R}X, \vec{S}Y, b')$ is our desired extension. ■

We are now ready to prove Theorem 7.4.10. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. Let h be the function induced by $\mathcal{F}$. By Lemma 7.4.12 and Lemma 7.4.13, h is a bijection from $M \cup S_0$ to $M \cup S_1$.

We claim that h is an isomorphism. We only prove the case of addition. Let $+_0$ and $+_1$ be the interpretation of the addition symbol in (M, S_0) and (M, S_1), respectively. Given $u, v \in M$, consider the Δ^0_0-formula

$$\delta(a, x, y, z) \equiv x + y = z$$

Let $w = u +_0 v$, and let $(\vec{r}, \vec{s}, \vec{R}, \vec{S}, b) \in \mathcal{F}$ be a condition such that $u, v, w \in \vec{r}$. Since the formula δ is standard, then $\ulcorner\delta\urcorner \in \omega < b$, so by definition of a condition, for each $i \in I$,

$$\mathcal{M}_0 \models \delta(a_i, u, v, w) \text{ iff } \mathcal{M}_1 \models \delta(a_i, h(u), h(v), h(w))$$

Since $u +_0 v = w$, then $\mathcal{M}_0 \models \delta(a_i, u, v, w)$, so $\mathcal{M}_1 \models \delta(a_i, h(u), h(v), h(w))$, and therefore $h(u) +_1 h(v) = h(w) = h(u +_0 v)$. This completes the proof of Theorem 7.4.10. ■

As an immediate consequence of Theorem 7.4.10, weak König's lemma is the maximal Π^1_2-problem which is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$.

Theorem 7.4.14 (Fiori-Carones et al [62])
Let P *be a* Π^1_2*-problem. Then* $\mathsf{RCA}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ *is* Π^1_1*-conservative over* $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ *iff* $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$.

Proof. First, by Theorem 7.4.7, $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$, so if $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$, $\mathsf{RCA}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$. We prove the other direction.

If $\mathsf{RCA}^*_0+\mathsf{P}+\neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0+\neg\mathsf{I}\Sigma^0_1$, then by Theorem 7.4.7 and a standard amalgamation argument (see Yokoyama [65]), $\mathsf{WKL}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$. Let $\mathcal{M} \models \mathsf{WKL}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ be a countable model. By Theorem 7.4.10, every coded ω-model of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ in $\mathcal{M}$ is elementarily equivalent to $\mathcal{M}$, hence satisfies P, so by Gödel's completeness theorem, $\mathsf{WKL}^*_0 + \mathsf{P} + \neg\mathsf{I}\Sigma^0_1$ proves that every coded ω-model of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ satisfies P. By Π^1_1-conservation, $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ proves the same statement.

Let $\mathcal{M}$ be a countable model of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ and $A \in \mathcal{M}$ witness $\neg\mathsf{I}\Sigma^0_1$. By Theorem 4.3.2, $\mathcal{M}$ contains a coded ω-model $\mathcal{N}$ of WKL^*_0 with $A \in \mathcal{N}$. In particular, $\mathcal{N} \models \mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$, so $\mathcal{N} \models \mathsf{P}$. Again by Theorem 7.4.10, $\mathcal{N}$ is an elementary submodel of $\mathcal{M}$, so $\mathcal{M} \models \mathsf{P}$. By Gödel's completeness theorem, $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$. ■

7.5 Conservation over $\mathsf{B}\Sigma^0_2$

The system $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ plays an important role in reverse mathematics for two reasons. First, it characterizes the first-order part of some statements related to Ramsey's theorem for pairs [66]. Second, it is the highest level in the hierarchy of induction which satisfies Hilbert's program. Indeed, $\mathsf{I}\Sigma^0_2$ is not finitistically reducible, as it proves the consistency of $\mathsf{I}\Sigma^0_1$, which is a Π_1 statement not provable over $\mathsf{I}\Sigma^0_1$ (see Hájek and Pudlák [50, Theorem 4.33]). On the other hand, by Parsons, Paris and Friedman (see [67]), $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is $\forall\Pi^0_3$-conservative over RCA_0.[29] In particular, $\mathsf{RCA}_0+\mathsf{B}\Sigma^0_2$ is a Π_2-conservative extension of PRA.

29: $\forall\Pi^0_n$ is the class of formulas starting with a universal set quantifier, followed by a Π^0_n formula. Every Π^1_1-formula is $\forall\Pi^0_n$ for some $n \in \mathbb{N}$.

Exercise 7.5.1. Let P be a Π^1_2 problem. Suppose that for every countable topped model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, and every $X \in S$ such that $\mathcal{M} \models X \in \operatorname{dom} \mathsf{P}$, there is a set $Y \subseteq M$ such that $\mathcal{M}[Y] \models \mathsf{RCA}_0+\mathsf{B}\Sigma^0_2+(Y \in \mathsf{P}(X))$. Adapt the proof of Proposition 7.3.2 to show that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{P}$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ★

Conservation over RCA_0 involved first-jump control to build sets while preserving $\mathsf{I}\Sigma^0_1$. One would therefore expect conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ to involve second-jump control to preserve $\mathsf{B}\Sigma^0_2$. However, as mentioned in Section 4.1, effectivization of first-jump control can often be used to obtain simple proofs of jump preservations. First-jump control being usually significantly simpler than second-jump control, one usually prefers to use the former technique. Actually, as a consequence of the isomorphism theorem for $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$, in the context of Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$, effective first-jump control can be used without loss of generality (see Fiori-Carones et al. [62]).

Exercise 7.5.2. Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set $Y \subseteq M$. Let $G \subseteq M$ be such that $(G \oplus Y)' \leq_T Y'$.[30] Use Exercise 7.4.3 and Exercise 7.4.5 to show that $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ★

30: $\mathsf{Q}+\mathsf{I}\Sigma^0_1$ is enough to prove the existence of a universal Σ^0_1-formula. From it, we can define a robust notion of Turing jump X' as the set of all codes of true $\Sigma^0_1(X)$ formulas.

Recall that the Turing reduction is robust in models of RCA^*_0 (see Groszek and Slaman [49]). If $\mathcal{M} = (M, S) \models RCA_0 + \mathsf{B}\Sigma^0_2$ then its jump model $\mathcal{N} = (M, \Delta^0_2\text{-Def}(\mathcal{M}))$ satisfies RCA^*_0, so the Turing reduction is robust between Δ^0_2 sets in models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Effective constructions in the context of weak arithmetic raise an issue that already occurs in higher computability theory. Many effectiveness constructions are done inductively along the integers, satisfying a requirement at each step.

In the case of a non-standard model of weak arithmetic, some steps are non-standard, hence are preceded by infinitely many other steps.[31] If induction fails, it might be the case that the set of steps of the construction forms a proper cut, and thus that some requirement at a non-standard step is never satisfied. Even if the model is countable, since the construction is internal, one cannot fix a countable enumeration of the integers.

31: Models of weak arithmetic have common similarities with ordinals. Indeed, one can reason inductively among both, and a non-standard integer, like an infinite ordinal, is infinite from an external point of view, but there is no infinite decreasing sequence starting from it.

Consider for example Cohen forcing over a non-standard model $\mathcal{M} = (M, S)$. Let $(D_a)_{a \in M}$ be a collection of dense sets. The naive approach to the construction of a $\vec{D}$-generic set G would consist in letting $\sigma_0 = \epsilon$, and σ_{a+1} be the lexicographically least extension of σ_a belonging to D_a. If the dense sets are to complex with respect to the level of induction in $\mathcal{M}$, the set $I = \{a \in M : \sigma_a \text{ is defined } \}$ might be a proper cut, while the set $\{|\sigma_a| : a \in I\}$ will be cofinal in M.

To circumvent this problem, one resorts to a technique from higher computability theory called *Shore blocking*.[32] Suppose one proves that the collection $(D_a)_{a \in M}$ is dense in a strong sense: for every $b \in M$ and every $\sigma \in 2^{<M}$, there exists an extension $\tau \geq \sigma$ intersecting every $(D_a)_{a<b}$ simultaneously. One can then build a $\vec{D}$-generic set G by letting $\sigma_0 = \epsilon$, and σ_{a+1} be the lexicographically least extension of σ_a intersecting $(D_c)_{c<|\sigma_a|}$ simultaneously. Then, even if the set $I = \{a \in M : \sigma_a \text{ is defined } \}$ is a proper cut, the resulting set G will be $\vec{D}$-generic, as for every $c \in M$, there is a stage $a \in I$ such that $|\sigma_a| > c$, hence σ_{a+1} intersects D_c. The main difficulty of conservation theorems over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ consists of proving the blocking lemma.

32: The "blocking" terminology might be confusing. It should be understood as satisfying blocks of requirements simultaneously instead of one by one.

Our first proof of Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is based on a formalization in weak arithmetic by Hájek [68] of the low basis theorem from Jockusch and Soare [9].

Theorem 7.5.3 (Hájek [68])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $P \in [T]$ such that $(P \oplus Y)' \leq_T Y'$ and $\mathcal{M}[P] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.[33]

33: The proof of Theorem 7.5.3 is slightly more verbose than necessary, but it is more modular, in that it is easy to interleave other blocking lemmas to satisfy more requirements. This will be useful for Theorem 7.6.16.

Proof. Consider the notion of forcing whose *conditions* are pairs (σ, T_1) where

- T_1 is a primitive Y-recursive infinite subtree of T ;
- $\sigma \in 2^{<M}$ is a *stem* of T_1, that is, every element in T_1 is comparable with σ.

The *interpretation* of a condition (σ, T_1) is $[\sigma, T_1] = [T_1]$. A condition (τ, T_2) *extends* (σ, T_1) (written $(\tau, T_2) \leq (\sigma, T_1)$) if $\sigma \preceq \tau$ and $T_2 \subseteq T_1$. A *code* of a condition (σ, T_1) is a pair $\langle \sigma, a \rangle$ such that a is a primitive Y-recursive code for T_1.

We need to satisfy the following requirements for every $b \in M$:

- $\mathcal{T}_b$: $G{\upharpoonright}_b$ is decided[34]
- $\mathcal{R}_b$: $(G \oplus Y)'{\upharpoonright}_b$ is decided

34: Technically, this requirement is not necessary, as deciding $(G \oplus Y)'$ implies deciding G. However, explicitly satisfying this requirement will be convenient for the construction.

For this, we prove a blocking lemma to decide the jump, Lemma 7.5.4. Given a condition (σ, T_1) and $e \in M$, let

- $(\sigma, T_1) \Vdash \Phi_e^{G \oplus Y}(e){\downarrow}$ if $\Phi_e^{\sigma \oplus Y}(e){\downarrow}$;
- $(\sigma, T_1) \Vdash \Phi_e^{G \oplus Y}(e){\uparrow}$ if for every $\tau \in T_1$, $\Phi_e^{\tau \oplus Y}(e){\uparrow}$;
- $(\sigma, T_1) \Vdash \rho \prec (G \oplus Y)'$ for some $\rho \in 2^{<M}$ if for every $e < |\rho|$, if $\rho(e) = 1$ then $(\sigma, T_1) \Vdash \Phi_e^{G \oplus Y}(e){\downarrow}$, and if $\rho(e) = 0$ then $(\sigma, T_1) \Vdash \Phi_e^{G \oplus Y}(e){\uparrow}$.

Note that the predicate $(\sigma, T_1) \Vdash \rho \prec (G \oplus Y)'$ is $\Pi^0_1(Y)$ uniformly in σ, T_1 and ρ.

Lemma 7.5.4. For every condition (σ, T_1) and $b \in M$, there is an extension (τ, T_2) and some M-coded $\rho \in 2^b$ such that $(\tau, T_2) \Vdash \rho \prec (G \oplus Y)'$. ★

Proof. Let U be the set of all $\rho \in 2^b$ such that the tree

$$T_\rho = \{\tau \in T_1 : (\forall e < b)(\rho(e) = 0 \to \Phi_e^{\rho \oplus Y}(e)\uparrow)\}$$

is infinite. U is $\Pi^1_0(Y)$ and hence M-finite, and it is non-empty as it contains the string $1111\dots$.

Let $\rho \in U$ be its lexicographically smallest element. For every $e < b$ such that $\rho(e) = 1$, the minimality of ρ implies that the set of $\tau \in T_\rho$ such that $\Phi_e^{\tau \oplus Y}(e)\uparrow$ is M-finite, so there is a level ℓ_e such that for every $\tau \in T_\rho \cap 2^{\ell_e}$, $\Phi_e^{\tau \oplus Y}(e)\downarrow$. The set $\{e < b : \rho(e) = 1\}$ is M-finite, so by $\mathsf{B}\Sigma^0_1$, there is an upper-bound ℓ of all the ℓ_e's. Finally, by Lemma 7.3.4, there is a node $\tau \in T_\rho \cap 2^\ell$ such that $T_2 = \{\mu \in T_\rho : \mu \textit{ is comparable with } \tau\}$ is M-infinite.

We claim that $(\tau, T_2) \Vdash \rho \prec (G \oplus Y)'$. Fix some $e < b$. Suppose $\rho(e) = 0$. Then $\Phi_e^{\mu \oplus Y}(e)\uparrow$ for every $\mu \in T_2$ since $T_2 \subseteq T_\rho$. Hence, $(\tau, T_2) \Vdash \Phi_e^{G \oplus Y}(e)\uparrow$. Suppose $\rho(e) = 1$. The definition of τ ensure that $\Phi_e^{\tau \oplus Y}(e)\downarrow$, so $(\tau, T_2) \Vdash \Phi_e^{G \oplus Y}(e)\downarrow$. ■

We are now ready to prove Theorem 7.5.3.

Construction. We build a decreasing sequence (σ_s, T_s) of conditions and then take G for the union of the σ_s. We also build an increasing sequence (ρ_s) such that $(G \oplus Y)'$ will be the union of the ρ_s. Initially, let $\sigma_0 = \sigma'_0 = \epsilon$ and $T_0 = T$. During the construction, we will ensure that $\langle \sigma_s, T_s \rangle, |\rho_s| \leq s$. Each stage will be either of type $\mathcal{T}$, or of type $\mathcal{R}$. The stage 0 is of type $\mathcal{T}$.

Assume that (σ_s, T_s) and ρ_s are already defined. Let $s_0 < s$ be the latest stage at which we switched the stage type. We have two cases.

Case 1: s is of type $\mathcal{T}$. If there a code $\langle \tau, \hat{T} \rangle \leq s$ such that $(\tau, \hat{T}) \leq (\sigma_s, T_s)$ and $|\tau| \geq s_0$, then let $\sigma_{s+1} = \tau$, $T_{s+1} = \hat{T}$, $\rho_{s+1} = \rho_s$ and let $s+1$ be of type $\mathcal{R}$. Otherwise, the elements are left unchanged and we go to the next stage.

Case 2: s is of type $\mathcal{R}$. If there a code $\langle \tau, \hat{T} \rangle \leq s$ such that $(\tau, \hat{T}) \leq (\sigma_s, T_s)$ and $(\sigma_s, \hat{T}) \Vdash \rho \prec (G \oplus Y)'$ for some $\rho \in 2^{s_0}$, then let $\sigma_{s+1} = \tau$, $T_{s+1} = \hat{T}$, $\rho_{s+1} = \rho$ and let $s+1$ be of type $\mathcal{T}$. Otherwise, the elements are left unchanged and we go to the next stage.

This completes the construction.

Verification. Since the size of σ_s, ρ_s and the index of T_s are bounded by s, there is a $\Delta^0_1(Y')$-formula $\phi(s)$ stating that the construction can be pursued up to stage s. Our construction implies that the set $\{s | \phi(s)\}$ is $\Delta^0_1(Y')$ and forms a cut, so by $\mathsf{I}\Delta^0_1(Y')$, the construction can be pursued at every stage.

Let $G = \bigcup_{s \in M} \sigma_s$. By Lemma 7.3.4 and Lemma 7.5.4, each type of stage changes M-infinitely often. Thus, $\{|\sigma_s| : s \in M\}$ and $\{|\rho_s| : s \in M\}$ are M-infinite. In particular, G is an M-regular path in T and $Y' \geq_T (G \oplus Y)'$. By Exercise 7.5.2, $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

This completes the proof of Theorem 7.5.3.

Corollary 7.5.5 (Hájek [68])
$\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2$ *is a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.[35]

35: Exercise 7.5.1 and Corollary 7.5.5 easily adapt to prove that for every $n \geq 2$ that $\mathsf{WKL}_0 + \mathsf{I}\Sigma^0_n$ and $\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_n$ are Π^1_1-conservative extensions of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n$, respectively.

Proof. Immediate by Theorem 7.5.3 and Exercise 7.5.1. ■

We have seen in Theorem 7.3.8 that Δ^0_2 sets are indistinguishable from arbitrary sets from the viewpoint of models of RCA_0, in that every countable model of RCA_0 can be ω-extended into another model of RCA_0 relative to which a fixed set becomes Δ^0_2. This is not true anymore when considering models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Indeed, by Theorem 7.2.11and Exercise 7.2.12, given a countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ and a non-M-regular set $A \subseteq M$, there is no ω-extension $\mathcal{N} \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ of $\mathcal{M}$ relative to which A is Δ^0_2, since it would imply M-regularity of A. On the other hand, Belanger [60] proved a formalized Friedberg jump inversion theorem with some extra assumptions on the set A.

Theorem 7.5.6 (Belanger [60])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ *be a countable model topped by a set* Y*, and* $A \subseteq M$ *be a set such that* $\mathcal{M}[A \oplus Y'] \models \mathsf{RCA}^*_0$*. Then there is a set* $G \subseteq M$ *such that* $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ *and* $A \oplus Y' \equiv_T (G \oplus Y)'$

Proof. Based on Shoenfield's limit lemma [8], we will construct a function $f : \mathbb{N}^2 \to 2$ such that for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists and equals $A(x)$. We are therefore going to build directly the function f by forcing, and let G be the graph of f.

Consider the notion of forcing whose *conditions* is a pairs (g, a)[36], such that

36: Contrary to Theorem 7.3.8, the set $A \oplus Y'$ is M-regular, so we can work with pairs (g, a) and lock a non-standard number of columns simultaneously.

- $g \subseteq M^2 \to \{0, 1\}$ is a partial function with two parameters whose domain is M-finite, representing an initial segment of the function f that we are building.
- $a \in M$ is the number of "locked" columns, meaning that from now on, when we extend the domain of g with a new pair (x, y), if $x < a$ then $g(x, y) = (A \oplus Y')(x)$.

The *interpretation* $[g, a]$ of a condition (g, a) is the class of all partial or total functions $h \subseteq M^2 \to 2$ such that

(1) $g \subseteq h$, i.e. $\operatorname{dom} g \subseteq \operatorname{dom} h$ and for all $(x, y) \in \operatorname{dom} g$, $g(x, y) = h(x, y)$;
(2) for all $(x, y) \in \operatorname{dom} h \setminus \operatorname{dom} g$, if $x < a$, then $h(x, y) = (A \oplus Y')(x)$.

A condition (h, b) *extends* (g, a) (denoted $(h, b) \leq (g, a)$) if $b \geq a$ and $h \in [g, a]$.

We will need to satisfy three kind of requirements for every $b \in M$:

- $\mathcal{T}_b$: $b^2 \subseteq \operatorname{dom} f$
- $\mathcal{R}_b$: $(f \oplus Y)' \upharpoonright_b$ is decided
- $\mathcal{S}_b$: $(\forall a < b) \lim_y f(a, y)$ exists

For this, we prove two lemmas, Lemma 7.5.7 and Lemma 7.5.8, stating that the set of conditions forcing $\mathcal{T}_b$ and $\mathcal{R}_b$ is dense for every $b \in M$. Density of the requirement $\mathcal{S}_b$ simply consists, given a condition (g, a), of taking the extension $(g, \max(a, b))$.

Lemma 7.5.7. For every condition (g, a) and $b \in M$, there is an extension $(h, a) \leq (g, a)$ such that $b^2 \subseteq \operatorname{dom} h$. ⋆

Proof. Since $A \oplus Y'$ is M-regular, the string $\sigma = (A \oplus Y')\upharpoonright_a$ is M-coded. By Δ^0_0-comprehension, the set $h = g \cup \{(x, y, \sigma(x)) \in b^2 \times 2 : (x, y) \notin \operatorname{dom} g\}$ is M-coded. By construction, $h \in [g, a]$ and $b^2 \subseteq \operatorname{dom} h$, so (h, a) is the desired extension. ■

Given a condition (g, a) and $e \in M$, let

- $(g, a) \Vdash \Phi^{f \oplus Y}_e(e){\downarrow}$ if $\Phi^{g \oplus Y}_e(e){\downarrow}$;
- $(g, a) \Vdash \Phi^{f \oplus Y}_e(e){\uparrow}$ if for every finite $h \in [g, a]$, $\Phi^{h \oplus Y}_e(e){\uparrow}$;
- $(g, a) \Vdash \rho \prec (f \oplus Y)'$ for some $\rho \in 2^{<M}$ if for every $e < |\rho|$, if $\rho(e) = 1$ then $(g, a) \Vdash \Phi^{f \oplus Y}_e(e){\downarrow}$, and if $\rho(e) = 0$ then $(g, a) \Vdash \Phi^{f \oplus Y}_e(e){\uparrow}$.

Note that the predicate $(g, a) \Vdash \rho \prec (f \oplus Y)'$ is $\Delta^0_2(Y)$ uniformly in g, a and ρ.

Lemma 7.5.8. For every condition (g, a) and $b \in M$, there is an extension $(h, a) \leq (g, a)$ and some M-coded $\rho \in 2^b$ such that $(h, a) \Vdash \rho \prec (f \oplus Y)'$.⋆

Proof. Let U be the set of all $\rho \in 2^b$ such that

$$(\exists h \in [g, a])(\exists t)(\forall e < b)(\rho(e) = 1 \rightarrow \Phi^{h \oplus Y}_e(e)[t]\downarrow)$$

Note that U is $\Sigma^0_1(Y)$, hence is M-finite. Moreover, U is non-empty, as it contains the string $000\dots$. Let $\rho \in U$ be the lexicographically maximal element, and let $h \in [g, a]$ witness that $\rho \in U$.

We claim that (h, a) forces $\rho \prec (G \oplus Y)'$. Fix some $e < b$. Suppose $\rho(e) = 1$. Then $\Phi^{h \oplus Y}_e(e){\downarrow}$, hence $(h, a) \Vdash \Phi^{f \oplus Y}_e(e){\downarrow}$. Suppose $\rho(e) = 0$. The maximality of ρ ensures that for every $\hat{h} \in [h, a]$, $\Phi^{\hat{h} \oplus Y}_e(e){\uparrow}$. It follows that $(h, a) \Vdash \Phi^{f \oplus Y}_e(e){\uparrow}$. ■

We are now ready to prove Theorem 7.5.6.

Construction. We will build a decreasing sequence (g_s, a_s) of conditions and then take for f the union of the g_s. We will also build an increasing sequence (ρ_s) such that $(f \oplus Y)'$ will be the union of the ρ_s. Initially, let $g_0 = \rho_0 = \epsilon$ and $a_0 = 0$. Each stage will be either of type $\mathcal{T}$, of type $\mathcal{R}$ or of type $\mathcal{S}$. The stage 0 is of type $\mathcal{T}$.

Assume that (g_s, a_s) and ρ_s are already defined. Let $s_0 < s$ be the latest stage at which we switched the stage type. We have three cases.

Case 1: s is of type $\mathcal{T}$. If there exists some $h \in 2^{\leq s \times \leq s}$ such that $(h, a_s) \leq (g_s, a_s)$ and $s_0 \times s_0 \subseteq \operatorname{dom} h$, then let $g_{s+1} = h$, $a_{s+1} = a_s$, $\rho_{s+1} = \rho_s$, and let $s + 1$ be of type $\mathcal{R}$. Otherwise, the elements are left unchanged and we go to the next stage.

Case 2: s is of type $\mathcal{R}$. If there exists some $h \in 2^{\leq s \times \leq s}$ and some $\mu \in 2^{s_0}$ such that $(h, a_s) \leq (g_s, a_s)$, and $(h, a_s) \Vdash \mu \prec (f \oplus Y)'$, then let $g_{s+1} = h$, $a_{s+1} = a_s$, $\rho_{s+1} = \mu$, and let $s + 1$ be of type $\mathcal{S}$. Otherwise, the elements are left unchanged and we go to the next stage.

Case 3: s is of type $\mathcal{S}$. Let $g_{s+1} = g_s$, $a_{s+1} = s$, $\rho_{s+1} = \rho_s$, and let $s + 1$ be of type $\mathcal{T}$. This completes the construction.

Verification. Since the size of g_s, a_s and ρ_s are bounded by s, there is a $\Delta^0_1(A \oplus Y')$-formula $\phi(s)$ stating that the construction can be pursued up to stage s. Our construction implies that the set $\{s|\phi(s)\}$ is a cut, so since $\mathcal{M}[A \oplus Y'] \models \mathsf{I}\Delta^0_1$, the construction can be pursued at every stage.

Let $f = \bigcup_{s \in M} g_s$. By Lemma 7.5.7 and Lemma 7.5.8, each type of stage changes M-infinitely often. Thus, $\operatorname{dom} f = M^2$, and $\{a_s : s \in M\}$ and $\{|\rho_s| : s \in M\}$ are both cofinal in M. It follows that f is stable and $A \oplus Y' \geq_T (f \oplus Y)'$. Since $\mathcal{M}[A \oplus Y'] \models \mathsf{RCA}^*_0$, then $\mathcal{M}[(f \oplus Y)'] \models \mathsf{RCA}_0$, so by Exercise 7.4.3, $\mathcal{M}[f] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Conversely, since $\lim_y f(\cdot, y) = A \oplus Y'$, then $A \oplus Y' \equiv_T (f \oplus Y)'$. This completes the proof of Theorem 7.5.6. ■

We now prove that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Recall that thanks to the characterization of COH in terms of Δ^0_2 approximations of paths through infinite Δ^0_2 binary trees (Exercise 3.4.3), there exist two main ways to build solutions to instances of COH: either picking a path, and constructing a Δ^0_2 approximation of it, or directly building a cohesive set through computable Mathias forcing. We shall start with the former approach. Belanger [60] proved that the above characterization holds over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Exercise 7.5.9 (Belanger [60]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$. Show that $\mathcal{M} \models \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ iff $(M, \Delta^0_2\text{-Def}(\mathcal{M})) \models \mathsf{WKL}^*_0$. ★

Theorem 7.5.10 (Chong, Slaman and Yang [66])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable topped model and $\vec{R} = R_0, R_1, \dots$ be a uniform sequence in S. Then there is an infinite $\vec{R}$-cohesive set $C \subseteq M$ such that $\mathcal{M}[C] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Proof. Say $\mathcal{M}$ is topped by a set Y. Given $\sigma \in 2^{<M}$, let

$$R_\sigma = \bigcap_{\sigma(n)=0} \overline{R}_n \bigcap_{\sigma(n)=1} R_n$$

Let $T = \{\sigma \in 2^{<M} : (\exists x > |\sigma|) x \in R_\sigma\}$. The tree T is infinite and $\Sigma^0_1(\mathcal{M})$. Since $(M, \Delta^0_2\text{-Def}(\mathcal{M})) \models \mathsf{RCA}^*_0$, by Theorem 7.4.7, there is a path $P \in [T]$ such that $\mathcal{M}[P \oplus Y'] \models \mathsf{RCA}^*_0$. By Theorem 7.5.6, there is a set $G \subseteq M$ such that $P \oplus Y' \leq_T (G \oplus Y)'$ and $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Let $(P_s)_{s \in M}$ be a Δ^0_2 approximation of P in $\mathcal{M}[G]$. Let $(x_a)_{a \in M}$ be inductively defined as follows: First, $x_0 = 0$. Given x_a, let $\langle s, x\rangle$ be the least tuple such that $s, x > x_a$ and $x \in R_{P_s \upharpoonright x_a}$. Such a tuple exists, since by $\mathsf{B}\Sigma^0_2$, there is some $s > x_a$ such that $P_s \upharpoonright x_a = P \upharpoonright x_a$, and that $R_{P \upharpoonright x_a}$ is infinite. Then let $x_{a+1} = x$. This completes the construction.

By Σ^0_1-induction, x_a is defined for every $a \in M$. Let $D = \{x_a : a \in M\}$. We claim that D is $\vec{R}$-cohesive. Indeed, given $a \in M$, by $\mathsf{B}\Sigma^0_2$, there is some $k > a$ such that for every $t > k$, $P_t \upharpoonright a = P \upharpoonright_a$. For every $t > k$, $x_{t+1} \in R_{P_s \upharpoonright x_t}$ for some $s > x_t$. Since $s > x_t > t > k > a$, $R_{P_s \upharpoonright x_t} \subseteq R_{P_s \upharpoonright a} = R_{P \upharpoonright a}$, so for all but finitely many $t \in M$, $x_t \in R_{P \upharpoonright a}$.

Since D is Σ^0_1, it contains an infinite Δ^0_1 subset $C \subseteq D$. In particular, $C \in \mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, so $\mathcal{M}[C] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ■

Corollary 7.5.11 (Chong, Slaman and Yang [66])
$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ *is a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Proof. Immediate by Theorem 7.5.10 and Exercise 7.5.1. ■

There exists another more direct construction of an $\vec{R}$-cohesive set by Mathias forcing, which does not involve the formalized Friedberg jump inversion theorem.

Exercise 7.5.12 (Le Houérou, Levy Patey and Yokoyama [69]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y, and let $\vec{R} = R_0, R_1, \dots$ be a uniform sequence in S. Let P be as in the proof of Theorem 7.5.10. A *condition* is a pair (σ, a) where $\sigma \in 2^{<M}$ and $a \in M$. The *interpretation* $[\sigma, a]$ of a condition (σ, a) is the class of all G such that $\sigma \prec G$ and $G \subseteq \sigma \cup R_{P\upharpoonright a}$. In other words, the interpretation of (σ, a) is the interpretation of the Mathias condition $(\sigma, R_{P\upharpoonright a} \setminus \{0, \dots, |\sigma|\})$. Build a $\Delta^0_1(P \oplus Y')$ infinite decreasing sequence of conditions while deciding the jump as in the proof of Theorem 7.5.6. ★

Recall that by Theorem 4.5.2, if a Σ^0_2 set A is co-hyperimmune, then it admits an infinite low subset. This theorem was then used by Hirschfeldt and Shore [23] to prove that every infinite computable stable linear order admits an infinite ascending or descending sequence of low degree (see Exercise 4.5.4). The proof of Theorem 4.5.2 does not seem to be formalizable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ because of Shore blocking. However, Chong, Slaman and Yang [66] used the transitive features of linear orders to prove that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{SADS}$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, where SADS is the Π^1_2-problem whose instances are stable linear orders, and solutions are infinite ascending or descending sequences.[37]

37: Actually, SADS implies $\mathsf{B}\Sigma^0_2$ over RCA_0, but the proof is non-trivial and involved a model-theoretic argument. See Hirschfeldt and Shore [23] and Chong, Lempp and Yang [70].

Exercise 7.5.13 (Chong, Slaman and Yang [66]). Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ be a countable model topped by a set Y. Let $\mathcal{L} = (M, <_{\mathcal{L}})$ be a computable stable linear order in $\mathcal{M}$.

1. Show that $\mathcal{M}$ does not contain any infinite descending sequence, then there is an M-regular infinite ascending sequence $G \subseteq M$ such that $(G \oplus Y)' \leq_T Y'$.
2. Deduce that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{SADS}$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. ★

7.6 Shore blocking and BME

The most naive way to prove a blocking lemma given a family $(D_a)_{a<b}$ of dense sets would be to start from a condition p_0, and then inductively letting p_{a+1} be an extension of p_a in D_a for every $a < b$. Then, p_b would be an extension simultaneously intersecting all the dense sets simultaneously. However, as explained above, in models of weak arithmetic, the set $I = \{a : p_a \text{ is defined }\}$ might be a proper cut bounded by b. We therefore used some combinatorial features of each construction to prove conservation theorems over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. As usual, these can often be formulated as properties of the forcing questions.

The main concern for Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ is to prove a blocking lemma to decide an initial segment of the jump. If an extension witnessing a positive answer to the forcing question can be found uniformly in the condition, then the naive sequential approach holds.

Definition 7.6.1. Let $(\mathbb{P}, \leq)$ be a notion of forcing and $n \geq 1$. A forcing question is *uniformly Σ^0_n-preserving* if for every Σ^0_n formula $\varphi(G, x, y)$, there is a Σ^0_n set $W \subseteq \mathbb{P} \times \mathbb{N} \times \mathbb{P} \times \mathbb{N}$ such that

- For every $(p, n, q, m) \in W$, $q \leq p$ and q forces $\varphi(G, m, n)$;
- For every condition $p \in \mathbb{P}$ and $n \in \mathbb{N}$, $p \mathrel{?}\!\vdash \exists x \varphi(G, x, n)$ if and only if $(p, n, q, m) \in W$ for some $q \leq p$ and $m \in \mathbb{N}$. ◊

Note that any uniformly Σ^0_n-preserving forcing question is Σ^0_n-preserving.[38]

38: Uniform Σ^0_n-preservation has two levels of uniformity: deciding a Σ^0_n-formula is Σ^0_n uniformly in the conditions, and if the forcing question holds, then one can find an extension witnessing the positive answer uniformly.

This assumes of course that there is a notion of computability over forcing conditions, which can be obtained by manipulating conditions through their codes.

Theorem 7.6.2
Let $\mathcal{M} = (M, S) \models \mathsf{Q}+\mathsf{I}\Sigma^0_1$ be a countable model topped by Y and let $(\mathbb{P}, \leq)$ be a notion of forcing with a uniformly Σ^0_1-preserving forcing question. For every condition $p \in \mathbb{P}$ and $b \in M$, there is an extension $q \leq p$ and some $\rho \in 2^{<M}$ of length b such that q forces $\rho \prec (G \oplus Y)'$.

Proof. Let $\varphi(G, F, y)$ be the following $\Sigma^0_1(\mathcal{M})$-formula, where F is a first-order variable coding a set

$$(\exists t)(F \subseteq \{0, \dots, b-1\} \wedge \operatorname{card} F = y \wedge (\forall e \in F)\Phi^{G\oplus Y}_e(e)[t]\downarrow)$$

Let W be the $\Sigma^0_1(\mathcal{M})$ set witnessing that the function is uniformly Σ^0_1-preserving. Let U be the $\Sigma^0_1(\mathcal{M})$ set of all $F \subseteq \{0, \dots, b-1\}$ such that there is some $k \in M$ and a sequence $\langle p_0, F_0, \dots, p_{k-1}, F_{k-1}, p_k\rangle$ satisfying

- $p_0 = p$; $F = F_{k-1}$;
- $(p_s, s, p_{s+1}, F_s) \in W$ for every $s < k$.

We claim that $\emptyset \in U$. Indeed, $p \mathrel{?}\!\vdash (\exists F)\varphi(G, F, 0)$, so there is some F such that $\operatorname{card} F = 0$ and some $q \leq p$ such that $(p, 0, q, F) \in W$. In particular, $F = \emptyset$, and the sequence $(p, \emptyset, q)$ witnesses that $\emptyset \in F$.

By Exercise 7.2.3, there is a maximal element $F \in U$ for inclusion. Let $\rho \in 2^b$ be such that $\{e < b : \rho(e) = 1\} = F$ and let $\langle p_0, F_0, \dots, p_{k-1}, F_{k-1}, p_k\rangle$ witness that $F \in U$. By definition of W, p_k forces $\varphi(G, F, k-1)$, and by maximality of F, $p_k \mathrel{?}\!\nvdash (\exists F)\varphi(G, F, k)$. By definition of the forcing question, there is an extension $q \leq p_k$ forcing $(\forall F)\neg\varphi(G, F, k)$.

We claim that q forces $\rho \prec (G \oplus Y)'$. By definition of φ, for every $e \in F$, p_k forces $\Phi^{G\oplus Y}_e(e)\downarrow$. Let $e < b$ be such that $e \notin F$. There is no extension of q forcing $\Phi^{G\oplus Y}_e(e)\downarrow$, otherwise $F \cup \{e\}$ would contradict the fact that q forces $\neg\varphi(G, F, k)$. Thus, q forces $\Phi^{G\oplus Y}_e(e)\uparrow$. This completes the proof of Theorem 7.6.2. ■

Exercise 7.6.3. Show that Cohen forcing admits a uniformly Σ^0_1-preserving forcing question. ★

Exercise 7.6.4. Let $(\mathbb{P}, \leq)$ be the notion of forcing of Theorem 7.5.6, and given $a \in M$, let $\mathbb{P}_a$ be the set of conditions of the form (g, a).

1. Show that for every $a \in M$, $(\mathbb{P}_a, \leq)$ admits a uniformly Σ^0_1-preserving forcing question.

2. Show that if a condition (g, a) forces a Σ^0_1 or a Π^0_1 property over $(\mathbb{P}_a, \leq)$, then so does it over $(\mathbb{P}, \leq)$.
3. Deduce the existence of a blocking lemma to decide the jump for $(\mathbb{P}, \leq)$. ★

Many forcing questions appearing in practice are not Σ^0_1-uniform. Thankfully, it often represents a dividing line at one of the extremes of Figure 7.2. In this case again, one can prove a blocking lemma to decide an initial segment of a the jump.

Definition 7.6.5. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*extremal* if for every formula $\varphi \in \Gamma$ and every condition $p \in \mathbb{P}$, if $p \,?\!\vdash \varphi(G)$ then p forces $\varphi(G)$. ◊

By extension, we say that a forcing question for Σ^0_n-formulas is Π^0_n-*extremal* if for every Σ^0_n-formula φ and every condition $p \in \mathbb{P}$, if $p \,?\!\nvdash \varphi(G)$, then p forces $\neg\varphi(G)$. Many notions of forcing considered in practice admit a Σ^0_1-preserving forcing question which is Π^0_1-extremal. In this case, one can obtain an abstract blocking lemma to decide the jump.

Theorem 7.6.6
Let $\mathcal{M} = (M, S) \models \mathsf{Q}+\mathsf{I}\Sigma^0_1$ be a countable model topped by Y and let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_1-preserving Π^0_1-extremal forcing question. For every condition $p \in \mathbb{P}$ and $b \in M$, there is an extension $q \leq p$ and some $\rho \in 2^{<M}$ of length b such that q forces $\rho \prec (G \oplus Y)'$.

Proof. Consider the following set

$$U = \{\rho \in 2^b : q \,?\!\vdash (\exists t)(\forall e < b)(\rho(e) = 1 \rightarrow \Phi_e^{G\oplus Y}(e)[t]\downarrow)\}$$

The set U is $\Sigma^0_1(\mathcal{M})$ since the forcing question is Σ^0_1-preserving. Moreover, U is non-empty, as it contains the string $000\dots$. By Exercise 7.2.3, there is a lexicographically maximal element $\rho \in U$. By maximality, for every $e' < |\sigma|$ such that $\sigma(e') = 0$,

$$p \,?\!\nvdash (\exists t)(\forall e < b)((\rho(e) = 1 \vee e = e') \rightarrow \Phi_e^{G\oplus Y}(e)[t]\downarrow)$$

so since the forcing question is Π^0_1-extremal, p forces

$$(\forall t)(\exists e < b)((\rho(e) = 1 \vee e = e') \wedge \Phi_e^{G\oplus Y}(e)[t]\uparrow)$$

Since $\rho \in U$, there is an extension $q \leq p$ and some $t \in \mathbb{N}$ such that q forces $(\forall e < b)(\rho(e) = 1 \rightarrow \Phi_e^{G\oplus Y}(e)[t]\downarrow)$. In particular, for every $e' < |\sigma|$ such that $\sigma(e') = 0$, q forces $\Phi_e^{G\oplus Y}(e)\uparrow$. It follows that q forces $\rho \prec (G \oplus Y)'$. This completes the proof of Theorem 7.6.6. ■

Exercise 7.6.7. Show that Theorem 7.6.6 also holds with a Σ^0_1-preserving Σ^0_1-extremal forcing question. ★

Recall that Ramsey's theorem for pairs can be decomposed into the cohesiveness principle (COH) and the pigeonhole principle for Δ^0_2 instances ($\mathsf{RT}^{1\prime}_2$). By Corollary 7.5.11 and an amalgamation theorem of Yokoyama [65], $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ iff so is $\mathsf{RCA}_0 + \mathsf{RT}^{1\prime}_2$. One would naturally want to adapt the proof that $\mathsf{RT}^{1\prime}_2$ admits a weakly low basis

(Theorem 4.7.5). However, the natural forcing question for the pigeonhole principle is neither uniformly Σ^0_1-preserving, nor extremal. It is therefore not clear how to prove a blocking lemma deciding the jump.

Question 7.6.8. Is $\mathsf{RCA}_0+\mathsf{RT}^2_2$ a Π^1_1-conservative extension of $\mathsf{RCA}_0+\mathsf{B}\Sigma^0_2$?★

As mentioned, the forcing question for the pigeonhole principle is not uniformly Σ^0_1-preserving, but enjoys a weaker uniformity property: if the answer to a Σ^0_1 question is positive, then one can effectively find a finite set of *pre-conditions*[39], one of each being a valid condition forcing the Σ^0_1 property. Successive applications of the forcing question to prove a blocking lemma then yields a c.e. tree of bounded depth, motivating the following definition.

Definition 7.6.9. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a c.e. tree.

- A *monotone enumeration* of T is a uniformly computable sequence of finite coded[40] trees $T_0, T_1, \dots$ such that $T_0 = \{\epsilon\}$, $\bigcup_s T_s = T$ and for every stage s such that $T_{s+1} \neq T_s$, every node in $T_{s+1} \setminus T_s$ is an immediate extension of a leaf in T_s.
- The tree T is k*-bounded* if every node in T has length at most k. A tree is *bounded* if it is k-bounded for some $k \in \mathbb{N}$.[41] ◇

A monotone enumeration of a tree is such that all the immediate successors of a node are enumerated in one block at the same stage. Therefore, it is not possible to add immediate children at a later stage. On the other hand, it is not possible to decide ahead of time whether a node is a leaf or not. An easy induction over k shows that every k-bounded Σ^0_1 tree with a monotone enumeration is finite. Let BME_* be the Π^1_2-problem whose instances are enumerations of k-bounded Σ^0_1 trees for some $k \in \mathbb{N}$, and whose solutions are canonical codes for the tree.[42]

Exercise 7.6.10 (Chong, Slaman and Yang [29]). Show that $\mathsf{Q} \vdash \mathsf{I}\Sigma^0_2 \to \mathsf{BME}_*$. ★

Over RCA_0, the Bounded Monotone Enumeration principle and $\mathsf{B}\Sigma^0_2$ are incomparable, and their conjunction is strictly weaker than $\mathsf{I}\Sigma^0_2$. In fact, BME_* happens to be equivalent to multiple existing principles, and therefore has an arguably natural proof-theoretic strength.

Exercise 7.6.11 (Kreuzer and Yokoyama [71]). A formula $\phi(x, y)$ represents a partial function if $(\forall x, y, z)(\phi(x, y) \wedge \phi(x, z) \to y = z)$. A string $\sigma \in \mathbb{N}^{<\mathbb{N}}$ is an *approximation*[43] of a partial function $\phi(x, y)$ if

$$(\forall i < |\sigma| - 1)(\forall x, y)[(x < \sigma(i) \wedge \phi(x, y)) \to y < \sigma(i+1)]$$

Given a collection of formulas Γ, let $\mathsf{P}\Gamma$ be the scheme "For every partial function $\phi \in \Gamma$ and every length $k \in \mathbb{N}$, there is an approximation of length k." Show that $\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \mathsf{BME}_* \leftrightarrow \mathsf{P}\Sigma^0_1$. ★

The Bounded Monotone Enumeration principle can also be understood in terms of well-foundedness of ordinals. It requires first to fix a representation of ordinals. By Cantor normal form, every ordinal α can be uniquely written as $\omega^{\beta_0} c_0 + \omega^{\beta_1} c_1 + \dots + \omega^{\beta_{k-1}} c_{k-1}$, where $c_0, \dots, c_{k-1}$ are non-zero natural numbers, and and $\beta_0 > \beta_1 > \dots > \beta_{k-1} > 0$ are ordinals. Based on this normal form, every ordinal less than ϵ_0[44] can be represented by a finite tree of

39: A *Mathias pre-condition* is a pair (σ, X), where X is not longer required to be infinite. Given a Turing ideal $\mathcal{M}$ coded by a set M, the set of all Mathias pre-conditions over $\mathcal{M}$ is M-computable, while the set of Mathias conditions over $\mathcal{M}$ is not.

40: A monotone enumeration can be represented as a sequence of integers, each of them being the canonical code of a finite tree. Thus, the complete information about each tree is known.

41: Technically, the tree being Σ^0_1, it may not belong to the model. However, a Σ^0_1 tree is k-bounded if at any stage, it contains nodes of length at most k.

42: Given a monotone enumeration $(T_s)_{s\in\mathbb{N}}$, a stage s is *expansionary* if $T_{s+1} \neq T_s$. Over RCA^*_0, BME_* is equivalent to stating that the expansionary stages of a bounded monotone enumeration are bounded. Indeed, letting $s \in \mathbb{N}$ be such a bound, then $T_s = T$, but T_s is finite, hence so is T. On the other direction, if T is finite, then for every $\sigma \in T$, there is a stage s such that $\sigma \in T_s$. By $\mathsf{B}\Sigma^0_1$, there is a uniform bound on such stages.

43: The notion was introduced by Paris and Hájek [72], who proved that $\mathsf{B}\Sigma^0_2$ and $\mathsf{P}\Sigma^0_1$ are incomparable over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

44: Recall that ϵ_0 is the least fixpoint of the operation $\alpha \mapsto \omega^\alpha$. In particular,

$$\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

coefficients. To simplify manipulation, it is more convenient to work with *regular trees*, that is, finite trees such that the set of immediate successors of a node is an initial segment of $\mathbb{N}$, together with an evaluation map which associates to each node a coefficient. Using this representation, the map $(\vec{\beta}, \vec{c}) \mapsto \sum \omega^{\beta_i} c_i$ and the order $\leq$ are provably Δ^0_1 in $\mathsf{Q} + \mathsf{I}\Sigma^0_1$. See Hájek and Pudlák [50, p. II.3] for a formal development of ordinals over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

Given an ordinal $\alpha \leq \epsilon_0$, let $\mathsf{WF}(\alpha)$ be the statement "α is well-founded", that is, there is no infinite decreasing sequence of ordinals smaller than α. Proving that α is well-founded for some large ordinals requires some non-trivial amount of induction.[45] Actually, $\mathsf{WF}(\omega^\omega)$ is equivalent to BME_* over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

45: The statement

$$\forall a(\mathsf{WF}(\omega^a) \to \mathsf{WF}(\omega^{a+1}))$$

is provable over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$. It follows that in any model $\mathcal{M} = (M, S) \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$, the set $I = \{a \in M : \mathcal{M} \models \mathsf{WF}(\omega^a)\}$ is a cut. Actually, in such models, I is an additive cut, that is, if $a \in I$, then $a + a \in I$, but there exists non-standard models of $\mathsf{Q} + \mathsf{I}\Sigma^0_1$ in which $I = \sup\{a \cdot n : n \in \omega\}$ for some non-standard integer a. In such models, I does not have any better closure property than additivity.

Theorem 7.6.12 (Kreuzer and Yokoyama [71])
$\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \mathsf{WF}(\omega^\omega) \to \mathsf{BME}_*$.

Proof. Given a k-bounded finite coded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we define an ranking $\zeta_T : T \to \omega^k$ inductively as follows:

$$\zeta_T(\sigma) = \begin{cases} 0 & \text{if } |\sigma| = k \\ \omega^{k-|\sigma|} & \text{if } \sigma \text{ is a leaf in } T \text{ and } |\sigma| < k \\ \sum_{\sigma \cdot a \in T} \zeta_T(\sigma \cdot a) & \text{if } \sigma \text{ is not a leaf.} \end{cases}$$

Note that $\zeta_T(\epsilon) < \omega^\omega$ for any such tree T. Given a monotone enumeration of a k-bounded Σ^0_1 tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, if $T_{s+1} \neq T_s$, then $\zeta_{T_{s+1}}(\epsilon) < \zeta_{T_s}(\epsilon)$[46], so by $\mathsf{WF}(\omega^\omega)$, there are only finitely such stages. Letting s be larger than all such stages. Then $T_s = T$, so T is finite coded. ■

46: Here, ϵ denotes the empty string, hence the root of the tree. It should not be confused with the ordinal ϵ_0.

Exercise 7.6.13 (Kreuzer and Yokoyama [71]). Fix $k \in \mathbb{N}$. Given a k-bounded finite coded tree T, let ζ_T be the function of Theorem 7.6.12.

1. Prove that for every ordinal $\alpha < \omega^k$, there is a k-bounded finite coded tree T such that $\zeta_T(\epsilon) = \alpha$.
2. Prove that for every k-bounded finite coded tree T and every $\alpha < \zeta_T(\epsilon)$, there is a a k-bounded finite coded tree $S \supseteq T$ which extends only leaves of T, and such that $\zeta_S(\epsilon) = \alpha$.
3. Deduce that $\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \mathsf{BME}_* \to \mathsf{WF}(\omega^\omega)$. ★

Working with a stronger base theory, namely, $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\alpha)$ for some ordinal $\alpha \leq \epsilon_0$, raises new complications, as one needs not only to prove a blocking lemma to control the jump, but also a blocking lemma to preserve $\mathsf{WF}(\alpha)$. For this, we shall use the natural (Hessenberg) sums and products over ordinals:

Definition 7.6.14 (Natural sum and product). Let α and β be two ordinals less than ϵ_0. Let $\alpha = \omega^{\gamma_1} n_1 + \dots + \omega^{\gamma_k} n_k$ and $\beta = \omega^{\gamma_1} m_1 + \dots + \omega^{\gamma_k} m_k$[47]. The *natural sum* $\alpha \dotplus \beta$ is defined as

$$\omega^{\gamma_1}(n_1 + m_1) + \dots + \omega^{\gamma_k}(n_k + m_k)$$

47: We allow the n_i and m_i to be equal to 0 in order to write α and β using the same exponents γ_i

Then, let $\alpha \dot{\times} k$ to be equal to be the natural sum of α with itself k times and $\alpha \dot{\times} \omega = \omega^{\gamma_1 + 1} n_1 + \dots + \omega^{\gamma_k + 1} n_k$.[48]

48: Note that the natural product differs from the natural sum. Indeed,

$$\alpha \times \omega = \omega^{\gamma_1 + 1} n_1$$

Thankfully, Shore blocking for preserving $\mathsf{WF}(\alpha)$ comes for free, in the sense that for every $k \in \mathbb{N}$, one can define a Turing functional Γ_k such that if Φ^X_e is an infinite, decreasing sequence of ordinals less than α for some $e < k$, then Γ_k is an infinite, decreasing sequence of ordinals less than $\alpha\dot{\times}k$. Since for any model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{WF}(\alpha)$ and any $k \in M$, $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{WF}(\alpha\dot{\times}k)$, then the natural product overhead is not a problem.[49] In what follows, a code $\langle\alpha\rangle$ for an ordinal $\alpha < \epsilon_0$ is any fixed representation of α as an integer such that the various operations on it are provably Δ^0_1 over $\mathsf{Q} + \mathsf{I}\Sigma^0_1$.

49: RCA_0 proves that the product of two well-orders is a well-order. Since $\alpha\dot{\times}k \leq \alpha \times \omega$ for every $k \in M$, it follows that $\mathsf{RCA}_0 \vdash \mathsf{WF}(\alpha) \to \mathsf{WF}(\alpha \times \omega)$.

Lemma 7.6.15 (Le Houérou, Levy Patey and Yokoyama [69]). Fix a model $\mathcal{M} = (M, S) \models \mathsf{Q}$. For every $k \in M$, there is a Turing functional Γ_k such that, letting $\alpha < \epsilon_0$ be the largest ordinal with $\langle\alpha\rangle < k$, for every $X \in 2^M$ such that $\mathcal{M} \cup \{X\} \models \mathsf{I}\Sigma^0_1$, if there is some $e < k$ such that Φ^X_e is an M-infinite decreasing sequence of elements smaller than α, then Γ^X_k is an M-infinite decreasing sequence of elements smaller than $\alpha\dot{\times}k$.

Moreover, an index of Γ_k can be found computably in k. ★

Proof. By twisting the Turing functionals, we can assume that for every $e, a \in M$, if $\Phi^\sigma_e(a)\downarrow$, then

(1) $a < |\sigma|$;
(2) $\Phi^\sigma_e(b)\downarrow$ for every $b < a$;
(3) $\Phi^\sigma_e(0), \Phi^\sigma_e(1), \dots, \Phi^\sigma_e(a)$ is a strictly decreasing sequence of elements smaller than α.

Given $\sigma \in 2^{<M}$ and $e < k$, let $\zeta(\sigma, e) = \Phi^\sigma_e(s)$ be the largest $s < |\sigma|$ such that $\Phi^\sigma_e(s)\downarrow$. If there is no such s, then $\zeta(\sigma, e) = \alpha$. Note that if $\sigma' \succeq \sigma$, then $\zeta(\sigma', e) \leq \zeta(\sigma, e)$.

Let $\sigma_{-1} = \epsilon$. Let Γ_k be the Turing functional which, on oracle X and input a, searches for some $x > |\sigma_{a-1}|$ and some $\sigma_a \prec X$ such that $\Phi^{\sigma_a}_e(x)\downarrow$ for some $e < k$. If found, it outputs $\zeta(\sigma, 0) \dotplus \dots \dotplus \zeta(\sigma, k-1)$. Note that if $\Gamma^X_k(a)\downarrow$, then by (3), $\Gamma^X_k(a)$ is an ordinal smaller than $\alpha\dot{\times}k$.

Suppose that X is such that $\mathcal{M} \cup \{X\} \models \mathsf{I}\Sigma^0_1$ and there is an $e < k$ is such that Φ^X_e is total. Since $\mathcal{M} \cup \{X\} \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$, then by Exercise 7.3.1, $\mathcal{M}[X] \models \mathsf{RCA}_0$, so Γ^X_k is total.

Moreover, since $x > |\sigma_{a-1}|$, then for $e < k$ such that $\Phi^{\sigma_a}_e(x)\downarrow$, by (1) we have $\Phi^{\sigma_{a-1}}_e(x)\uparrow$. Thus, by (2) and (3), $\zeta(\sigma_{a+1}, e) < \zeta(\sigma_a, e)$, hence $\Gamma^X_k(a+1) < \Gamma^X_k(a)$. It follows that Γ^X_k is an M-infinite decreasing sequence of ordinals smaller than $\alpha\dot{\times}k$. ∎

All the previous conservation theorems over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ also hold while preserving $\mathsf{WF}(\alpha)$ for any fixed ordinal $\alpha \leq \epsilon_0$. We give the details for formalized low basis theorem, and leave the other conservation theorems as exercises.

Theorem 7.6.16 (Le Houérou, Levy Patey and Yokoyama [69])
Fix $\alpha \leq \epsilon_0$. Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\alpha)$ be a countable model topped by a set Y and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $P \in [T]$ such that $(P \oplus Y)' \leq_T Y'$ and $\mathcal{M}[P] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\alpha)$.

Proof. The proof is very similar to Theorem 7.5.3, with an extra requirement for every $b \in \mathbb{N}$:

- $\mathcal{S}_b$: Let $\beta < \alpha$ be the $<_{\epsilon_0}$-largest ordinal with $\langle\beta\rangle < b$. For every $e < b$, $\Phi^{G\oplus Y}_e$ is not an infinite $<_{\epsilon_0}$-decreasing sequence of ordinals smaller than β.

For this, we need to prove a blocking lemma:

Lemma 7.6.17. Let (σ, T_1) be a condition. For every $b \in M$, letting Γ_b be the functional of Lemma 7.6.15, there is an extension $(\sigma, T_2) \leq (\sigma, T_1)$ and an $a \in M$ such that $(\sigma, T_2) \Vdash \Gamma_b^{G \oplus Y}(a)\uparrow$. ★

Proof. We have two cases.

Case 1: there exists some $a \in M$ such that the tree $T_2 = \{\tau \in T_1 : \Gamma_b^{\tau \oplus Y}(a)\uparrow\}$ is infinite. Note that the set T_2 is a primitive Y-recursive, as the set T_1 and the predicate $\Gamma_k^{\tau \oplus Y}(n)\uparrow$ are primitive Y-recursive. Then $(\sigma, T_2) \leq (\sigma, T_1)$ and $(\sigma, T_2) \Vdash \Gamma_k^{G \oplus Y}(a)\downarrow$.

Case 2: for every $a \in M$, there is some $\ell_a \in M$ such that for every $\tau \in T$ of length ℓ_a, $\Gamma_b^{\tau}(a)\downarrow$. For every $a \in M$, let

$$\alpha_a = \max\{\Gamma_b^{\tau}(a) : \tau \in T_1 \wedge |\tau| = \ell_a\}$$

We claim that for every $a \in M$, $\alpha_{a+1} <_{\epsilon_0} \alpha_a$. Indeed, for every $\tau \in T_1$ such that $|\tau| = \ell_{a+1}$, $\Gamma_b^{\tau}(a+1) <_{\epsilon_0} \Gamma_b^{\tau \upharpoonright \ell_a}(a)$, so

$$\max\{\Gamma_b^{\tau}(a+1) : \tau \in T_1 \wedge |\tau| = \ell_{a+1}\} <_{\epsilon_0} \max\{\Gamma_b^{\tau}(a) : \tau \in T_1 \wedge |\tau| = \ell_a\}$$

So $\mathcal{M} \not\models$ WF$(\alpha \dot{\times} b)$. However, since $\mathcal{M} \models$ BΣ_2^0+WF(α), then $\mathcal{M} \models$ WF$(\alpha \dot{\times} b)$. Contradiction. ■

The construction is the same as in Theorem 7.5.3, except that there is a third type of stage, $\mathcal{S}$. Suppose a stage s is of type $\mathcal{S}$ and $s_0 < s$ is the latest stage at which we switched the stage type. If there exists some $\langle \tau, \hat{T} \rangle, a \leq s$ such that $(\tau, \hat{T}) \leq (\sigma_s, T_s)$ and $(\tau, \hat{T}) \Vdash \Gamma_{s_0}^{G \oplus Y}(a) \uparrow$, then let $\sigma_{s+1} = \tau$, $T_{s+1} = \hat{T}$, $\rho_{s+1} = \rho_s$ and let $s+1$ be of the next type. Otherwise, the elements are left unchanged and we go to the next stage. By Lemma 7.6.17, the construction eventually switches stage type.

The remainder of the proof is left unchanged. This completes the proof of Theorem 7.6.16. ■

Exercise 7.6.18. Fix $\alpha \leq \epsilon_0$. Let $\mathcal{M} = (M, S) \models$ RCA$_0$ + BΣ_2^0 + WF(α) be a countable model topped by a set Y, and $A \subseteq M$ be a set such that $\mathcal{M}[A \oplus Y'] \models$ RCA$_0^*$. Adapt the proof of Theorem 7.5.6 to show the existence of a set $G \subseteq M$ such that $\mathcal{M}[G] \models$ RCA$_0$ + BΣ_2^0 + WF(α) and $A \oplus Y' \equiv_T (G \oplus Y)'$ ★

Exercise 7.6.19 (Le Houérou, Levy Patey and Yokoyama [69]). Fix $\alpha \leq \epsilon_0$. Let $\mathcal{M} = (M, S) \models$ RCA$_0$ + BΣ_2^0 + WF(α) be a countable topped model, and $\vec{R} = R_0, R_1, \ldots$ be a uniform sequence in S. Adapt the proof of Theorem 7.5.10 to show the existence of an infinite $\vec{R}$-cohesive set $C \subseteq M$ such that $\mathcal{M}[C] \models$ RCA$_0$ + BΣ_2^0 + WF(α). ★

With a similar technique, but a much more involved disjunctive construction, Le Houérou, Levy Patey and Yokoyama [69] prove that RCA$_0$ + WF(ϵ_0) + RT$_2^2$ is a Π_1^1-conservative extension of RCA$_0$ + BΣ_2^0 + WF(ϵ_0).[50] The proof is based on the decomposition of RT$_2^2$ into COH and RT$_2^{1\prime}$. The proof of following theorem goes beyond the scope of this book:

50: Based on the equivalence between BME$_*$ and WF(ω^ω), one would expect to work with models of WF(ω^ω) instead of WF(ϵ_0). However, in order to preserve WF(ω_k^ω) in the extended model, one seems to need WF(ω_{k+1}^ω), where

$$\omega_0^\alpha = \alpha \text{ and } \omega_{k+1}^\alpha = \omega^{\omega_k^\alpha}$$

Theorem 7.6.20 (Le Houérou, Levy Patey and Yokoyama [69])
Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\epsilon_0)$ *be a countable topped model. For every* Δ^0_2 *set* $A \subseteq M$*, there is an infinite set* $H \subseteq A$ *or* $H \subseteq M \setminus A$ *such that* $\mathcal{M}[H] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{WF}(\epsilon_0)$.

Forcing design | 8

As emphasized throughout the previous chapters, the computability-theoretic analysis of combinatorial theorems is closely related to the combinatorial features of the corresponding forcing questions. This analysis therefore depends on the choice of an appropriate notion of forcing to build solutions to the problem. So far, the preliminary step of designing a good notion of forcing was given for granted. In this chapter, we fill in the gap by explaining the key ideas behind the design of such notion of forcing. These core concepts will be exemplified with the analysis of the Erdős-Moser theorem and the free set theorem.

Prerequisites: Chapters 2 and 3

8.1 Core concepts

We focus on theorems coming from Ramsey theory. Indeed, as explained in Section 6.2, most theorems are equivalent in reverse mathematics to one of five systems of axioms with a well-understood computability-theoretic strength. The few exceptions to this empirical observation almost come exclusively from Ramsey theory, and require the design of a specific machinery. Ramsey theory deals with many kind of mathematical structures. Here, we consider statements about sets, that is, with no additional structure than cardinality. Furthermore, classical reverse mathematics being formulated in the language of second-order arithmetic, we shall focus on statements about the existence of an infinite subset of $\mathbb{N}$. [1]

1: The considerations in this section are rather abstract, and might make sense only after having considered a few examples. The reader might choose to skip this section, and directly learn by examples, with the Erdős-Moser and free set theorems.

The takeway of this discussion is that there is some tension between the structural properties imposed on the forcing conditions to build a solution to the instance of a combinatorial problem, and the necessity to add elements by block to the stem by satisfying only a Σ^0_1 predicate.

Stem. Turing functionals being continuous functions over Cantor space, computability-theoretic properties of the constructed object G are naturally forced by fixing initial segments of G. It follows that the forcing conditions usually contain a *stem*, represented as a finite binary string. This stem is supposed to grow over condition extension, and every sufficiently generic filter $\mathcal{F}$ will contain conditions with stems of arbitrary length, yielding a binary sequence $G_\mathcal{F}$ defined as the limit of these stems. The notion of focring with stems, partially ordered by the prefix relation, is nothing but Cohen forcing.

Structural properties. Given an instance I of a problem P, the goal is to build a P-solution to I. One therefore needs to impose structural constraints on the stem. The most basic such constraint is that the stem is a finite P-solution to I. For instance, in the case of Ramsey's theorem for pairs, one wants σ to code a finite homogeneous set. Thus, for every filter $\mathcal{F}$, the (finite or infinite) sequence $G_\mathcal{F}$ yields a homogeneous set.

Extendibility. One can think of a condition as an invariant property of the construction. Usually, being a finite P-solution to I is not a sufficiently strong invariant, in that some finite solution might not be extendible into an infinite solution. For instance, if P is Ramsey's theorem for pairs and two colors, given finite homogeneous set F for color 0, there might be an element $x \in F$ which, paired with cofinitely many other elements, has color 1. The extendibility constraint is usually formulated in terms of an infinite reservoir satisfying some additional structural properties. For instance, for Ramsey's theorem for pairs, one works with triples (σ_0, σ_1, X), where σ_0 and σ_1 are two stems, homogeneous for color 0 and 1, respectively, and $X \subseteq \mathbb{N}$ is an infinite reservoir with $\min X > |\sigma_i|$, such that for every $i < 2$, every $x \in \sigma_i$ and $y \in X$, $\{x, y\}$ has color i. To see that, given a condition (σ_0, σ_1, X), at least one of the stems is extendible into an infinite solution, apply Ramsey's theorem for pairs within X, to obtain an infinite homogeneous subset $Y \subseteq X$ for some color $i < 2$. Then, by the structural properties of the reservoir, $\sigma_i \cup Y$ is again homogeneous for color i.

Block extendibility. Extendibility yields a classical proof of the problem P, in that for every sufficiently generic filter $\mathcal{F}$, the set $G_{\mathcal{F}}$ is an infinite P-solution to I. However, in order to obtain a good forcing question for Σ^0_1-formulas, yielding a computationally weak solution, one must be able to add elements by block, and not only one by one. Indeed, the natural forcing question for Σ^0_1-formulas is of the form "Is there a block of elements from the reservoir such that, if I add them to the stem, it will satisfy the Σ^0_1-formula?" Because being a finite P-solution to I is usually not a sufficiently strong invariant to ensure extendibility, one must choose a block which will maintain the stronger extendibility property. The extendibility property being usually Π^0_1, the main difficulty lies in finding a sufficient Σ^0_1 property that must satisfy a block to preserve the extendibility property.

Computational properties. Because of the use of a reservoir, a Mathias condition is an infinite object. Given a Mathias-like condition (σ, X), the forcing question will ask for a finite subset $\rho \subseteq X$ with additional structural properties. It follows that the complexity of the forcing question involves the one of the reservoir. In order to obtain a diagonalization theorem such as Theorem 3.3.4, one must therefore impose some computational weakness to the reservoir. The usual requirement is that the reservoir satisfies the weakness property being studied. For instance, in cone avoidance of a set C, one will usually work with reservoirs $X \not\geq_T C$.

8.2 Erdős-Moser theorem

The Erdős-Moser was introduced and studied in Section 6.4, with a notion of forcing coming out of the blue. We recall the basic definitions, and give a step-by-step explanation of the process yielding to the design of its notion of forcing.

A *tournament* over an infinite domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation $T \subseteq D^2$ such that for every $a, b \in D$ with $a \neq b$, $T(a, b)$ iff $\neg T(b, a)$. The tournament T is *transitive* if for all $a, b, c \in D$, if $T(a, b)$ and $T(b, c)$ hold, then $T(a, c)$ also holds.[2] A *sub-tournament* of T is the restriction of T to a subdomain $D_1 \subseteq D$. Thus, given T, a sub-tournament is fully specified by the sub-domain D_1, and is therefore identified with it, and we say that D_1 is T-*transitive* if T is transitive on D_1. The Erdős-Moser theorem states that every infinite tournament admits an infinite transitive sub-tournament.

2: It is important to note that transitivity is a property over $[D]^3$. Thus, if a tournament is not transitive, then it is witnessed by a 3-tuple of elements of D.

Fix a computable tournament T over $\mathbb{N}$. In order to design a good notion of forcing to build an infinite T-transitive subtournament, one starts with Mathias forcing, that is, the notion of forcing whose conditions are pairs (σ, X), where $\sigma \in 2^{<\mathbb{N}}$ is the *stem*[3] and and $X \subseteq \mathbb{N}$ is an infinite *reservoir*. A condition (τ, Y) *extends* (σ, X) if $\sigma \preceq \tau$ (a longer initial segment of the solution is specified), $Y \subseteq X$ (the reservoir is restricted), and $\tau \setminus \sigma \subseteq X$ (the new elements of the stem come from the reservoir).

3: Think of the stem as an initial segment of the object being built.

Step 1: Extendibility. Of course, pure Mathias forcing does not produce infinite T-transitive sub-tournaments. One must therefore put a first restriction by asking the stem σ to be a finite T-transitive sub-tournament. This restriction structurally ensures that for every filter $\mathcal{F}$, the set $G_{\mathcal{F}}$ (defined as the limit of the stems of conditions in $\mathcal{F}$) is T-transitive. However, this restriction comes with a price: even with sufficiently generic filters $\mathcal{F}$, the set $G_{\mathcal{F}}$ might not be infinite. Indeed, there might be conditions (σ, X) where the stem is not extendible into an infinite solution. For instance, there might be some $x, y \in [\sigma]^2$ such that for all but finitely many $z \in X$, $\{x, y, z\}$ forms a 3-cycle. There might be an even more subtle situation: for almost every $z \in X$, there is some $x, y \in [\omega]^2$ (which depend on z) such that $\{x, y, z\}$ forms a 3-cycle.

One must therefore identify a stronger structural property which will ensure extendibility of the stem, and play the role of an invariant. Thankfully, there is a simple empirical criterion to identify this invariant: Given a condition (σ, X), by the classical Erdős-Moser theorem, there is an infinite T-transitive subset $Y \subseteq X$. The structural invariant is obtained by identifying sufficient hypothesis to ensure that $\sigma \cup Y$ is again T-transitive.

As mentioned, if $\sigma \cup Y$ is not T-transitive, then there exists a 3-cycle $\{x, y, z\} \in [\sigma \cup Y]^2$. Say $x < y < z$. Because σ and Y are T-transitive, one cannot have $x, y, z \in \sigma$ or $x, y, z \in Y$. There are only two possibilities remaining.

- Case 1: $x \in \sigma$ and $y, z \in Y$. This can be avoided by ensuring that each $x \in \sigma$ has the same behavior with respect to every element of X. We say that σ *is stabilized by* X if for every $x \in \sigma$, either $\forall y \in X$, $T(x, y)$, or $\forall y \in X, T(y, x)$. Given a condition (σ, X), one can always find an infinite X-computable subset $Y \subseteq X$ such that σ is stabilized by Y, as follows: Given a condition (σ, X), let $f : X \to 2^{|\sigma|}$ be defined by $f(y) = \rho$, where ρ is the binary string of length $|\sigma|$ such that for every $x < |\sigma|$, $\rho(x) = 1$ iff $T(x, y)$.[4] Since the pigeonhole principle is computably true, one can find an infinite X-computable f-homogeneous subset $Y \subseteq X$. One easily sees that σ is stabilized by Y. Thus, the condition (σ, Y) avoids every 3-cycle with one element in σ and two elements in Y.
- Case 2: $x, y \in \sigma$, $z \in Y$. This cannot be avoided for free by restricting the reservoir. One must therefore explicitely forbid this behavior. Because σ is T-transitive, one can equivalently ask that every element $y \in X$ is a *one-point extension*, that is, $\sigma \cup \{y\}$ is T-transitive.

4: Another way to see this is to consider each element x of σ, and successively apply RT^1_2 by considering the 2-partition $\{y \in X : T(x, y)\}$ and $\{y \in X : T(y, x)\}$. This yields a finite decreasing sequence of infinite sets, stabilizing the behavior of more and more elements of σ. The last set is the desired reservoir.

The previous analysis reveals two structural extendibility properties, the former being optional. A condition is a Mathias pair (σ, X) such that σ is stabilized by X, and every element of X is a one-point extension. In other words,

(a) $\forall x \in \sigma$, either $(\forall y \in X)T(x, y)$ or $(\forall y \in X)T(y, x)$
(b) $\forall y \in X$, $\sigma \cup \{y\}$ is T-transitive[5]

5: Note that this property encompasses the fact that σ is T-transitive. Thus, there is no need to add explicitly this constraint on the stem.

As mentioned, the first property is optional, as given a Mathias condition (σ, X), one can always find an infinite X-computable subset $Y \subseteq X$ such that (σ, Y) satisfies (a). On the other hand, the second property truly imposes a constraint on the stem σ. Because of this, one must check that property (b) can be preserved by adding new elements to the stem. The following extendibility lemma states that it is the case.

Lemma 8.2.1. Let (σ, X) be a condition, and $x \in X$. There is an X-computable infinite set $Y \subseteq X$ such that $(\sigma \cup \{x\}, Y)$ is a valid extension.[6] ⋆

6: Note how in this proof, the optional property (a) is useful to preserve property (b).

Proof. Fix $x \in X$ and let Y be either $\{y \in X : T(x,y)\}$ or $\{y \in X : T(y,x)\}$, depending on which one is infinite. We claim that $(\sigma \cup \{x\}, Y)$ is a valid extension. It is by construction a Mathias extension of (σ, X), so one only needs to check that properties (a) and (b) are satisfied. Property (a) of $(\sigma \cup \{x\}, Y)$ is satisfied by property (a) of (σ, X) and the choice of Y. We now prove (b). Suppose for the contradiction that $\sigma \cup \{x\} \cup \{y\}$ is not T-transitive, for some $y \in Y$. By definition, there is a 3-cycle $\{a,b,c\} \in [\sigma \cup \{x\} \cup \{y\}]^3$. Say $a < b < c$. Because of property (b) of (σ, X), one cannot have $\{a,b,c\} \in [\sigma \cup \{x\}]^3$ or $\{a,b,c\} \in [\sigma \cup \{y\}]^3$, so $a \in \sigma$, $b = x$ and $c = y$. In particular, a does not have the same behavior with respect to b and c, contradicting property (a) of (σ, X). ■

Step 2: Block extendibility. We now have a notion of forcing to build solutions to a given computable instance of the Erdős-Moser theorem. However, additional work is required to design a good forcing question for Σ^0_1-formulas. Consider the forcing question for Mathias forcing:

> **Definition 8.2.2.** Given a Mathias condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \mathrel{?\!\vdash} \varphi(G)$ iff there is some finite set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◇

An Erdős-Moser condition being a Mathias condition, one should expect to have a similar forcing question, by replacing "finite set $\rho \subseteq X$" with "finite T-transitive set $\rho \subseteq X$". This definition raises two difficulties. First, one wants the forcing question for Σ^0_1-formulas to be Σ^0_1-preserving, but given a Mathias condition (σ, X), the forcing question for a Σ^0_1-formula is $\Sigma^0_1(X)$. We shall ignore this difficulty until Step 3. Second, the property (b) of a condition is not preserved by adding blocks simultaneously.

Example 8.2.3. Let (σ, X) be a condition, and $\rho = \{x, y\} \subseteq X$ be a finite set. The set ρ is vacuously T-transitive. Moreover, by choice of properties (a) and (b), $\sigma \cup \rho$ is again T-transitive. However, suppose that $T(x,y)$ holds, but for all but finitely many $z \in X$, $T(y,z)$ and $T(z,x)$ both hold. Then there is no infinite subset $Y \subseteq X$ such that $(\sigma \cup \rho, Y)$ satisfies property (b).

The previous example shows the importance of some "compatibility" property between the elements of ρ. Suppose first for simplicity that T is *stable*, that is, for every x, either $(\forall^\infty y)T(x,y)$, or $(\forall^\infty y)T(y,x)$. Such tournament induces a $\emptyset'$-computable coloring of singletons $f : \mathbb{N} \to 2$ defined by $f(x) = 1$ iff $(\forall^\infty y)T(x,y)$.[7]

7: One can see a tournament $T \subseteq \mathbb{N}^2$ as a function $h : [\mathbb{N}]^2 \to 2$ defined for $x < y$ by $h(x,y) = 1$ iff $T(x,y)$ and $h(x,y) = 0$ otherwise. The tournament is stable iff h is stable, and $f(x) = \lim_y h(x,y)$. is the limit function.

> **Definition 8.2.4.** A set ρ is *f-compatible* if for every $x, y \in \rho$, if $T(x,y)$ holds, then $f(x) \geq f(y)$. ◇

Note that every f-homogeneous set is f-compatible. We leave as an exercise the fact that f-compatibility is a sufficient notion to preserve property (b).

Exercise 8.2.5. Suppose T is stable, with limit function $f : \mathbb{N} \to 2$. Let (σ, X) be a condition, and $\rho \subseteq X$ be a finite f-compatible set. Show that $(\sigma \cup \rho, X \cap (\max \rho, \infty))$ satisfies property (b). ★

Even among stable tournaments, the naive definition of the forcing question is too complex definitionally. Indeed, given a condition (σ, X), the following statement

"There is some finite f-compatible and T-transitive subset $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds."

is $\Sigma^0_1(X \oplus \emptyset')$, since the coloring f is $\emptyset'$-computable. In order to decrease the complexity of the statement, we use a standard trick of over-approximation by considering all the candidate limit colorings over an effectively compact space.

Definition 8.2.6. Given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let (σ, X) $?\vdash \varphi(G)$ iff for every coloring $g : \mathbb{N} \to 2$, there is some finite T-transitive and g-compatible set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◇

At first sight, this yields a statement of much stronger complexity, as it contains a universal second-order quantification. However, thanks to compactness, the statement is actually $\Sigma^0_1(X)$.

Exercise 8.2.7. Let (σ, X) be a condition and $\varphi(G)$ be a Σ^0_1-formula. Show that $(\sigma, X) ?\vdash \varphi(G)$ iff there is some $\ell \in \mathbb{N}$ such that for every coloring $g : \ell \to 2$, there is some finite T-transitive and g-compatible[8] set $\rho \subseteq X{\upharpoonright}_\ell$ such that $\varphi(\sigma \cup \rho)$ holds. ★

8: One can actually replace "g-compatible" with "g-homogeneous", and obtain a valid forcing question. Although less familiar, the notion of g-compatibilty is more natural in this context, as it contains the least necessary hypothesis to preserve property (b).

Because this forcing question is an over-approximation of the naive forcing question, if it holds, then there is an extension forcing the Σ^0_1-formula. On the other hand, if the forcing question does not hold, the witness of failure might be a function $g : \mathbb{N} \to 2$ which is not related to the true limit function $f : \mathbb{N} \to 2$. We shall then exploit the Ramseyan nature of the statements[9] by working with sets which are simultaneously f and g-compatible. With a little bit more work, one can actually show that this forcing question works even for non-stable tournaments, by stabilizing the set ρ *a posteriori*.

9: A common denominator of many Ramseyan statements is the existence, given multiple instances, of a singlet set which is simultaneously a solution to each instances. Consider Ramsey's theorem for example. Given two colorings $f : [\mathbb{N}]^n \to k$ and $g : [\mathbb{N}]^m \to \ell$, apply Ramsey's theorem to obtain an infinite f-homogeneous set $X \subseteq \mathbb{N}$. Then, within X, apply again Ramsey's theorem to obtain an infinite g-homogeneous subset $Y \subseteq X$. The set Y is simultaneously g-homogeneous and f-homogeneous.

Lemma 8.2.8. Let $p = (\sigma, X)$ be a condition and $\varphi(G)$ be a Σ^0_1-formula.

1. If $p ?\vdash \varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\varphi(G)$.
2. If $p ?\nvdash \varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\neg\varphi(G)$.

Moreover, every set P of PA degree over X computes such a set Y. ★

Proof. Suppose first $p ?\vdash \varphi(G)$. Then, by Exercise 8.2.7, there is some threshold $\ell \in \mathbb{N}$ such that for every coloring $g : \ell \to 2$, there is finite T-transitive and g-compatible set $\rho \subseteq X{\upharpoonright}_\ell$ such that $\varphi(\sigma \cup \rho)$ holds. Let $Y \subseteq X$ be an X-computable subset stabilizing $[0, \ell)$. This induces an X-computable coloring $g : \ell \to 2$ defined by $g(x) = 1$ iff $(\forall y \in Y)T(x, y)$. Let $\rho \subseteq X{\upharpoonright}_\ell$ be a finite T-transitive and g-compatible set such that $\varphi(\sigma \cup \rho)$ holds. We claim that $(\sigma \cup \rho, Y)$ is the desired extension. First, it is a Mathias condition, and by choice of Y, it satisfies property (a). By Exercise 8.2.5, it satisfies property (b). By choice of ρ, it forces $\varphi(G)$.

Suppose now $p ?\nvdash \varphi(G)$. Let $\mathscr{C}$ be the $\Pi^0_1(X)$ class of all $g : \mathbb{N} \to 2$ such that for every finite T-transitive and g-compatible set $\rho \subseteq X$, $\varphi(\sigma \cup \rho)$ does not hold. By assumption, the class $\mathscr{C}$ is non-empty. Pick any $g \in \mathscr{C}$ and let $Y \subseteq X$ be an infinite g-homogeneous subset. As mentioned, every g-homogeneous set is g-compatible, and the pigeonhole principle is computably true, so Y can be chosen $X \oplus g$-computably. The condition (σ, Y) is an extension of p forcing $\neg\varphi(G)$. Note that any PA degree over X computes member of $\mathscr{C}$, hence computes such a set Y. ■

Step 3: Computational property.

As mentioned, given a condition (σ, X), the forcing question for a Σ^0_1-formula is $\Sigma^0_1(X)$. In order to obtain a diagonalization theorem such as Theorem 3.3.4, one must impose some computational constraint on the reservoir X. In the most general case, one will add the following property to the definition of a condition (σ, X):

(c) $X \in \mathscr{W}$

where $\mathscr{W}$ is a weakness property[10] whose additional closure properties are identified by looking at the operations on the reservoir that appear in the use of the forcing question.

10: Recall from Section 6.1 that a *weakness property* is a class of sets downward-closed under the Turing reduction. The reader might be more familiar with the notion of Turing ideal, which is closed under effective join. However, most natural weakness properties, such as being low, avoiding a cone, or preserving hyperimmunies, are not closed under effective join.

In our case, all the operations on the reservoir are computable transformations (finite truncation, stabilization of the stem), except in the case where the forcing question does not hold. One then obtain a Π^0_1 class of 2-partitions, and take any infinite homogeneous set for any of these partitions as the new reservoir. Thus, the previous lemmas hold for any weakness property $\mathscr{W}$ preserved[11] by RT^1_2 and WKL.[12] The pigeonhole principle being computably true, it preserves every weakness property, so one can simply require $\mathscr{W}$ to be preserved by WKL, that is, for every $X \in \mathscr{W}$, there is some set $P \in \mathscr{W}$ of PA degree over X. In most cases, the weakness property $\mathscr{W}$ is nothing but the property that one wants the resulting set G to satisfy.

11: Recall that a problem P *preserves* a weakness property $\mathscr{W}$ if for every $Z \in \mathscr{W}$ and every Z-computable instance X, there is a solution Y to X such that $Z \oplus Y \in \mathscr{W}$.

12: One can actually be even more cautious, and only ask $\mathscr{W}$ to be closed under the Rasmey-type weak König's lemma (RWKL). However, over-optimization is not always desirable, and it sometimes yields unnecessary additional complexity.

Example 8.2.9. Suppose one wants to prove that EM admits cone avoidance. Any non-computable set C induces a weakness property $\mathscr{W}_C = \{Z : C \not\leq_T Z\}$. By the cone avoidance basis theorem (Theorem 3.2.6), $\mathscr{W}_C$ is closed under PA degrees, so one can impose $X \in \mathscr{W}_C$, in other words, $C \not\leq_T X$.

Exercise 8.2.10 (Wang ; Patey [73]). Recall that a problem P admits *strong cone avoidance*[13] if for every set Z and every non-Z-computable set C, every instance X of P admits a solution Y such that C is not $Z \oplus Y$-computable. Fix a non-computable set C and an arbitrary tournament $T \subseteq \mathbb{N}^2$. Consider the same notion of condition above, that is, pairs (σ, X) satisfying properties (a), (b) and (c).

13: The difference between cone avoidance and strong cone avoidance is that the instance X of P is not asked to be Z-computable in the latter case.

1. Use strong cone avoidance of RT^1_2 (Theorem 3.4.6) to prove that for every condition (σ, X) and $x \in X$, there is an infinite set $Y \subseteq X$ such that $(\sigma \cup \{x\}, Y)$ is a valid extension.

Given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \operatorname{?\vdash} \varphi(G)$ if for every tournament $S \subseteq \mathbb{N}^2$ and every coloring $g : \mathbb{N} \to 2$, there is some finite S-transitive and g-compatible set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds.

2. Show that the relation $(\sigma, X) \operatorname{?\vdash} \varphi(G)$ is $\Sigma^0_1(X)$.
3. Use strong cone avoidance of RT^1_2 to prove that if $(\sigma, X) \operatorname{?\vdash} \varphi(G)$, then there is an extension forcing $\varphi(G)$.
4. Use cone avoidance of EM and the cone avoidance basis theorem to prove that if $(\sigma, X) \operatorname{?\nvdash} \varphi(G)$, then there is an extension forcing $\neg\varphi(G)$.
5. Deduce that EM admits strong cone avoidance. ⋆

8.3 Free set theorem

The free set theorem is a combinatorial statement introduced by Friedman [74] which provides another good illustration of the forcing design process. Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, an infinite set $H \subseteq \mathbb{N}$ is *f-free* if for every $\sigma \in [\mathbb{N}]^n$, if $f(\sigma) \in H$, then $f(\sigma) \in \sigma$. The *free set theorem* for n-tuples (FS^n) is the problem whose instances are colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite f-free sets. This problem might seem artificial at first sight, but it can be reformulated as a strong version of the thin set theorem.[14] An infinite set $H \subseteq \mathbb{N}$ is *f-thin* if $f[H]^n \neq \mathbb{N}$, that is, at least one color does not appear on $[H]^n$.

14: Another way to think of the free set theorem is that any n-tuple $\sigma \in [\mathbb{N}]^n$ can optionally "choose" a forbidden element $f(\sigma)$, so that if σ belongs so the solution, then $f(\sigma)$ must be excluded. Setting $f(\sigma) \in \sigma$ is a way to refuse to choose.

Exercise 8.3.1. Let $f : [\mathbb{N}]^n \to \mathbb{N}$ be a coloring. Show that an infinite set $H \subseteq \mathbb{N}$ is f-free iff for every $x \in \mathbb{N}$, $H \setminus \{x\}$ is f-thin with witness color x. ★

Similar to Ramsey's theorem, the free set theorem induces a hierarchy of statements based on the size of the colored tuples. However, while Ramsey's theorem hierarchy collapses and is equivalent to ACA_0 for $n \geq 3$, Wang [15] surprisingly proved that the free set theorem admits strong cone avoidance for any size of tuples. The proof goes by induction over n.

In this section, we shall design a notion of forcing for computable instances of FS^3 with a Σ^0_1-preserving forcing question for Σ^0_1-formulas. This provides a good example of a statement which is not about colorings of pairs, but still admits a good first-jump control. For this, we follow the same steps as for the Erdős-Moser theorem. Fix a computable coloring $f : [\mathbb{N}]^3 \to \mathbb{N}$, and start with Mathias forcing.

Step 1: Extendibility. As before, we refine Mathias forcing by asking the stem to be a finite solution, that is, we work with Mathias conditions (σ, X) such that σ is a finite f-free set. Of course, there might be conditions (σ, X) such that the set σ is f-free, but not extendible into an infinite f-free set. For instance, it might be that for almost every $\{x, y, z\} \in [X]^3$, $f(x, y, z) \in \sigma$. There might also also be some $x \in \sigma$ such that for almost every $\{y, z\} \in [X]^2$, $f(x, y, z) \in \sigma \setminus \{x\}$. These are only a few examples of the possible issues.

In order to identify the stronger structural property ensuring extendibility, we apply the same criterion as before: Given a condition (σ, X), let $Y \subseteq X$ be an infinite f-free set. Suppose that $\sigma \cup Y$ is not f-free. There is therefore some $\{x, y, z\} \in [\sigma \cup Y]^3$ such that $f(x, y, z) \in (\sigma \cup Y) \setminus \{x, y, z\}$. Say $x < y < z$. Because σ and Y are both f-free, one cannot have x, y, z, and $f(x, y, z)$ in σ or Y. There are multiple possibilities remaining:

- Case 1: $x, y, z \in \sigma$; $f(x, y, z) \in Y$. This case can be simply avoided by removing the range of $f \upharpoonright [\sigma]^3$ from the reservoir. This range is finite, so this can be obtained for free by finite truncation of the reservoir.
- Case 2: $x, y \in \sigma$; $z, f(x, y, z) \in Y$. Fixing $\{x, y\} \in \sigma$ induces a coloring $f_{x,y} : \mathbb{N} \to \mathbb{N}$ defined by $f_{x,y}(z) = f(x, y, z)$. This coloring can be seen as an instance of FS^1. Given a condition (σ, X), one can use the induction hypothesis, and apply FS^1 on $f_{x,y}$ for every $\{x, y\} \in [\sigma]^2$ to obtain an infinite sub-reservoir $Y \subseteq X$ which is $f_{x,y}$-free simultaneously. Case 2 cannot happen with (σ, Y). It follows that Case 2 can be avoided without putting constraints to the stem σ.

- Case 3: $x, y, f(x, y, z) \in \sigma$; $z \in Y$. This cannot be avoided for free by restricting the reservoir. One must therefore explicitely forbid this behavior.
- Case 4: $x \in \sigma$; $y, z, f(x, y, z) \in Y$. This case is similar to Case 2. Fixing some $x \in \sigma$ induces a coloring $f_x : [\mathbb{N}]^2 \to \mathbb{N}$ defined by $f_x(y, z) = f(x, y, z)$. One can again use the induction hypothesis, and apply FS^2 finitely many times to avoid this case.
- Case 5: $x, f(x, y, z) \in \sigma$; $y, z \in Y$. This case is similar to Case 3. In particular, it cannot be avoided simply by restricting the reservoir, so this must be explicitly ruled out.
- Case 6: $f(x, y, z) \in \sigma$; $x, y, z \in Y$. This case is once again similar to Case 3 and Case 5.

These 6 cases can therefore be divided into two categories: the optional structural properties, which can be ensured by restricting the reservoir, with no constraint on the stem, and the required structural properties, which are really necessary to ensure extendibility. A condition is a Mathias pair (σ, X) satisfying the following two properties:

(a) $\forall \{x, y, z\} \in [\sigma \cup X]^3$ with $x \in \sigma$, $f(x, y, z) \notin X \setminus \{y, z\}$
(b) $\forall \{x, y, z\} \in [\sigma \cup X]^3$, $f(x, y, z) \notin \sigma \setminus \{x, y, z\}$.[15]

Property (a) encompasses f-freeness of σ together with the optional properties, namely, Case 1, Case 2 and Case 4, while property (b) covers Case 3, Case 5 and Case 6. We must now show that these structural properties provide a good invariant by proving an extendibility lemma. More precisely, the difficulty is to add new elements to the stem while preserving property (b). Given a condition (σ, X) and $x \in X$, property (b) on $(\sigma \cup \{x\}, X \setminus [0, x])$ is almost inherited from properties (a) and (b) on (σ, X), except one case: there might be some $\{a, b, c\} \in [X \setminus [0, x]]^3$ such that $f(a, b, c) = x$. This corresponds to Case 6, which must receive some special attention.

Given $x_0 \in X$, by Ramsey's theorem for triples, there is an infinite subset $Y \subseteq X$ such that either $(\forall \{a, b, c\} \in [Y]^3) f(a, b, c) \neq x_0$ or $(\forall \{a, b, c\} \in [Y]^3) f(a, b, c) = x_0$. In the former case, $(\sigma \cup \{x_0\}, Y)$ satisfies property (b), while in the latter case, for any $x_1 \in X$ with $x_0 \neq x_1$, $(\sigma \cup \{x_1\}, Y)$ satisfies property (b). Thus, combinatorially, it suffices to pick two elements in X, and at least one of them can be added to the stem while preserving the structural invariant. From a computational viewpoint however, Ramsey's theorem for triples is very strong, and is even applied of an f-computable coloring, which is of arbitrary complexity. Thankfully, one does not need the full power of Ramsey's theorem, and can weaken the statement by considering more than two elements in the reservoir.

Given $n, \ell \geq 1$, let $\mathsf{RT}^n_{<\infty,\ell}$ be the problem[16] whose instances are colorings $f : [\mathbb{N}]^n \to k$ for some $k \in \mathbb{N}$, and whose solutions are infinite sets $H \subseteq \mathbb{N}$ such that $\operatorname{card} f[H]^n \leq \ell$. In particular, $\mathsf{RT}^n_{<\infty,1}$ is nothing but Ramsey's theorem for n-tuples. Wang [15] proved that when ℓ is sufficiently large with respect to n, then $\mathsf{RT}^n_{<\infty,\ell}$ looses all its coding power and admits strong cone avoidance. In our case, fix some sufficiently large bound ℓ_n with respect to n so that $\mathsf{RT}^n_{<\infty,\ell_n}$ preserves our desired computational property.[17]

15: As for the Erdős-Moser theorem, property (a) could be technically removed from the definition of a condition, and one would still obtain a structural invariant. However, property (a) is very convenient to preserve property (b), and can be added for free by restricting further the reservoir, so we include it in the definition.

16: This problem admits many names in the reverse mathematics literature. In Wang [15], it is called the *achromatic Ramsey theorem* and is written $\mathsf{ART}^n_{<\infty,\ell}$. In Dorais et al. [75] or Patey [14], it is considered as a strong version of the *thin set theorem*, and is written $\mathsf{TS}^n_{\ell+1}$. In Patey [76], it is seen as a generalization of Ramsey's theorem, and is written $\mathsf{RT}^n_{<\infty,\ell}$.

17: For $n = 1$, we can take $\ell_1 = 1$, as the pigeonhole principle is computably true, hence preserves any weakness property.

Lemma 8.3.2. Let (σ, X) be a condition, and $x_0, \dots, x_{\ell_3}$ be distinct elements of X. There is some $i \leq \ell_3$ and some infinite subset $Y \subseteq X$ such that $(\sigma \cup \{x_i\}, Y)$ is a valid extension. ★

Proof. Let $g : [X \setminus \{x_0, \dots, x_{\ell_3}\}]^3 \to \{x_0, \dots, x_{\ell_3}\}$ be defined by

$$g(a,b,c) = \begin{cases} f(a,b,c) & \text{if } f(a,b,c) \in \{x_0, \dots, x_{\ell_3}\} \\ x_0 & \text{otherwise.} \end{cases}$$

By $\mathsf{RT}^3_{<\infty,\ell_3}$, there is some $i \leq \ell_3$ and an infinite subset $Z \subseteq X$ such that $x_i \notin g[Z]^3$. We claim that $(\sigma \cup \{x_i\}, Z)$ satisfies property (b). Indeed, let $\{a,b,c\} \in [\sigma \cup \{x_i\} \cup Z]^3$ be such that $f(a,b,c) \in (\sigma \cup \{x_i\}) \setminus \{a,b,c\}$. By property (b) of (σ, X), $f(a,b,c) \notin \sigma \setminus \{a,b,c\}$, hence $f(a,b,c) = x_i$ and $x_i \notin \{a,b,c\}$. By property (a) of (σ, X), if $a \in \sigma$, $f(a,b,c) \notin X \setminus \{b,c\}$, so $a \notin \sigma$, hence $a,b,c \in Y \setminus \{x_i\}$. But then, $g(a,b,c) = f(a,b,c) = x_i$, contradicting the choice of Z and x_i. Let $Y \subseteq Z$ be an infinite subset such that $(\sigma \cup \{x_i\}, Y)$ satisfies property (a). Then $(\sigma \cup \{x_i\}, Y)$ is the desired extension. ■

Step 2: Block extendibility. We now want to design a good forcing question for this notion of forcing. For this, we restart with the standard forcing question for Mathias forcing.

Definition 8.3.3. Given a Mathias condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \,?\!\vdash \varphi(G)$ iff there is some finite set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◊

As for the Erdős-Moser theorem, one wants to modify this definition by asking for a finite f-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. Because of the combinatorics of the extendibility lemma, one needs to ask for $\ell_3 + 1$ many pairwise disjoint f-free sets $\rho_0, \dots, \rho_{\ell_3} \subseteq X$ such that for every $i \leq \ell_3$, $\varphi(\sigma \cup \rho_i)$ holds. However, even with this modification, property (b) might not hold over $(\sigma \cup \rho_i, Y)$ for any $i \leq \ell_3$ and any infinite set $Y \subseteq X$.

Example 8.3.4. Let (σ, X) be a condition, and $\rho = \{x, y, z\} \subseteq X$ be a finite set. The set ρ is vacuously f-free. Even putting aside Case 6, it might be that for all but finitely many $w \in X$, $f(x,y,w) = z$, or for all but finitely many $\{u, w\} \in [X]^2$, $f(x,u,w) = y$. Then there is no infinite subset $Y \subseteq X$ such that $(\sigma \cup \rho, Y)$ satisfies property (b).

One needs to find the appropriate notion of compatibility so that property (b) is preserved when adding blocks of elements. The issue usually comes from some hidden non-computable constraint between the elements of the block ρ and the limit behavior of the coloring. In order to reveal this constraint, one must first consider the appropriate notion of stability. In the case of the Erdős-Moser theorem, stability was obtained by multiple applications of the pigeonhole principle. In the case of the free set theorem, we shall use $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$.

Definition 8.3.5. An infinite set X *stabilizes* a finite set σ if there are finite sets $I \in [\sigma]^{\leq \ell_3}$, $\langle I_x \in [\sigma]^{\leq \ell_2} : x \in \sigma \rangle$ and $\langle I_{x,y} \in [\sigma]^{\leq \ell_1} : \{x,y\} \in [\sigma]^2 \rangle$ such that[18]

(i) $f[X]^3 \cap \sigma \subseteq I$;
(ii) for every $x \in \sigma$, $f_x[X]^2 \cap \sigma \subseteq I_x$;
(iii) for every $\{x,y\} \in [\sigma]^2$, $f_{x,y}[X]^1 \cap \sigma \subseteq I_{x,y}$.[19] ◊

18: Given a finite or infinite set Z and some $k \in \mathbb{N}$, we write $[Z]^{\leq k}$ for the collection of all subsets of Z of size at most k. In particular, $[Z]^{\leq k}$ contains the empty set.

19: Recall that $f_x : [\mathbb{N}]^2 \to \mathbb{N}$ and $f_{x,y} : \mathbb{N} \to \mathbb{N}$ are the functions obtained by fixing the parameters x and y.

We leave as an exercise the proof that every finite set can be stabilized by restricting the reservoir.

Exercise 8.3.6. Let σ be a finite set and $X \subseteq \mathbb{N}$ an infinite set. Use $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$ to show that there exists an infinite subset $Y \subseteq X$ stabilizing σ. ⋆

Suppose X stabilizes an initial segment $[0,k]$ for some $k \in \mathbb{N}$. Then this induces a coloring $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ defined by $g(\emptyset) = I$, $g(\{x\}) = I_x$ and $g(\{x,y\}) = I_{x,y}$. Note that for every $v \in [k]^{\leq 2}$, $\operatorname{card} g(v) \leq \ell_{3-|v|}$. A set $H \subseteq k$ is *g-free* if for every $v \in [H]^{\leq 3}$, $g(v) \cap H \subseteq v$.

Exercise 8.3.7. Let (σ, X) be a condition, and $Y \subseteq X$ be an infinite subset stabilizing some initial segment $[0,k]$. Let $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ be the corresponding limit function. Show that if $\rho \subseteq X$ is f-free and g-free, then $(\sigma \cup \rho, Y)$ satisfies property (b). ⋆

The previous exercise motivates the following definition of the forcing question.

Definition 8.3.8. Given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, let $(\sigma, X) \mathrel{?\vdash} \varphi(G)$ iff there is some $k \in \mathbb{N}$ such that for every coloring $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ such that for every $v \in [k]^{\leq 2}$, $\operatorname{card} g(v) \leq \ell_{3-|v|}$, there is some finite f-free and g-free set $\rho \subseteq X{\upharpoonright}_k$ such that $\varphi(\sigma \cup \rho)$ holds. ◊

Note that the previous definition is in explicit Σ^0_1 form. In order to handle the case where the forcing question does not hold, one would like to also state the same forcing question in the form of a second-order quantification. Let $\mathcal{F}$ be the class of all functions $g : [\mathbb{N}]^{\leq 2} \to [\mathbb{N}]^{<\mathbb{N}}$ such that for every $v \in [\mathbb{N}]^{\leq 2}$, $\operatorname{card} g(v) \leq \ell_{3-|v|}$. Contrary to the class of all tournaments, the class $\mathcal{F}$ is not compact. Thankfully, given a function $g \in \mathcal{F}$ and finite set ρ, the predicate "ρ is g-free" does not require to have a complete information about $g{\upharpoonright}[\rho]^{\leq 2}$, but only to decide $\{(v,z) : v \in [\rho]^{\leq 2}, z \in g(v)\}$. It follows that one can represent g by the relation $R_g = \{(v,z) : v \in [\mathbb{N}]^{\leq 2}, z \in g(v)\}$. Given such a set R_g and some v, g-freeness is decidable, but one cannot know for example the cardinality of $g(v)$ in general. Let $\mathcal{R}$ be the class of all relations R over $[\mathbb{N}]^{\leq 2} \times \mathbb{N}$ such that for every $v \in [\mathbb{N}]^{\leq 2}$, $\operatorname{card}\{z : (v,z) \in R\} \leq \ell_{3-|v|}$. The class $\mathcal{R}$ forms an effectively compact set, and there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{R}$. Given a relation $R \in \mathcal{R}$, we write g_R for the corresponding function in $\mathcal{F}$.

Exercise 8.3.9. Let (σ, X) be a condition, and $\varphi(G)$ be a Σ^0_1-formula. Show that $(\sigma, X) \mathrel{?\vdash} \varphi(G)$ iff for every $R \in \mathcal{R}$, there is some finite f-free and g_R-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ⋆

We are now ready to prove that the forcing question meets its specification.

Lemma 8.3.10. Let $p = (\sigma, X)$ be a condition and $\varphi(G)$ be a Σ^0_1-formula.

1. If $p \mathrel{?\vdash} \varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\varphi(G)$.
2. If $p \mathrel{?\nvdash} \varphi(G)$, then there is an extension $(\tau, Y) \leq p$ forcing $\neg\varphi(G)$. ⋆

Proof. Suppose first $p \mathrel{?\vdash} \varphi(G)$. Let $k \in \mathbb{N}$ witness the definition of the forcing question. By Exercise 8.3.6, there is an infinite subset $Y_0 \subseteq X$ stabilizing $[0,k]$. Let $g : [k]^{\leq 2} \to [k]^{<\mathbb{N}}$ be the corresponding function, and let $\rho \subseteq X{\upharpoonright}_k$ be a finite f-free and g-free subset such that $\varphi(\sigma \cup \rho)$ holds. By Exercise 8.3.7, $(\sigma \cup \rho, Y_0)$ satisfies property (b). Let $Y \subseteq Y_0$ be an infinite subset such that $(\sigma \cup \rho, Y)$ satisfies property (a). Then $(\sigma \cup \rho, Y)$ is a valid extension forcing $\varphi(G)$.

Suppose now $p \operatorname{?\nvdash} \varphi(G)$. Let $\mathscr{C}$ be the $\Pi^0_1(X)$ class of all $R \in \mathscr{R}$ such that for every finite f-free and g_R-free set $\rho \subseteq X$, $\varphi(\sigma \cup \rho)$ does not hold. By Exercise 8.3.9, the class $\mathscr{C}$ is non-empty. Pick any $g \in \mathscr{C}$. By finitely many applications of FS^1 and FS^2, there is an infinite g-free subset $Y \subseteq X$. The condition (σ, Y) is an extension of p forcing $\neg\varphi(G)$. ■

Step 3: Computational property. As before, given a condition (σ, X) and a Σ^0_1-formula $\varphi(G)$, the forcing question $(\sigma, X) \operatorname{?\vdash} \varphi(G)$ is $\Sigma^0_1(X)$. One must therefore impose some computability-theoretic constraints to the set X to obtain diagonalization theorems. A condition (σ, X) must therefore also satisfy the following property

(c) $X \in \mathscr{W}$

where $\mathscr{W}$ is a weakness property. Looking at the various lemmas, many preservation assumptions are used on $\mathscr{W}$: in the extendibility lemma, one used X-computable instances of FS^1 and FS^2 to satisfy property (a), and $\mathsf{RT}^3_{<\infty,\ell_3}$ to satisfy property (b). In the forcing question, one used X-computable instances of $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$ for stabilizing initial segments if the forcing question holds, and X-computable instances of WKL to pick a coloring $g : [\mathbb{N}]^{\leq 2} \to [\mathbb{N}]^{<\mathbb{N}}$ and $X \oplus g$-computable instances of FS^1 and FS^2 to thin out the reservoir and obtain an infinite g-free subset. Thus, overall, we required $\mathscr{W}$ to be preserved by FS^1, FS^2, $\mathsf{RT}^1_{<\infty,\ell_1}$, $\mathsf{RT}^2_{<\infty,\ell_2}$ and $\mathsf{RT}^3_{<\infty,\ell_3}$.

Note that there is some degree of freedom in the choice of ℓ_1, ℓ_2 and ℓ_3. These integers can be chosen to be arbitrarily large, depending on the property one wants to preserve.

Example 8.3.11. If one wants to prove cone avoidance, we shall use $\ell_1 = 1$, $\ell_2 = 1$ and $\ell_3 = 2$, as Wang [15] proved that these statements admit cone avoidance. If one wants to preserve k hyperimmunities simultaneously, we shall use larger values depending on k, based on Patey [45].

Exercise 8.3.12 (Wang [15]). Assume that for every $n \in \mathbb{N}$, there is some $\ell_n \in \mathbb{N}$ such that $\mathsf{RT}^n_{<\infty,\ell_n}$ admits cone avoidance.

1. Design a notion of forcing for FS^n.
2. Prove by induction on n that FS^n admits cone avoidance. ★

Exercise 8.3.13 (Wang [15]). A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is k-*bounded* if for every $c \in \mathbb{N}$, $f^{-1}(c)$ has size at most k. A set $H \subseteq \mathbb{N}$ is an f-*rainbow* if f is injective on $[H]^n$. The *rainbow Ramsey theorem* for n-tuples and k-bounded functions RRT^n_k is the problem whose instances are k-bounded colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite f-rainbows.

1. Design a notion of forcing for RRT^3_2.
2. Prove that RRT^3_2 admits cone avoidance.[20] ★

20: Actually, Wang proved that RRT^n_k is strongly computably reducible to FS^n, hence RRT^n_k admits strong cone avoidance for every $n, k \geq 2$.

Exercise 8.3.14 (Patey [45]). A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is *left (right) trapped* if for every $v \in [\mathbb{N}]^n$, $f(v) < \max v$ ($f(v) \geq \max v$). Fix a weakness property $\mathscr{W}$.

1. Show that if FS^n for left trapped and right trapped functions preserve $\mathscr{W}$, then so does FS^n.
2. Use Proposition 5.7.1 to show that for every right trapped function $f : [\mathbb{N}]^n \to \mathbb{N}$, every DNC function[21] relative to f computes an infinite f-free set.
2. Given a set X, construct a left trapped coloring $f : \mathbb{N} \to \mathbb{N}$ such that every infinite f-free set is effectively X-immune.
3. Deduce that if FS^n for left trapped functions preserves $\mathscr{W}$, then so does FS^n. ★

21: Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *DNC* relative to X if for every e, $f(e) \neq \Phi_e^X(e)$. This notion admits many computability-theoretic characterizations, in terms of effective X-immunity, and escaping bounded X-c.e. sets. See Sections 5.7 and 6.2.

Exercise 8.3.15. Given a coloring $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$, a set $H \subseteq \mathbb{N}$ if *f-free* if for every $v \in [H]^n$, $f(v) \cap H \subseteq v$. The coloring f is *h-constrained* for a function $h : \mathbb{N} \to \mathbb{N}$ if for every $v \in [\mathbb{N}]^n$, $\operatorname{card} f(v) \leq h(\min v)$. If h is the constant function k, we say that f is k-constrained.

1. Show that there exists an $(x \mapsto x)$-constrained coloring $f : \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}$ with no infinite f-free set.
2. Use FS^n to show that for every k-constrained coloring $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$, there is an infinite f-free set.

A coloring $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$ is *progressive* if for every $v \in [\mathbb{N}]^n$, $\min f(v) \geq \min v$.

3. Design a notion of forcing to build infinite f-free sets for $(x \mapsto x)$-constrained progressive colorings $f : [\mathbb{N}]^n \to [\mathbb{N}]^{<\mathbb{N}}$. ★

Higher jump control

Jump cone avoidance 9

From many perspectives, second-jump control is the same as first-jump control, *mutatis mutandis*: it consists of constructing a set G while controlling its $\Sigma^0_2(G)$ properties. To achieve this, one defines again a forcing question for the class of Σ^0_2 formulas, with the same abstract theorems. In practice, however, there is a strong technical gap from first-jump control to second-jump control. This is merely due to the fact that, unlike Turing functionals, jump functionals are not continuous functions in Cantor space. The forcing question therefore becomes a density statement, which often does not yield the appropriate definitional complexity. The main task of the design of a good second-jump control consists in finding the most effective notion of forcing to build solutions to a given problem. As a byproduct, this often yields insights about the structural nature of the problem.

Prerequisites: Chapters 2 to 4

9.1 Context and motivation

Second-jump control received much less attention than first-jump control in computability theory, and reverse mathematics in particular. One of the reasons is that the vast majority of statements studied in reverse mathematics could be separated using first-jump properties. Moreover, as we shall see in the next section, many second-jump properties can be obtained from effectivization of first-jump properties. Besides reverse mathematics, second-jump control can be used in computability theory to construct sets of low$_2$ degree. Such sets occur naturally in computability theory, but often using the following characterization, rather than directly using a second-jump control: a set X is of low$_2$ degree iff $\emptyset'$ is of high degree over X. There are however a few examples where second-jump control naturally occurs in reverse mathematics.

In the study of Ramsey's theorem and more generally combinatorial hierarchies, the cohesiveness principle quickly became an unavoidable tool, as a bridge between computable instances for $(n+1)$-tuples and arbitrary instances of n-tuples. For example, COH reduces computable instances of Ramsey's theorem for pairs to arbitrary instances of the pigeonhole principle (see Theorem 3.4.1). Recall from Section 3.4 that an infinite set $C \subseteq \mathbb{N}$ is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R}_n$, where $\subseteq^*$ means "included up to finite changes". The *cohesiveness principle* is the problem COH whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets. Jockusch and Stephan [13] [1] proved that COH is equivalent to the problem "For every Δ^0_2 infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, there is a Δ^0_2-approximation of an infinite path." The cohesiveness principle is therefore a statement about jump computation and separating principles from COH over reverse mathematics requires to use second-jump control [78].

1: Jockusch and Stephan [13] actually proved that the sequence of all primitive recursive sets is maximally difficult for COH, and the degrees of its cohesive sets are exactly those whose jump is PA over $\emptyset'$. Bratka, Hendtlass and Kreuzer [77] refined it to obtain an instance-wise correspondence.

Ramsey's theorem for n-tuples induces a hierarchy of statements based on n. From a reverse mathematical perspective, this hierarchy is known to collapse at level 3 and RT^n_2 is equivalent to ACA_0 for every $n \geq 3$. [5, 16]. On the other hand, some consequences of Ramsey's theorem, such as the free set (FS^n) [79] and the rainbow Ramsey (RRT^n_2) [80] theorems are not known to collapse [15]. The most promising approach to prove the strictness of these hierarchies is using iterated jump control [81].

In this section, we shall focus on the unability, for a given problem, to code a fixed set in the jump of its solutions. This is the notion of jump cone avoidance. This is one of the simplest applications of second-jump control, and already illustrates the core problematics of the techniques.

Definition 9.1.1. A problem P admits *jump cone avoidance* if for every set Z and every non-$\Delta^0_2(Z)$ set C, every Z-computable instance X of P admits a solution Y such that C is not $\Delta^0_2(Z \oplus Y)$. ◇

Here again, one can drop the Z-computability restriction of the P-instance, to yield *strong jump cone avoidance*. By letting $Z = \emptyset$ and $C = \emptyset''$, if a problem P admits jump cone avoidance, then even computable instance admits a solution of non-high degree.

9.2 Use first-jump control

Second-jump control aims at proving theorems about the jump of solutions to mathematical problems. However, an effectivization of first-jump control is sometimes sufficient to obtain the same results. Indeed, if a problem admits a low basis, or a weakly low basis[2], it admits jump cone avoidance, a low$_2$ basis, and many other properties.

2: Recall that a problem P admits a *weakly low basis* if for every set Z every PA degree P over Z', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)' \leq_T P$. For example, Ramsey's theorem for pairs admits a weakly low basis.

Proposition 9.2.1. If a problem P admits a weakly low basis, then it admits jump cone avoidance. ★

Proof. Fix a set Z, a non-$\Delta^0_2(Z)$ set C and a Z-computable instance X of P. By the cone avoidance basis theorem relativized to Z' (see Theorem 3.2.6), there is a set Q of PA degree over Z' such that $C \not\leq_T Q$. Since P admits a weakly low basis, then there is a solution Y such that $(Y \oplus Z)' \leq_T Q$. In particular, C is not $\Delta^0_2(Z \oplus Y)$. ■

The strong technical gap between first-jump and second-jump control gives a strong incentive to use first-jump control to prove second-jump properties when possible. This should be the first consideration is the decisional process of the choice of jump-control techniques.

Exercise 9.2.2. A problem P admits *preservation of 1 jump hyperimmunity* if for every set Z and every Z'-hyperimmune function f, every Z-computable instance X of P admits a solution Y such that f is $(Y \oplus Z)'$-hyperimmune. Use the computably dominated basis theorem to prove that if P admits a weakly low basis, then it admits preservation of 1 jump hyperimmunity. ★

Exercise 9.2.3. A problem P admits *jump DNC avoidance* if for every set Z and every set D such that Z' is not of DNC degree over D, every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)'$ is not of DNC degree over D.

1. Show that if P admits a low basis, then it admits jump DNC avoidance.
2. Give an example of a problem which admits a weakly low basis, but not jump DNC avoidance. ★

9.3 Forcing and density

First-jump control using forcing constructions can be really thought of as a straightforward generalization of the finite extension method. On the other hand, the full power of the forcing framework is unleashed when deciding properties at higher levels on the arithmetic hierarchy, and it is already witnessed with Π^0_2 properties. Consider Cohen forcing for the sake of simplicity, that is, the set of finite binary strings $2^{<\mathbb{N}}$ partially ordered by the prefix relation $\preceq$.[3] The *interpretation* of a Cohen condition σ is the class $[\sigma] = \{X \in 2^{\mathbb{N}} : \sigma \prec X\}$, that is, the class of all infinite binary sequences starting with σ.

3: Traditionally, the order relation is reversed in forcing, that is, a condition q *extends* p if $q \leq p$. This order is justified by the fact that the condition q seen as an approximation the constructed set G is more precise than p, hence the class $[q]$ of candidate sets satisfying the approximation q is a subclass of $[p]$.

In the case of Cohen forcing, the relation "σ is a prefix of τ" is denoted $\sigma \preceq \tau$, which might cause some confusion with the usual forcing notation. In particular, an infinite descending sequence of Cohen conditions is an infinite ascending sequence of strings $\sigma_0 \preceq \sigma_1 \preceq \dots$

Intuitively, a condition p forces a property $\varphi(G)$ if p, seen as an approximation of the constructed set G, already contains the information that $\varphi(G)$ will hold. One would be therefore tempted to use the following definition:

> **Definition 9.3.1.** A condition p *strongly forces* a property $\varphi(G)$ if $\varphi(G)$ holds for every $G \in [p]$. ◊

In the case of Cohen forcing, σ strongly forces $\varphi(G)$ if $\varphi(G)$ holds for every infinite binary sequence starting with σ. The strong forcing relation ensures that whatever the remainder of the construction, even if the construction is very degenerate, then the property will hold. For example, if σ strongly forces $\varphi(G)$, then $\varphi(G)$ will hold even for $G = \sigma 00000\cdots$ or $G = \sigma 11111\cdots$, which can both be considered as very degenerate constructions since at any stage, one could decide to include any arbitrary finite binary sequence. This strong forcing relation is suitable for Σ^0_1 and Π^0_1 properties, and therefore sufficient for first-jump control.

Lemma 9.3.2. For every Σ^0_1 formula $\varphi(G)$, the set of all Cohen conditions strongly forcing either $\varphi(G)$ or $\neg\varphi(G)$ is dense. ★

Proof. Say $\varphi(G) \equiv (\exists x)\psi(G\restriction x)$ for some Δ^0_0-formula ψ. Let σ be a Cohen condition. If there is some $\tau \succeq \sigma$ and some $x < |\tau|$ such that $\psi(\tau\restriction x)$ holds, then for every $G \in [\tau]$, $\psi(G\restriction x)$ holds, hence τ strongly forces $\varphi(G)$. Otherwise, for every $\tau \succeq \sigma$ and every $x < |\tau|$, $\neg\psi(\tau\restriction x)$ holds, hence for every $G \in [\sigma]$ and every x, $\neg\psi(G\restriction x)$ holds, so σ strongly forces $\neg\varphi(G)$. ■

The previous lemma can be thought of as stating the completeness of the strong forcing relation for Σ^0_1 and Π^0_1 formulas in Cohen forcing. In particular, it follows that every such property about the constructed set can be decided at a finite stage of the construction. We loose completeness of the strong forcing relation when dealing with Σ^0_2 and Π^0_2 formulas. Consider for example the Π^0_2 formula $\varphi(G) \equiv$ "G is infinite", which can be written as $\forall x \exists y(y > x \wedge y \in G)$. Then no Cohen condition σ strongly forces either $\varphi(G)$ or $\neg\varphi(G)$ since $[\sigma]$ contains the finite set $G = \sigma 00000\cdots$ and the infinite set $G = \sigma 11111\cdots$. On the other hand, there is an asymmetry between the two cases, as there are many ways to construct an infinite set, while any construction of a finite set must be degenerate. For every condition σ, there is an extension $\tau \succeq \sigma$ such that $\operatorname{card} \tau > \operatorname{card} \sigma$[4], hence every sufficiently generic filter yields an infinite set.

4: Here, we distinguish the length $|\sigma|$ of a string σ, and the cardinality $\operatorname{card} \sigma$ which is the cardinality of the finite set $\{x < |\sigma| : \sigma(x) = 1\}$.

Let us now consider an arbitrary Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, where ψ is a Π^0_1 formula. Given a Cohen condition σ, either there exists an extension $\tau \succeq \sigma$ strongly forcing $\psi(G, x)$ for some x, in which case τ forces $\varphi(G)$, or for every x and every extension $\tau \geq \sigma$, τ does not strongly force $\psi(G,x)$. In the latter case, by Lemma 9.3.2, for every x and every $\tau \geq \sigma$, there is an extension ρ strongly forcing $\neg\psi(G,x)$. In other words, for every x, the set of conditions strongly forcing $\neg\psi(G,x)$ is dense below σ. Then, if $\mathcal{F}$ is a sufficiently generic filter containing σ, it will contain for every x a condition strongly forcing $\neg\psi(G,x)$, hence $(\forall x)\neg\psi(G_{\mathcal{F}},x)$ will hold. This motivates the following definition of the forcing relation.

Definition 9.3.3. A condition p *forces* a property $\varphi(G)$ if $\varphi(G_{\mathcal{F}})$ holds for every sufficiently generic filter $\mathcal{F}$ containing p. ◇

With this definition, every Cohen condition forces G to be infinite. For any reasonable notion of forcing, one can prove that for every arithmetic formula $\varphi(G)$, the set of conditions forcing either $\varphi(G)$ or $\neg\varphi(G)$ is dense.

The previous explanation induced a forcing question for Σ^0_2 formulas in Cohen forcing.

Definition 9.3.4. Let σ be a Cohen condition, and $\varphi(G) \equiv \exists x\psi(G,x)$ be a Σ^0_2 formula. Define $\sigma\ ?\!\vdash \varphi(G)$ to hold if there exists some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that τ strongly forces $\psi(G,x)$, that is, for every $\rho \geq \tau$, $\psi(\rho,x)$ holds.[5] [6] ◇

5: Recall that Cohen forcing admits a Σ^0_1-preserving forcing question for Σ^0_1 formulas defined as $\sigma\ ?\!\vdash \varphi(G)$ if there is some $\tau \geq \sigma$ such that $\varphi(\tau)$ holds. It induces a forcing question for Π^0_1 formulas by taking its negation. In the following of this chapter, it might be better to think of the forcing question for a Σ^0_2 formula $\varphi(G) \equiv \exists x\psi(G,x)$ as $\sigma\ ?\!\vdash \varphi(G)$ if there is some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau\ ?\!\vdash \psi(G,x)$.

6: Note that with this forcing question, either there exists an extension strongly forcing $\varphi(G)$, or an extension forcing $\neg\varphi(G)$. In general, the forcing relation for Σ^0_2 formulas can be chosen to be the strong version, while the general definition is needed for Π^0_2 formulas.

A simple analysis on the definition of the forcing question shows that it is Σ^0_2-preserving. The existence of a Σ^0_2-preserving forcing question for Σ^0_2 formulas yields jump cone avoidance, with the same proof of Theorem 3.3.4, *mutatis mutandis*

Theorem 9.3.5
Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_2-preserving forcing question. For every non-Δ^0_2 set C and every sufficiently generic filter $\mathcal{F}$, C is not $\Delta^0_2(G_{\mathcal{F}})$.

Proof. It suffices to prove the following lemma:

Lemma 9.3.6. For every condition $p \in \mathbb{P}$ and every Turing index $e \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\Phi_e^{G'} \neq C$. ★

Proof. Consider the following set[7]

$$U = \{(x,v) \in \mathbb{N} \times 2 : p\ ?\!\vdash \Phi_e^{G'}(x){\downarrow}= v\}$$

7: By Post's theorem, the property $\Phi_e^{G'}(x){\downarrow}= v$ is Σ^0_2, although the translation is not straightforward. It can be written as

$$\exists\rho\exists t[\Phi_e^{\rho}(x){\downarrow}= v \wedge \forall s\ \rho \leq G'_{t+s}]$$

where $\{G'_s\}_{s\in\mathbb{N}}$ is a fixed G-c.e. enumeration of G'.

Since the forcing question is Σ^0_2-preserving, the set U is Σ^0_2. There are three cases:

- Case 1: $(x, 1-C(x)) \in U$ for some $x \in \mathbb{N}$. By Property (1) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^{G'}(x){\downarrow}= 1 - C(x)$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. By Property (2) of the forcing question, there is an extension $q \leq p$ forcing $\Phi_e^{G'}(x){\uparrow}$ or $\Phi_e^{G'}(x){\downarrow}\neq C(x)$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_2 graph of the characteristic function of C, hence C is Δ^0_2. This contradicts our hypothesis. ■

We are now ready to prove Theorem 9.3.5. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $q \in \mathbb{P}$ forcing $\Phi_e^{G'} \neq C$. It follows from Lemma 9.3.6 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $C \not\leq_T G'_{\mathcal{F}}$. This completes the proof of Theorem 9.3.5. ■

In particular, since Cohen forcing admits a Σ^0_2-preserving forcing question for Σ^0_2 formulas, we obtain our first jump cone avoidance theorem using a direct second-jump control.

Theorem 9.3.7
Let C be a non-Δ^0_2 set. For every sufficiently Cohen generic filter $\mathcal{F}$, C is not $\Delta^0_2(G_\mathcal{F})$.

Exercise 9.3.8. Consider Cohen forcing. Recall from Section 3.6 that a forcing question is Σ^0_n*-compact* if for every $p \in \mathbb{P}$ and every Σ^0_n formula $\varphi(G, x)$, if $p \,?\!\vdash \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ such that $p \,?\!\vdash \exists x \in F\, \varphi(G, x)$.

1. Show that the forcing question for Σ^0_2 formulas is Σ^0_2-compact
2. Adapt Theorem 3.6.4 to prove that for every $\emptyset'$-hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and every sufficiently Cohen generic filter $\mathcal{F}$, the function f is $G'_\mathcal{F}$-hyperimmune. ★

9.4 Weak König's lemma

As explained in the previous section, the forcing relation for a Π^0_2 formula $\forall x \psi(G, x)$ is a density statement for a countable family of Σ^0_1 formulas $\{\psi(G, x) : x \in \mathbb{N}\}$. Density statements require to quantify over the partial order, which is not an issue when dealing with Cohen forcing, but can be very complicated if the partial order is not computable as it is often the case. One will then need to define a custom forcing question with the desired properties.

Our first non-trivial example concerns weak König's lemma, for which we prove it admits simultaneously cone and jump cone avoidance.[8]

8: By the cone avoidance basis theorem (Theorem 3.2.6), given a non-computable set C, every non-empty Π^0_1 class admits a member G such that $C \not\leq_T G$. By the low basis theorem (Theorem 4.4.6), given a non-Δ^0_2 set D, every non-empty Π^0_1 class admits a member G of low degree, in which case D is not $\Delta^0_2(G)$. One cannot however abstractly deduce from these theorems that WKL admits simultaneously cone and jump cone avoidance.

Lawton (see [47]) proved that one can actually combine the low and the cone avoidance basis theorem, by showing that if C is Δ^0_2 and non-computable, then every non-empty Π^0_1 class admits a member G of low degree such that $C \not\leq_T G$. The case where C is non-Δ^0_2 follows directly from the low basis theorem. Thus, as stated, Theorem 9.4.1 follows from Lawton's theorem, but its proof generalizes to countable cones avoidance, while Lawton's proof does not.

Theorem 9.4.1 (Wang [82])
Let C be a non-computable set and D be a non-Δ^0_2 set. For every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^\mathbb{N}$, there exists a member $G \in \mathcal{P}$ such that $C \not\leq_T G$ and $D \not\leq_T G'$.

Proof. Recall that Jockusch-Soare forcing is the notion of forcing whose conditions are infinite computable binary trees $T \subseteq 2^{<\mathbb{N}}$, partially ordered by the subset relation. In this proof, we shall actually restrict the partial order to infinite *primitive recursive* binary trees. Indeed, as mentioned before, the complexity of the partial order is relevant in second-jump control. The index set of all total computable sets is Π^0_2-complete, while all primitive recursive sets can be computably listed. The restriction to primitive recursive trees is without loss of generality, as shows the following lemma:

Lemma 9.4.2. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite co-c.e. tree. There is a primitive recursive tree $S \supseteq T$ such that $[S] = [T]$. ★

Proof. Say $T = \{\sigma \in 2^{<\mathbb{N}} : \Phi_e(\sigma)\uparrow\}$ for some partial computable function Φ_e. Let $S = \{\sigma \in 2^{<\mathbb{N}} : \forall s < |\sigma|\ \Phi_e(\sigma{\upharpoonright}s)[s]\uparrow\}$. Note that the predicate $\Phi_e(x)[s]\uparrow$ is primitive recursive, and primitive recursion is closed under bounded quantification. We first show that $S \supseteq T$. If $\sigma \in T$, then T being a tree, for every $s < |\sigma|$, $\sigma{\upharpoonright}s \in T$, so by definition of T, $\Phi_e(\sigma{\upharpoonright}s)[s]\uparrow$, hence $\sigma \in S$. Thus $S \supseteq T$, and in particular $[S] \supseteq [T]$. We now prove that $[S] \subseteq [T]$.

Let $P \in [S]$ and $\sigma \prec P$. Suppose for the contradiction that $\Phi_e(\sigma)\downarrow$. Then, letting $t > |\sigma|$ be such that $\Phi_e(\sigma)[t]\downarrow$, $P \upharpoonright t \notin S$, contradicting $P \in [S]$. It follows that $\Phi_e(\sigma)\uparrow$, and this for every $\sigma \prec P$, so $P \in [T]$. ■

In particular, there exists a primitive recursive tree T such that $[T] = \mathcal{P}$. The *interpretation* $[T]$ of a tree T is the class of its paths. Every sufficiently generic filter $\mathcal{F}$ for this notion of forcing induces a path $G_{\mathcal{F}}$ which is the unique element of $\bigcap\{[T] : T \in \mathcal{F}\}$. The forcing question for Σ^0_1 formulas of Exercise 3.3.7 also holds when working with primitive recursive trees.

Definition 9.4.3. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_1 formula $\varphi(G)$, define $T \operatorname{?\vdash} \varphi(G)$ to hold if there is some level $\ell \in \mathbb{N}$ such that $\varphi(\sigma)$ holds for every node σ at level ℓ in T. ◊

One easily sees that this forcing question is Σ^0_1-preserving.

Lemma 9.4.4. Let $T \subseteq 2^{<\mathbb{N}}$ be a condition and $\varphi(G)$ be a Σ^0_1 formula.

1. If $T \operatorname{?\vdash} \varphi(G)$, then T forces $\varphi(G)$
2. If $T \operatorname{?\nvdash} \varphi(G)$, then there is an extension $S \leq T$ forcing $\neg\varphi(G)$. ★

Proof. Suppose first $T \operatorname{?\vdash} \varphi(G)$. Let $\ell \in \mathbb{N}$ be the level witnessing it. For every $P \in [T]$, $P \upharpoonright \ell \in T$, so $\varphi(P \upharpoonright \ell)$ holds, hence $\varphi(P)$ holds. Thus T forces $\varphi(G)$. Suppose now $T \operatorname{?\nvdash} \varphi(G)$. Say $\varphi(G) \equiv \exists x \psi(G, x)$ for some Δ^0_0 formula ψ. Then $S = \{\sigma \in T : \forall x < |\sigma| \neg\psi(\sigma, x)\}$ is an infinite primitive recursive[9] subtree of T forcing $\neg\varphi(G)$. ■

9: Every Δ^0_0 formula is primitive recursive. On this other hand, there exist primitive recursive predicates which are not Δ^0_0.

Since this notion of forcing admits a Σ^0_1-preserving forcing question for Σ^0_1 formulas, by Theorem 3.3.4 for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$. Until now, the proof was only a rewriting of Theorem 3.2.6 with primitive recursive trees, using the more abstract framework of the forcing question. We now turn to second jump control.

Definition 9.4.5. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $T \operatorname{?\vdash} \varphi(G)$ to hold if there is some $x \in \mathbb{N}$ and an extension $S \leq T$ such that $S \operatorname{?\vdash} \psi(G, x)$.[10] [11] ◊

10: In this definition, ψ is a Π^0_1 formula, so the relation $S \operatorname{?\vdash} \psi(G, x)$ is the forcing question for Π^0_1 formulas induced by the forcing question for Σ^0_1 formulas by taking the negation. Note the similarity with the forcing question for Σ^0_2 formulas in Cohen forcing.

11: Although the partial order is not computable, the complexity of finding an extension is "absorbed" in the overall complexity of the forcing question for Σ^0_2 formulas, yielding a Σ^0_2-preserving forcing question. Because of this, the forcing questions at higher levels of the arithmetic hierarchy will be similar to the ones for Cohen forcing.

Looking at the complexity of the forcing question for Σ^0_2 formulas, the relation $S \operatorname{?\vdash} \psi(G, x)$ is Π^0_1 since it is the negation of the Σ^0_1-preserving forcing question for Σ^0_1 formulas. Being an infinite primitive recursive tree and being a subset of another primitive recursive tree is a Π^0_1 predicate, so the overall formula is Σ^0_2. We now show that this relation satisfies the specifications of a forcing question.

Lemma 9.4.6. Let $T \subseteq 2^{<\mathbb{N}}$ be a condition and $\varphi(G)$ be a Σ^0_2 formula.

1. If $T \operatorname{?\vdash} \varphi(G)$, then there is an extension $S \leq T$ forcing $\varphi(G)$
2. If $T \operatorname{?\nvdash} \varphi(G)$, then T forces $\neg\varphi(G)$. ★

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$. Suppose first $T \operatorname{?\vdash} \varphi(G)$. Let $x \in \mathbb{N}$ and $S \leq T$ be such that $S \operatorname{?\vdash} \psi(G, x)$. By Lemma 9.4.4, there is an extension $S_1 \leq S$ forcing $\psi(G, x)$. In particular, $S_1 \leq T$ and S_1 forces $\varphi(G)$. Suppose now $T \operatorname{?\nvdash} \varphi(G)$. Let $x \in \mathbb{N}$. We claim that the set of all conditions forcing $\neg\psi(G, x)$ is dense below T. Indeed, given a condition $S \leq T$, $S \operatorname{?\nvdash} \psi(G, x)$, so by Lemma 9.4.4, there is an extension $S_1 \leq S$ forcing $\neg\psi(G, x)$. Thus, for every sufficiently generic filter $\mathcal{F}$ containing T and every $x \in \mathbb{N}$, there is a condition $S_1 \in \mathcal{F}$ forcing $\neg\psi(G, x)$, thus $\neg\varphi(G_{\mathcal{F}})$ holds. ■

Since this notion of forcing admits a Σ^0_2-preserving forcing question for Σ^0_2 formulas, by Theorem 9.3.5 for every sufficiently generic filter $\mathcal{F}$, $D \not\leq_T G'_{\mathcal{F}}$. To conclude the theorem, by Lemma 9.4.2, there is a condition T such that $[T] = \mathcal{P}$, so for every sufficiently generic filter $\mathcal{F}$ containing T, $G_{\mathcal{F}} \in \mathcal{P}$. This completes the proof of Theorem 9.4.1. ■

Exercise 9.4.7 (Le Houérou, Levy Patey and Mimouni [83]). Recall the notion of Σ^0_n-compactness from Section 3.6. Consider the Jockusch-Soare notion of forcing restricted to primitive recursive trees (Theorem 9.4.1).

1. Show that the forcing questions for Σ^0_1 and Σ^0_2 formulas are Σ^0_1-compact and Σ^0_2-compact, respectively.
2. Fix a hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and a $\emptyset'$-hyperimmune function $g : \mathbb{N} \to \mathbb{N}$. Prove that every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ has a member G such that f is G-hyperimmune and g is G'-hyperimmune. ⋆

9.5 Cohesiveness principle

As mentioned before, because of its equivalence with the statement "every Δ^0_2 infinite binary tree admits a Δ^0_2-approximation of a path", the cohesiveness principle is a statement about jump computation. By Toswner's theorem (Theorem 7.3.8) Δ^0_2-approximations of a path can be added to a model of RCA_0 without affecting its first-jump properties. Thus, one should expect from a natural notion of forcing for COH to have a trivial first-jump control, and a second-jump control resembling the one of weak König's lemma. This is actually the case.

Consider a uniformly computable sequence of sets $R_0, R_1, \dots$ The usual notion of forcing to build $\vec{R}$-cohesive sets with a good first-jump control is computable Mathias forcing, that is, Mathias forcing whose reservoirs are computable. The first-jump control of such a notion of forcing is very similar to Cohen forcing, and preserves the same first-jump properties. On the other hand, even when working with computable reservoirs, Mathias forcing does not admit a good second-jump control. In particular, every sufficiently generic filter for computable Mathias forcing yields a set of high degree. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *dominating* if it eventually dominates every total computable function. By Martin's domination theorem [84], a set X is of high degree iff it computes a dominating function.

Proposition 9.5.1. Let $\mathcal{F}$ be a sufficiently generic filter for computable Mathias forcing. Then the principal function of $G_{\mathcal{F}}$ is dominating, hence $G_{\mathcal{F}}$ is of high degree. ⋆

Proof. Let f be a total computable function. We can assume without loss of generality that f is strictly increasing. Let us shows that the class $\mathcal{D}_f$ of all computable Mathias conditions (τ, Y) forcing the principal function of G to eventually dominate f is dense. Fix a computable Mathias condition (σ, X), and say $X = \{x_0 < x_1 < \dots\}$. Let $a = \mathrm{card}\{x < |\sigma| : \sigma(x) = 1\}$. Then the set $Y = \{x_{f(a+s)} : s \in \mathbb{N}\}$ is a computable subset of X and (σ, Y) forces the principal function of G to eventually dominate f. ■

There are multiple ways to explain why computable Mathias forcing does not admit a good second-jump control, each of them yielding the same conclusion:

the problem comes from the permissiveness of the reservoirs, which can be arbitrary computable sets.[12]

12: The general takeway of this discussion is that when trying to design a notion of forcing with a good second-jump control, consider a notion of forcing with a good first-jump control, then restrict the partial order to be the less permissive possible, allowing only the conditions produced by the first-jump control. This usually yields a partial order with better complexity, and hopefully enables to define a Σ^0_2-preserving forcing question.

1. *Sparsity of the reservoirs*. Proposition 9.5.1 shows that computable Mathias forcing allows to take extensions with sparse reservoirs and then produce dominant functions. However, the only operations needed to produce cohesive sets is to split the reservoir according to computable partitions and pick any infinite part. The first condition is $(\epsilon, \mathbb{N})$ with a non-sparse reservoir. Then, intuitively, if a reservoir X is not too sparse, then for every 2-partition $X_0 \sqcup X_1 = X$, at least one of the parts is not too sparse either. One could therefore maintain non-sparsity as an invariant by asking the reservoirs to be boolean combinations of $R_0, R_1, \dots$
2. *Complexity of the partial order*. When trying to design a forcing question for Σ^0_2 formulas in computable Mathias forcing, one needs to quantify over the partial order, and therefore quantify over infinite computable subsets of the reservoir. This quantification is too complex and cannot be "absorbed" in the complexity of the general formula to produce a Σ^0_2-preserving question. One must therefore adopt a more efficient way to represent forcing conditions, such as only keeping track of the boolean choices of partitions induced by the sets $R_0, R_1, \dots$

In the following theorem, we restrict computable Mathias forcing to conditions obtained from boolean combinations of computable partitions, and take advantage of this additional structure to design a forcing question with a good second-jump control. This yields that COH admits simultaneously cone and jump cone avoidance.

Theorem 9.5.2
Let C be a non-computable set and D be a non-Δ^0_2 set. For every uniformly computable sequence of sets $R_0, R_1, \dots$, there exists an infinite cohesive set G such that $C \not\leq_T G$ and $D \not\leq_T G'$.

Proof. Given $\rho \in 2^{<\mathbb{N}}$, let

$$R_\rho = \bigcap_{\rho(n)=0} R_n \bigcap_{\rho(n)=1} \overline{R}_n$$

and let $T = \{\rho \in 2^{<\mathbb{N}} : \exists x > |\rho|\ x \in R_\rho\}$. Note that T is a Σ^0_1 tree, and for every extendible node $\rho \in T$, R_ρ is infinite. By the cone avoidance basis theorem (Theorem 3.2.6) relativized to $\emptyset'$, there is a path $P \in [T]$ such that $D \not\leq_T P \oplus \emptyset'$.

Consider the notion of forcing whose conditions[13] are pairs (σ, n). One can think of such a condition as computable Mathias condition $(\sigma, R_{P\upharpoonright n})$. Note that since $P \in [T]$, $R_{P\upharpoonright n}$ is infinite. The *interpretation* of a condition (σ, n) is the interpretation of the associated computable Mathias condition, that is

13: Note the similarity with the notion of forcing in Theorem 3.2.4. In both cases, we build a cone avoiding set G whose jump computes a fixed degree. Indeed, if G is $\vec{R}$-cohesive, then for every n, there is exactly one ρ of length n such that $G \subseteq^* R_\rho$, and such a ρ can be found G'-computably. By construction, $\rho \prec P$, so $G' \geq_T P$.

$$[\sigma, n] = \{G : \sigma \preceq G \subseteq \sigma \cup R_{P\upharpoonright n}\}$$

A condition (τ, m) *extends* (σ, n) if $\sigma \preceq \tau$, $m \geq n$, and $\tau \setminus \sigma \subseteq R_{P\upharpoonright n}$. Every sufficiently generic filter $\mathcal{F}$ for this notion of forcing induces a path $G_\mathcal{F}$ defined as $\bigcup\{\sigma : (\sigma, n) \in \mathcal{F}\}$. Alternatively, $G_\mathcal{F}$ is the unique element of $\bigcap_{(\sigma,n)\in\mathcal{F}}[\sigma, n]$. The forcing question for Σ^0_1 formulas is induced from the forcing question in computable Mathias forcing:

Definition 9.5.3. Given a condition (σ, n) and a Σ^0_1 formula $\varphi(G)$, define $(\sigma, n) \operatorname{?\vdash} \varphi(G)$ to hold if there is some $\tau \in [\sigma, n]$ such that $\varphi(\tau)$ holds. $\diamond$

One easily sees that this forcing question is Σ^0_1-preserving, although not uniformly in the condition, since one needs to hard-code the initial segment of P of length n.

Lemma 9.5.4. Let (σ, n) be a condition and $\varphi(G)$ be a Σ^0_1 formula.

1. If $(\sigma, n) \operatorname{?\vdash} \varphi(G)$, then there is an extension $(\tau, n) \leq (\sigma, n)$ forcing $\varphi(G)$;
2. If $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$, then (σ, n) forces $\neg\varphi(G)$. $\star$

Proof. Suppose first $(\sigma, n) \operatorname{?\vdash} \varphi(G)$. Let $\tau \in [\sigma, n]$ be such that $\varphi(\tau)$ holds. Then (τ, n) is a valid extension and for every $G \in [\tau, n]$, $\tau \leq G$, so $\varphi(G)$ holds. It follows that (τ, n) forces $\varphi(G)$. Suppose now $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$. Then for every extension $(\tau, m) \leq (\sigma, n)$, $\tau \in [\sigma, n]$, so $\neg\varphi(\tau)$ holds. It follows that (σ, n) forces $\neg\varphi(G)$. ■

Since this notion of forcing admits a Σ^0_1-preserving forcing question for Σ^0_1 formulas, by Theorem 3.3.4 for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_\mathcal{F}$. We now turn to second jump control.

Definition 9.5.5. Given a condition (σ, n) and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $(\sigma, n) \operatorname{?\vdash} \varphi(G)$ to hold if there is some $x \in \mathbb{N}$ and an extension $(\tau, m) \leq (\sigma, n)$ such that $(\tau, m) \operatorname{?\vdash} \psi(G, x)$.[14] $\diamond$

14: As before, ψ is a Π^0_1 formula, so we consider the forcing question for Π^0_1 induced by the forcing question for Σ^0_1 formulas by taking the negation.

The extension relation $(\tau, m) \leq (\sigma, n)$ is computable uniformly in P. Moreover, the relation $(\tau, m) \operatorname{?\vdash} \psi(G, x)$ is Π^0_1 since the forcing question for Σ^0_1 formulas is Σ^0_1-preserving. It follows that the forcing question for Σ^0_2 formulas is $\Sigma^0_1(P \oplus \emptyset')$.

Lemma 9.5.6. Let (σ, n) be a condition and $\varphi(G)$ be a Σ^0_2 formula.

1. If $(\sigma, n) \operatorname{?\vdash} \varphi(G)$, then there is an extension $(\tau, m) \leq (\sigma, n)$ forcing $\varphi(G)$;
2. If $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$, then (σ, n) forces $\neg\varphi(G)$. $\star$

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$. Suppose first $(\sigma, n) \operatorname{?\vdash} \varphi(G)$. Then there exists some $x \in \mathbb{N}$ and an extension $(\tau, m) \leq (\sigma, n)$ such that $(\tau, m) \operatorname{?\vdash} \psi(G, x)$. By Lemma 9.5.4, (τ, m) forces $\psi(G, x)$, hence forces $\varphi(G)$. Suppose now $(\sigma, n) \operatorname{?\nvdash} \varphi(G)$. Fix some $x \in \mathbb{N}$. We claim that the set of all conditions forcing $\neg\psi(G, x)$ is dense below (σ, n). Indeed, given a condition $(\tau, m) \leq (\sigma, n)$, $(\tau, m) \operatorname{?\nvdash} \psi(G, x)$, so by Lemma 9.5.4, there is an extension for (τ, m) forcing $\neg\psi(G, x)$. Thus, for every sufficiently generic filter $\mathcal{F}$ containing (σ, n) and every $x \in \mathbb{N}$, there is a condition in $\mathcal{F}$ forcing $\neg\psi(G, x)$, so $\neg\varphi(G_\mathcal{F})$ holds. ■

Exercise 9.5.7. Using the fact that the forcing question for Σ^0_2 formulas is $\Sigma^0_1(P \oplus \emptyset')$ and that $D \not\leq_T P \oplus \emptyset'$, adapt Theorem 3.3.4 to show that for every sufficiently generic filter $\mathcal{F}$, $D \not\leq_T G'_\mathcal{F}$. $\star$

Thus, for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_\mathcal{F}$ and $D \not\leq_T G'_\mathcal{F}$. Since $P \in [T]$, then for every n, $R_{P \upharpoonright n}$ is infinite, hence for every sufficiently generic filter $\mathcal{F}$, $G_\mathcal{F}$ is infinite. Last, for every condition (σ, n), the condition $(\sigma, n+1)$ is a valid extension, so for every sufficiently generic filter $\mathcal{F}$, $G_\mathcal{F}$ is cohesive for $R_0, R_1, \dots$ This completes the proof of Theorem 9.5.2. ■

The second-jump control in the proof of Theorem 9.5.2 was in two steps: first, one picked the sequence of boolean decisions $P \in [T]$ by a relativized first-jump control for WKL, then one built an infinite cohesive set G with a $\Sigma^0_1(P \oplus \emptyset')$ forcing question for Σ^0_2 formulas. One can actually define a notion of forcing doing both at once, as shows the following exercise.

Exercise 9.5.8 (Patey [85]). Fix a uniformly computable sequence of sets $R_0, R_1, \dots$ and define R_ρ and T as in Theorem 9.5.2. Consider the notion of forcing whose *conditions* are tuples (σ, ρ, S), where σ is a finite string, S is an infinite $\emptyset'$-primitive recursive subtree of T[15], and ρ is an extendible node in S. One can think of a condition as a computable Mathias condition (σ, R_ρ), together with a $\emptyset'$-primitive recursive Jockusch-Soare forcing condition S. A condition (τ, μ, V) *extends* a condition (σ, ρ, S) if $\sigma \preceq \tau$, $\rho \preceq \mu$, $V \subseteq S$ and $\tau \setminus \sigma \subseteq R_\rho$.

1. Define a Σ^0_1-preserving forcing question for Σ^0_1 formulas.[16]
2. Define a Σ^0_2-preserving forcing question for Σ^0_2 formulas.[17] ★

15: Note that by restricting the tree T, one restricts the possible reservoirs R_ρ with $\rho \in T$, so one restricts the forced negative information. Thus, the third component of a condition forces positive information. This shall be explained in the next section in further details.

16: Note that given a condition (σ, ρ, S), the forcing question does not involve S, and the answers leave ρ and S unchanged. First-jump control can therefore "ignore" the components responsible of higher jump control.

17: Hint: combine the forcing question for Σ^0_2 formulas in Definition 9.5.5 and the forcing question for Σ^0_1 formulas in Definition 9.4.3.

9.6 Partition regularity

Most theorems from Ramsey theory are proven using variants of Mathias forcing. However, as shows Proposition 9.5.1, generic Mathias filters tend to produce sets of high degree, even when working with computable reservoirs. In order to construct solutions to theorems from Ramsey theory with a good second-jump control, one must therefore refine this notion of forcing to be less permissive about reservoirs. In the case of the cohesiveness principle, the solution was restricting the reservoirs to boolean combinations of a uniformly computable sequence of sets. In this section, we generalize the approach by allowing to split the reservoirs based on *any* finite partition of the integers. This yields the notion of partition regularity.

Definition 9.6.1. A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is *partition regular*[18] if

1. $\mathcal{P}$ is non-empty ;
2. For all $X \in \mathcal{P}$ and $Y \supseteq X$, $Y \in \mathcal{P}$;
3. For every $X \in \mathcal{P}$ and every 2-cover $Y_0 \cup Y_1 \supseteq X$, there is some $i < 2$ such that $Y_i \in \mathcal{P}$. ◊

18: By the upward-closure of a partition regular class, $\mathcal{P}$ is non-empty iff $\mathbb{N} \in \mathcal{P}$, and the last property can be restricted to 2-partitions of X, that is, where $Y_0 \cap Y_1 = \emptyset$ and $Y_0 \cup Y_1 = X$. By iterating the splitting, if $\mathcal{P}$ is partition regular, then for every k, for every $X \in \mathcal{P}$ and every k-cover $Y_0 \cup \dots \cup Y_{k-1} \supseteq X$, there is some $i < k$ such that $Y_i \in \mathcal{P}$.

There exist many examples of partition regularity statements in combinatorics.

Example 9.6.2. The following classes are partition regular:

1. $\{X : X \text{ is infinite }\}$ by the infinite pigeonhole principle ;
2. $\{X : n \in X\}$ for a fixed $n \in \mathbb{N}$;
3. $\{X : \limsup_{n\to\infty} \frac{|\{1,2,\dots,n\} \cap X|}{n} > 0\}$;
4. $\{X : \sum_{n \in X} \frac{1}{n} = \infty\}$.

Among these examples, the second is considered as degenerate, as it contains finite sets. A partition regular class is *principal* if it is of the form $\{X : n \in X\}$ for a fixed $n \in \mathbb{N}$. We shall work only with partition regular classes containing only infinite sets. A class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *non-trivial* if it contains only sets with at least two elements. If $\mathcal{A}$ is partition regular, then it is non-trivial iff it contains only infinite sets.[19] The following operator is an easy way to define non-trivial

19: Note that a non-trivial partition regular class does not contain any principal partition regular subclass.

partition regular classes:

Definition 9.6.3. Given an infinite set X, let $\mathcal{L}_X = \{Y : X \cap Y \text{ is infinite }\}$.◇

In the computability-theoretic realm, many statements of the form "Every set A has an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ satisfying some weakness property" can be rephrased in terms of partition regularity.

Example 9.6.4. The following classes are partition regular:

1. $\{X : \exists Y \in [X]^\omega\ Y \not\geq_T C\}$ for any $C \not\leq_T \emptyset$ (Theorem 3.4.6);
2. $\{X : \exists Y \in [X]^\omega\ Y \text{ is not of PA degree }\}$ (Theorem 5.4.3);

One can think of non-trivial partition regular classes as generalizations of the notion of infinity, satisfying some basic operations that one expects of infinite sets, that is, if a set is infinite, then any superset is again infinite, and when splitting an infinite set in two parts, at least one of the parts is infinite.[20] Looking at the proof of strong cone avoidance of RT^1_2 (Theorem 3.4.6), splitting and finite truncation are the only operations on the reservoir to obtain a good first-jump control. One can therefore fix a partition regular class $\mathcal{P}$ and work with conditions whose reservoir belongs to $\mathcal{P}$.

20: Partition regular classes contain every "typical set". In particular, if $\mathcal{P}$ is partition regular and measurable, then its measure is 1 (see Monin and Patey [86]). Moreover, if $\mathcal{P}$ satisfies the Baire property, then it is co-meager.

Exercise 9.6.5 (Flood [87]). Adapt the proof of Theorem 3.4.6 to show that for every non-computable set C and every set A, there is a set $H \subseteq A$ or $H \subseteq \overline{A}$ such that $C \not\leq_T H$ and $\limsup_{n\to\infty} \frac{|\{1,2,\dots,n\} \cap X|}{n} > 0$. ★

Exercise 9.6.6. Let $\mathcal{P}$ be a non-trivial partition regular class. Show that if $X \in \mathcal{P}$ and $Y =^* X$, then $Y \in \mathcal{P}$. In other words, $\mathcal{P}$ is closed under finite changes. ★

Exercise 9.6.7 (Monin and Patey [86]). Let $\{\mathcal{P}_i\}_{i\in I}$ be an arbitrary union of partition regular classes. Show that $\bigcup_{i \in I} \mathcal{P}_i$ is partition regular. ★

Exercise 9.6.8. Given an infinite set X, let $\mathcal{L}_X = \{Z : Z \cap X \text{ is infinite }\}$. Prove that for every partition regular class $\mathcal{P}$, the following class is partition regular:

$$\{X : \mathcal{L}_X \cap \mathcal{P} \text{ is partition regular }\}$$

Positive and negative information. One can understand the restriction of the reservoirs to partition regular classes in terms of *positive* and *negative* information. In a Mathias condition (σ, X), the stem σ fixes an initial segment of the constructed set G. It specifies that G must contain $\{n : \sigma(n) = 1\}$ and must avoid $\{n : \sigma(n) = 0\}$. Thus, σ forces a finite amount of positive and negative information. On the other hand, the reservoir X forces an infinite amount of negative information since G must avoid any new element outside the reservoir, but does not force any positive information, as for every $n \in X$, one can construct a set G such that $n \notin G$.

It is useful to think as a Σ^0_1 property as a positive information and therefore a Π^0_1 property as a negative one. When constructing a set using a variant of Mathias forcing with the first-jump control, one usually increases the stem to force Σ^0_1 properties, and decrease the reservoir to force Π^0_1 properties. The situation becomes more complicated when forcing Π^0_2 properties $\forall x \psi(G, x)$, as it becomes a density statement about a countable collection of Σ^0_1 properties $\{\psi(G,x) : x \in \mathbb{N}\}$. It therefore requires to maintain some positive information over all future conditions. A partition regular class is therefore a "reservoir of reservoirs", as it restricts the possible choices of reservoirs, hence restricts the future negative information, which is a way of forcing positive information.

9.6.1 Largeness

One should expect from a notion of largeness that it is upward-closed under inclusion, that is, if $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is a largeness notion and $\mathcal{B} \supseteq \mathcal{A}$, then so is $\mathcal{B}$. The collection of all partition regular classes is not closed upward. For instance, pick any non-trivial partition regular class $\mathcal{P}$ which does not contain some infinite set X. Then the $\mathcal{P} \cup \{Z : Z \supseteq X\}$ is an upward-closed superset of $\mathcal{P}$, but is not partition regular. The following notion of largeness is more convenient to work with:

21: Note that a large class is necessarily non-empty, as $\mathbb{N} \in \mathcal{A}$.

Definition 9.6.9. A class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *large*[21] if

1. For all $X \in \mathcal{A}$ and $Y \supseteq X$, $Y \in \mathcal{A}$;
2. For every $k \in \mathbb{N}$ and every k-cover $Y_0 \cup \cdots \cup Y_{k-1} = \mathbb{N}$, there is some $i < k$ such that $Y_i \in \mathcal{A}$. ◊

There exists a formal relationship between largeness and partition regularity: a class is large iff it contains a partition regular subclass. The union of a family of partition regular classes being again partition regular, every large class contains a maximal partition regular subclass for inclusion. This subclass admits the following explicit syntactic definition.

Proposition 9.6.10 (Monin and Patey [31]). Given a large class $\mathcal{A} \subseteq 2^{\mathbb{N}}$, the class

$$\mathcal{L}(\mathcal{A}) = \{X \in 2^{\mathbb{N}} : \forall k \forall X_0 \cup \cdots \cup X_{k-1} \supseteq X \ \exists i < k \ X_i \in \mathcal{A}\}$$

is the maximal partition regular subclass of $\mathcal{A}$. ★

Proof. We first prove that $\mathcal{L}(\mathcal{A})$ is a partition regular subclass of $\mathcal{A}$. First, note that $\mathcal{L}(\mathcal{A})$ is upward-closed. Moreover, by definition of $\mathcal{A}$ being large, $\mathbb{N} \in \mathcal{L}(\mathcal{A})$, so $\mathcal{L}(\mathcal{A})$ is non-empty. Let $X \in \mathcal{L}(\mathcal{A})$ and $X_0 \cup \cdots \cup X_{k-1} \supseteq X$. Suppose for the contradiction that $X_i \notin \mathcal{L}(\mathcal{A})$ for every $i < k$. Then, for every $i < k$, there is some $k_i \in \mathbb{N}$ and some k_i-cover $Y_i^0 \cup \cdots \cup Y_i^{k_i-1} \supseteq X_i$ such that $Y_i^j \notin \mathcal{A}$ for every $j < k_i$. Then $\{Y_i^j : i < k, j < k_i\}$ is a cover of X contradicting $X \in \mathcal{L}(\mathcal{A})$. Therefore, $\mathcal{L}(\mathcal{A})$ is partition regular. Moreover, $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ as witnessed by taking the trivial cover of X by itself.

We now prove that $\mathcal{L}(\mathcal{A})$ is the maximal partition regular subclass of $\mathcal{A}$. Let $\mathcal{B}$ be a partition regular subclass of $\mathcal{A}$. Then for every $X \in \mathcal{B}$, every $X_0 \cup \cdots \cup X_{k-1} \supseteq X$, there is some $i < k$ such that $X_i \in \mathcal{B} \subseteq \mathcal{A}$. Thus $X \in \mathcal{L}(\mathcal{A})$, so $\mathcal{B} \subseteq \mathcal{L}(\mathcal{A})$. ■

Recall that a class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *non-trivial* if it contains only sets with at least two elements. Note that contrary to partition regular classes, a non-trivial large class may contain finite sets, but its maximal partition regular subclass $\mathcal{L}(\mathcal{A})$ contains only infinite sets.

Exercise 9.6.11 (Monin and Patey [86] ; Mimouni).

1. Show that if $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is a non-trivial partition regular class and $X \in \mathcal{P}$, then $\mathcal{P} \cap \mathcal{L}_X$ is large.
2. Construct a non-trivial partition regular class $\mathcal{P}$ and a set $X \in \mathcal{P}$ such that $\mathcal{P} \cap \mathcal{L}_X$ is not partition regular. ⋆

Exercise 9.6.12 (Monin and Patey [86]). Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a non-trivial large class. Show that $\mathcal{L}(\mathcal{A}) = \{X : \mathcal{A} \cap \mathcal{L}_X \text{ is large }\}$. ⋆

Exercise 9.6.13 (Monin and Patey [31]). Show that if $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence of large classes, then $\bigcap_n \mathcal{A}_n$ is large. ⋆

Exercise 9.6.14. Consider the following relations[22] between a set $X \subseteq \mathbb{N}$ and a non-trivial large class $\mathcal{A} \subseteq 2^{\mathbb{N}}$.

(1) $X \in \mathcal{A}$
(2) $X \in \mathcal{L}(\mathcal{A})$
(3) $\mathcal{A} \cap \mathcal{L}_X$ is large
(4) $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}_X$
(5) $\overline{X} \notin \mathcal{A}$

1. What are the implications between these relations? Which ones are strict?
2. When fixing $\mathcal{A}$, these relations induces classes of sets. Which ones are large? partition regular? ⋆

22: Monin and Patey [86] defined another relation, called *partition genericity*. Although arguably less natural, it can be appropriate when considering non-effective constructions.

9.6.2 Effective classes

The class of all infinite sets is Π^0_2. Actually, this is the first level of the effective Borel hierarchy containing a non-trivial partition regular class, as there is no non-trivial $\mathbf{\Sigma}^0_2$ partition regular class [86].[23] Moreover, Π^0_2 classes is the first level satisfying some stability, in the sense that if a Σ^0_1 class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is large, then $\mathcal{L}(\mathcal{A})$ is Π^0_2, while if $\mathcal{A}$ is Π^0_2, then so is $\mathcal{L}(\mathcal{A})$. Actually, we shall work with a slightly more general family of partition regular classes: arbitrary intersections of Σ^0_1 classes over a Scott ideal.

23: We write boldface $\mathbf{\Sigma^0_n}$ for the levels of the Borel hierarchy, and lightface Σ^0_n for the levels of its effective hierarchy.

In what follows, fix a uniform sequence of all c.e. sets of strings $W_0, W_1, \dots \subseteq 2^{<\mathbb{N}}$. It induces an enumeration of all upward-closed Σ^0_1 classes $\mathcal{U}_0, \mathcal{U}_1, \dots$ defined by $\mathcal{U}_e = \{X \in 2^{\mathbb{N}} : \exists \rho \in W_e\ \rho \subseteq X\}$. These enumerations admit immediate relativizations to oracles. We therefore let $\mathcal{U}^Z_0, \mathcal{U}^Z_1, \dots$ be an enumeration of all upward-closed $\Sigma^0_1(Z)$ classes. From now on, fix a Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$ with Scott code M.[24] Given a set $C \subseteq \mathbb{N}^2$, we let

$$\mathcal{U}^{\mathcal{M}}_C = \bigcap_{(e,i) \in C} \mathcal{U}^{Z_i}_e$$

24: Recall that a Scott ideal is a Turing ideal which satisfies weak König's lemma, that is, for every infinite binary tree $T \in \mathcal{M}$, then $[T] \cap \mathcal{M} \neq \emptyset$. A *Scott code* for a Turing ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$ is a set $M = \bigoplus_i Z_i$ such that the basic operations on the M-indices are computable.

From now on, we shall work exclusively with classes of the form $\mathcal{U}^{\mathcal{M}}_C$, and give a particular focus on the complexity of the set C of indices. Thanks to Exercise 9.6.13, if $\mathcal{U}^{\mathcal{M}}_C$ is not large, then there is a finite set $F \subseteq C$ such that $\mathcal{U}^{\mathcal{M}}_F$ is not large either. Note that the latter class is $\Sigma^0_1(\mathcal{M})$. This pseudo-compactness phenomenon plays a key role in the computability-theoretic features of large classes.

Lemma 9.6.15 (Monin and Patey [81]). Let $C \subseteq \mathbb{N}^2$ be a set. The statement "$\mathcal{U}_C^{\mathcal{M}}$ is large" is $\Pi_1^0(C \oplus M')$ uniformly in C and M. ⋆

Proof. Let us first show that the statement "$\mathcal{U}_e^Z$ is large" is $\Pi_2^0(Z)$ uniformly in e and Z. Indeed, by compactness, $\mathcal{U}_e^Z$ is large iff for every $k \in \mathbb{N}$, there is some $\ell \in \mathbb{N}$ such that for every k-partition $Y_0 \cup \cdots \cup Y_{k-1} = \{0, \dots, \ell\}$, there is some $i < k$ and some $\rho \in W_e$ such that $\rho \subseteq Y_i$. This statement is $\Pi_2^0(Z)$ uniformly in e and Z. Then, by Exercise 9.6.13, $\mathcal{U}_C^{\mathcal{M}}$ is large iff for every finite set $F \subseteq C$, $\mathcal{U}_F^{\mathcal{M}}$ is large. The resulting statement is therefore $\Pi_1^0(C \oplus M')$. ■

The following lemma shows that classes of the form $\mathcal{U}_C^{\mathcal{M}}$ are robust, in the sense that if a large class is of this form, then so is its maximum partition regular subclass. Moreover, the translation of the index sets is computable.

Lemma 9.6.16 (Monin and Patey [81]). Let $C \subseteq \mathbb{N}^2$ be a set. Then there exists a set $D \subseteq \mathbb{N}^2$ computable uniformly in C such that $\mathcal{U}_D^{\mathcal{M}} = \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. ⋆

Proof. We first claim that $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$. Indeed, for some finite $F \subseteq C$, $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{U}_F^{\mathcal{M}}$, so $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$ is a partition regular subclass of $\mathcal{U}_F^{\mathcal{M}}$. By maximality of $\mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$, we have $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$. Since it is the case for every $F \subseteq_{\mathtt{fin}} C$, we have $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) \subseteq \bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$.

We next claim that $\bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}}) \subseteq \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. Suppose that $X \notin \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. Then there is some k and some k-cover $Y_0 \cup \cdots \cup Y_{k-1} = X$ such that for every $i < k$, $Y_i \notin \mathcal{U}_C^{\mathcal{M}}$. Then there is a finite set $F \subseteq_{\mathtt{fin}} C$ such that for every $i < k$, $Y_i \notin \mathcal{U}_F^{\mathcal{M}}$, so $X \notin \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$. This proves our claim.

For every $F \subseteq_{\mathtt{fin}} C$, let $h(F)$ be an M-index of the set $\bigoplus_{(e,i) \in F} Z_i$. For every $F \subseteq_{\mathtt{fin}} C$ and $k \in \mathbb{N}$, let $g(F, k)$ be an index of the $Z_{h(F)}$-c.e. set of all $\rho \in 2^{<\mathbb{N}}$ such that for every k-partition $\rho_0 \cup \cdots \cup \rho_{k-1} = \rho$, there is some $i < k$ such that for each $(e, i) \in F$, $W_e^{Z_i}$ enumerates a subset of ρ_i. In other words,

$$\mathcal{U}_{g(F,k)}^{Z_{h(F)}} = \{X : \forall Y_0 \cup \cdots \cup Y_{k-1} = X \ \exists i < k \ Y_i \in \mathcal{U}_F^{\mathcal{M}}\}$$

Then, letting $D = \{(g(F, k), h(F)) : k \in \mathbb{N}, F \subseteq_{\mathtt{fin}} C\}$, the class $\mathcal{U}_D^{\mathcal{M}}$ equals $\bigcap_{F \subseteq_{\mathtt{fin}} C} \mathcal{L}(\mathcal{U}_F^{\mathcal{M}})$, which is nothing but $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. ■

Exercise 9.6.17 (Monin and Patey [86]). Let $\mathcal{P}$ be a Π_2^0 large class and X be co-hyperimmune. Show that $X \in \mathcal{P}$. ⋆

9.6.3 $\mathcal{M}$-minimal classes

As mentioned above, to obtain a variant of Mathias forcing with a good second-jump control, one needs to maintain some positive information over all the reservoirs. This is achieved by restricting the reservoirs to a fixed partition regular class. Given the computability-theoretic nature of the $\Sigma_2^0(G)$ and $\Pi_2^0(G)$ statements needed to be forced, the appropriate partition regular class does not admit a nice explicit combinatorial definition. Seeing a partition regular class as a "reservoir of reservoirs", if $\mathbb{Q} \subseteq \mathcal{P}$ are two partition regular classes, $\mathbb{Q}$ will impose more restrictions on the possible choices of reservoirs than $\mathcal{P}$. Considering a reservoir forces negative information about the set G, $\mathbb{Q}$ will force more positive information than $\mathcal{P}$. With this intuition, minimal partition regular classes will ensure as much positive information as possible, while allowing the reservoirs to be split.

Definition 9.6.18. A large class $\mathcal{A}$ is *$\mathcal{M}$-minimal*[25] if for every set $X \in \mathcal{M}$ and $e \in \mathbb{N}$, either $\mathcal{A} \subseteq \mathcal{U}_e^X$, or $\mathcal{A} \cap \mathcal{U}_e^X$ is not large. ◊

25: This notion of minimality is effective and not combinatorial, in the sense that there might exist large subclasses $\mathcal{B} \subsetneq \mathcal{A}$, but not of the form $\mathcal{U}_C^{\mathcal{M}}$.

Every large class containing a partition regular subclass, every $\mathcal{M}$-minimal large class of the form $\mathcal{U}_C^{\mathcal{M}}$ is also partition regular. There exists a natural greedy algorithm to build a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is non-trivial and $\mathcal{M}$-minimal.

Proposition 9.6.19 (Le Houérou, Levy Patey and Mimouni [83]). Let $D \subseteq \mathbb{N}^2$ be a set such that $\mathcal{U}_D^{\mathcal{M}}$ is large. Then $(D \oplus M')'$ computes a set $C \supseteq D$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-minimal. ⋆

Proof. By the padding lemma, there is a total computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, s \in \mathbb{N}$ and every set X, $\mathcal{U}_{g(e,s)}^X = \mathcal{U}_e^X$ and $g(e,s) > s$. By uniformity of the properties of a Scott code, there is another total computable function $h : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, s \in \mathbb{N}$ and every Scott code M, $h(e,s)$ and e are both M-indices of the same set, and $h(e,s) > s$.

We build a $(D \oplus M')'$-computable sequence of D-computable sets $C_0 \subseteq C_1 \subseteq \dots$ such that, letting $C = \bigcup_s C_s$, $\mathcal{U}_C^M$ is $\mathcal{M}$-minimal and for every s, $C \upharpoonright s = C_s \upharpoonright s$. Start with $C_0 = D$. Then, given a set $C_s \subseteq \mathbb{N}^2$ such that $\mathcal{U}_{C_s}^{\mathcal{M}}$ is large, and a pair (e,i), define $C_{s+1} = C_s \cup \{(g(e,s), h(i,s))\}$ if $\mathcal{U}_{C_s}^{\mathcal{M}} \cap \mathcal{U}_e^{Z_i}$ is large, and $C_{s+1} = C_s$ otherwise. The set $C = \bigcup_s C_s$ is the desired set. Note that by choice of g and h, in the former case, $\mathcal{U}_{C_{s+1}}^{\mathcal{M}} = \mathcal{U}_{C_s}^{\mathcal{M}} \cap \mathcal{U}_e^{Z_i}$. By Lemma 9.6.15, the statement "$\mathcal{U}_{C_s}^{\mathcal{M}} \cap \mathcal{U}_e^{Z_i}$ is large" is $\Pi_1^0(C_s \oplus M')$, so it can be decided $(D \oplus M')'$-computably since $C_s \leq_T D$. The use of g and h ensures that $C_{s+1} \upharpoonright s = C_s \upharpoonright s$. ∎

Suppose M is of low degree by the low basis theorem (Theorem 4.4.6). One can start with a non-trivial class $\mathcal{U}_D^{\mathcal{M}}$ for some computable set D, and apply Proposition 9.6.19 to obtain a $\emptyset''$-computable set $C \supseteq D$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-minimal. However, $\emptyset''$-computability is too complex for our purpose. Thankfully, one does not need to explicitly have access to the set of indices of the $\mathcal{M}$-minimal class, but only to be able to check that a class is "compatible" with it. This yields the notion of $\mathcal{M}$-cohesive class.

9.6.4 $\mathcal{M}$-cohesive classes

In general, if $\mathcal{A}$ and $\mathcal{B}$ are two large classes, then $\mathcal{A} \cap \mathcal{B}$ is not necessarily large. For instance, consider the class $\mathcal{A} = \mathcal{L}_X$ and $\mathcal{B} = \mathcal{L}_{\overline{X}}$ for some bi-infinite set X. Thus, in the algorithm of Proposition 9.6.19, the order in which one considers the pairs (e,i) matters. Therefore, there exist many $\mathcal{M}$-minimal classes of the form $\mathcal{U}_C^{\mathcal{M}}$, depending on the ordering of the pairs. The following notion of $\mathcal{M}$-cohesiveness is a way of choosing an $\mathcal{M}$-minimal class without explicitly giving its set of indices.

Definition 9.6.20. A large class $\mathcal{A}$ is *$\mathcal{M}$-cohesive*[26] if for every set $X \in \mathcal{M}$, either $\mathcal{A} \subseteq \mathcal{L}_X$, or $\mathcal{A} \subseteq \mathcal{L}_{\overline{X}}$. ◊

26: By Le Houérou, Levy Patey and Mimouni [83], for every countable Turing ideal $\mathcal{M}$, there exists a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive but not $\mathcal{M}$-minimal.

This definition may seem out of the blue, so let us start with a few manipulations which will give some intuition.

Exercise 9.6.21. Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be $\mathcal{M}$-cohesive.

1. Show that for every $X \in \mathcal{M}$, $X \in \mathcal{A}$ iff $\mathcal{A} \subseteq \mathcal{L}_X$.
2. Deduce that $\mathcal{A} \cap \mathcal{M}$ is an ultrafilter on $\mathcal{M}$. ★

The following exercise justifies the cohesiveness terminology.

Exercise 9.6.22 (Le Houérou, Levy Patey and Mimouni [83]). Recall that an infinite set H is *cohesive* for a sequence of sets $R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, either $H \subseteq^* R_n$, or $H \subseteq \overline{R}_n$. Show that for every infinite set H cohesive for the Turing ideal $\mathcal{M}$ seen as a sequence of sets, the class $\mathcal{L}_H$ is partition regular and $\mathcal{M}$-cohesive. ★

The following lemma is the most important combinatorial feature of $\mathcal{M}$-cohesive classes. It actually says that an $\mathcal{M}$-cohesive class already contains the information of an $\mathcal{M}$-minimal class, in the sense that in the greedy algorithm of Proposition 9.6.19, the ordering on the pairs does not matter.

Lemma 9.6.23 (Monin and Patey [81]). Let $\mathcal{U}_C^{\mathcal{M}}$ be an $\mathcal{M}$-cohesive class. Let $\mathcal{U}_D^{\mathcal{M}}$ and $\mathcal{U}_E^{M}$ be such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}}$ and $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ are both large. Then so is $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$.[27] ★

27: Note that in this proof, we exploit the fact that all these classes are intersections of $\Sigma^0_1(\mathcal{M})$ classes, and the fact that $\mathcal{M}$ is a Scott ideal.

Proof. Suppose for the contradiction that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ is not large. Then, by Exercise 9.6.13, there are some finite sets $C_1 \subseteq C$, $D_1 \subseteq D$ and $E_1 \subseteq E$ such that $\mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$ is not large. For every $k \in \mathbb{N}$, let $\mathcal{C}_k$ be the collection of all sets $Y_0 \oplus \dots \oplus Y_{k-1}$ such that $Y_0 \sqcup \dots \sqcup Y_{k-1} = \mathbb{N}$ and for every $i < k$, $Y_i \notin \mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$. Note that for every k, $\mathcal{C}_k$ is $\Pi^0_1(\mathcal{M})$ since $\mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$ is $\Sigma^0_1(\mathcal{M})$. Moreover, there is some k such that $\mathcal{C}_k \neq \emptyset$. Since $\mathcal{M}$ is a Scott ideal, there is such a set $Y_0 \oplus \dots \oplus Y_{k-1} \in \mathcal{C}_k \cap \mathcal{M}$. Since $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive, there is some $i < k$ such that $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{Y_i}$. In particular, $Y_i \in \mathcal{U}_C^{\mathcal{M}}$, so either $Y_i \notin \mathcal{U}_D^{\mathcal{M}}$, or $Y_i \notin \mathcal{U}_E^{\mathcal{M}}$. Suppose $Y_i \notin \mathcal{U}_D^{\mathcal{M}}$, as the other case is symmetric. Since $Y_j \cap Y_i = \emptyset$ for every $j \neq i$, then $Y_j \notin \mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{Y_i}$ for every $j \neq i$. It follows that $Y_0, \dots, Y_{k-1}$ witnesses that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}}$ is not large. Contradiction. ■

It follows that every $\mathcal{M}$-cohesive class of the form $\mathcal{U}_C^{\mathcal{M}}$ admits a unique $\mathcal{M}$-minimal large subclass.

Lemma 9.6.24 (Monin and Patey [81]). For every $\mathcal{M}$-cohesive class $\mathcal{U}_C^{\mathcal{M}}$, there exists a unique $\mathcal{M}$-minimal large subclass:

$$\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \bigcap_{e \in \mathbb{N}, X \in \mathcal{M}} \{ \mathcal{U}_e^X : \mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^X \text{ is large } \}$$

Proof. We first prove that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is large. Let $(e_0, X_0), (e_1, X_1), \dots$ be an enumeration of all pairs $(e, X) \in \mathbb{N} \times \mathcal{M}$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^X$ is large. By induction on n, using Lemma 9.6.23, $\bigcap_{i<n} \mathcal{U}_{e_i}^{X_i}$ is large for every n. Thus, by Exercise 9.6.13, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is large. Next, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle \subseteq \mathcal{U}_C^{\mathcal{M}}$ as for every $(e, i) \in C$, $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^{Z_i}$ is trivially large. Last, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is $\mathcal{M}$-minimal by construction. ■

Contrary to $\mathcal{M}$-minimal classes, one can build a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive computably in any PA degree over M'.

Proposition 9.6.25 (Le Houérou, Levy Patey and Mimouni [83]). Let $D \subseteq \mathbb{N}^2$ be a set such that $\mathcal{U}_D^{\mathcal{M}}$ is large and non-trivial. Then any PA degree over $D \oplus M'$ computes a set $C \supseteq D$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive. ⋆

Proof. Fix P a PA degree over $D \oplus M'$.[28] First, consider two M-computable enumerations of sets $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\mathcal{U}_{E_n}^{Z_n} = \mathcal{L}_{Z_n}$ and $\mathcal{U}_{F_n}^{Z_n} = \mathcal{L}_{\overline{Z}_n}$. By the padding lemma, one can suppose that $\min E_n, \min F_n \geq n$. The set C will be defined as $\bigcup_{n\in\mathbb{N}} C_n$ for $C_0 \subseteq C_1 \subseteq \dots$ a P-computable sequence of $M \oplus D$-computable sets satisfying:

- $C_0 = D$,
- $\mathcal{U}_{C_k}^{\mathcal{M}}$ is large for every $k \in \mathbb{N}$,
- $C_k \upharpoonright k = C \upharpoonright k$ for every $k \in \mathbb{N}$, and thus C will be P-computable.

Let $C_0 = D$, then, by assumption, $\mathcal{U}_{C_0}^{\mathcal{M}}$ is large.

Assume C_k has been defined for some $k \in \mathbb{N}$. Then, as $\mathcal{U}_{C_k}^{\mathcal{M}}$ is large, one of the two following $\Pi_1^0(D \oplus M')$ statements must hold: "$\mathcal{U}_{C_k}^{\mathcal{M}} \cap \mathcal{L}_{Z_k}$ is large" or "$\mathcal{U}_{C_k}^{\mathcal{M}} \cap \mathcal{L}_{\overline{Z}_k}$ is large". Hence, P is able to choose one that is true. If $\mathcal{U}_{C_k}^{\mathcal{M}} \cap \mathcal{L}_{Z_k}$ is large, let $C_{k+1} = C_k \cup E_k$, and if $\mathcal{U}_{C_k}^{\mathcal{M}} \cap \mathcal{L}_{\overline{Z}_k}$ is large, let $C_{k+1} = C_k \cup F_k$. By our assumption that $\min E_n, \min F_n \geq n$ for all n, the value of $C_k \upharpoonright k$ will be left unchanged in the rest of the construction. ■

28: Recall that by Exercise 4.6.5, P is able to choose, among two $\Pi_1^0(D \oplus M')$ formulas such that at least one is true, a valid one.

Exercise 9.6.26 (Le Houérou, Levy Patey and Mimouni [83]). Let $\mathcal{U}_C^{\mathcal{M}}$ be an $\mathcal{M}$-cohesive class. Show that $C \oplus M'$ is of PA degree over X' for every $X \in \mathcal{M}$. ⋆

Exercise 9.6.27. Let $\mathcal{M} \subseteq 2^{\mathbb{N}}$ be a Scott ideal coded by a set M of low degree and $C \subseteq \mathbb{N}^2$ be a Δ_2^0 set such that $\mathcal{U}_{\mathcal{M}}^C$ is non-trivial and large. Show that for every computable instance $R_0, R_1, \dots$ of COH with no computable solution, there exists some $n \in \mathbb{N}$ such that $\mathcal{U}_{\mathcal{M}}^C \cap \mathcal{L}_{R_n}$ and $\mathcal{U}_{\mathcal{M}}^C \cap \mathcal{L}_{\overline{R}_n}$ are both large.[29] ⋆

29: Hint: use Exercise 3.4.3 and Exercise 9.6.11.

9.7 Pigeonhole principle

By Jockusch and Dzhafarov's theorem (Theorem 3.4.6), RT^1_2 admits strong cone avoidance, the only sets that can be encoded by all the infinite subsets and co-subsets of an arbitrary set are the computable ones. Using the framework of largeness and partition regularity, we can now prove the counterpart for jump computation, known as strong jump cone avoidance of RT^1_2. It follows that for every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ of non-high degree.

Theorem 9.7.1 (Monin and Patey [31])
Let C be a non-Δ_2^0 set. For every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that C is not $\Delta_2^0(H)$.

Proof. Fix C and A. As in Theorem 3.4.6, we shall construct two sets $G_0 \subseteq A$ and $G_1 \subseteq \overline{A}$ using a disjunctive notion of forcing. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

By the low basis theorem (Theorem 4.4.6) and Theorem 4.3.2, there exists a set M of low degree coding a Scott ideal $\mathcal{M}$. By the cone avoidance basis theorem (Theorem 3.2.6) relativized to $\emptyset'$ and Theorem 4.3.2, there is a code N for a Scott ideal $\mathcal{N}$ containing $\emptyset'$ such that $C \not\leq_T N$. By Proposition 9.6.25, $\mathcal{N}$ contains a set $D \subseteq \mathbb{N}^2$ such that $\mathcal{U}_D^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive class.

Notion of forcing. The two sets G_0 and G_1 will be constructed using a variant of Mathias forcing whose conditions are triples (σ_0, σ_1, X), where

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_i \subseteq A_i$; $X \in \langle \mathcal{U}_D^{\mathcal{M}} \rangle$;
3. $X \in \mathcal{N}$.[30]

30: This notion of forcing ressembles the one of Theorem 3.4.6, with two main differences. First, the reservoir must belong to the $\mathcal{M}$-minimal partition regular subclass of $\mathcal{U}_D^{\mathcal{M}}$, which ensures that it maintains a lot of positive information. Second, one usually requires that the reservoir satisfies the desired property, that is, C is not $\Delta_2^0(X)$. However, because of the forcing question for Σ_2^0 formulas, the reservoir only satisfies that $C \not\leq_T X \oplus D \oplus \emptyset'$. In particular, X can compute $\emptyset'$, or can even be of PA degree over $\emptyset'$.

One must really think of a condition as a pair of Mathias conditions which share a same reservoir. The *interpretation* $[\sigma_0, \sigma_1, X]$ of a condition (σ_0, σ_1, X) is the class

$$[\sigma_0, \sigma_1, X] = \{(G_0, G_1) : \forall i < 2\ \sigma_i \preceq G_i \subseteq \sigma_i \cup X\}$$

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X) for each $i < 2$. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$ defined by $G_{\mathcal{F},i} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$. Note that $(G_{\mathcal{F},0}, G_{\mathcal{F},1}) \in \bigcap\{[\sigma_0, \sigma_1, X] : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$.

The goal is therefore to build two infinite sets G_0, G_1, satisfying the following requirements for every $e_0, e_1 \in \mathbb{N}$:

$$\mathcal{R}_{e_0,e_1} : \Phi_{e_0}^{G_0'} \neq C \vee \Phi_{e_1}^{G_1'} \neq C$$

If every requirement is satisfied, then an easy pairing argument shows that either $C \not\leq_T G_0'$, or $C \not\leq_T G_1'$. However, in general, it is not possible to ensure that G_0 and G_1 are both infinite. For example, A could be finite or co-finite.

Validity. In the proof of Theorem 3.4.6, we used as a hypothesis that there is no set satisfying the statement of the theorem, which implies in particular that for every reservoir X, both $X \cap A$ and $X \cap \overline{A}$ are infinite. In this proof, we will need to consider a stronger property.

Definition 9.7.2. We say that part i of (σ_0, σ_1, X) is *valid* if $X \cap A_i \in \mathcal{U}_D^{\mathcal{M}}$. Part i of a filter $\mathcal{F}$ is *valid* if part i is valid for every condition in $\mathcal{F}$. ◇

Since $X \in \langle \mathcal{U}_D^{\mathcal{M}} \rangle$, then by partition regularity, either $A_0 \cap X$ or $A_1 \cap X$ belongs to $\langle \mathcal{U}_D^{\mathcal{M}} \rangle$. It follows that every condition has at least a valid part.[31] Moreover, if q extends p and part i of q is valid, then so is part i of p. Thus, every filter admits a valid part.

31: Also note that by Exercise 9.6.6, if part i is valid in $p = (\sigma_0, \sigma_1, X)$ and $q = (\tau_0, \tau_1, Y) \leq p$ with $Y =^* X$, then part i is valid in q.

We shall first prove that for every sufficiently generic filter $\mathcal{F}$ with valid part i, not only $G_{\mathcal{F},i}$ is infinite, but it furthermore belongs to $\langle \mathcal{U}_D^{\mathcal{M}} \rangle$.

Lemma 9.7.3. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and let $\mathcal{V} \supseteq \langle \mathcal{U}_D^{\mathcal{M}} \rangle$ be a large $\Sigma_1^0(\mathcal{M})$ class. There is an extension (τ_0, τ_1, Y) of p such that $[\tau_i] \subseteq \mathcal{V}$. ★

Proof. Since part i of p is valid, then $X \cap A_i \in \langle \mathcal{U}_D^{\mathcal{M}} \rangle \subseteq \mathcal{V}$. Moreover, $\mathcal{V}$ is $\Sigma_1^0(\mathcal{M})$, so there is some $\rho \subseteq X \cap A_i$ such that $[\rho] \subseteq \mathcal{V}$. Last, by upward-closure of $\mathcal{V}$, $[\sigma_i \cup \rho] \subseteq \mathcal{V}$, so letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, (τ_0, τ_1, Y) is the desired extension. ■

Forcing question for Σ^0_1-formulas. We now design a forcing question for Σ^0_1 formulas. Note that this forcing question is not Σ^0_1-preserving, and therefore does not yield a good first-jump control. This is due to the fact that the reservoir X is too complex, so the only way to access it is to approximate it by a large class, yielding a $\Pi^0_1(\mathcal{N})$ statement. On the bright side, the forcing question is not disjunctive, and can be applied on every valid part.

> **Definition 9.7.4.** Given a string $\sigma \in 2^{<\mathbb{N}}$ and a Σ^0_1 formula $\varphi(G)$, define $\sigma \mathrel{?{\vdash}} \varphi(G)$ to hold if the following class is large[32]:
>
> $$\mathcal{U}^{\mathcal{M}}_D \cap \{Z : \exists \rho \subseteq Z\ \varphi(\sigma \cup \rho)\}$$

32: Note that this forcing question is not defined over conditions, but over strings. Given a condition (σ_0, σ_1, X), it is intended to be applied on σ_0 or σ_1, depending on which part is valid. Also note that, surprisingly, since the forcing question does not involve the reservoir, its answer only depends on the stem.

By Lemma 9.6.15, the forcing question is $\Pi^0_1(D \oplus M')$ uniformly in σ and φ. Since M is of low degree, $M' \in \mathcal{N}$ and by assumption, $D \in \mathcal{N}$, so the forcing question is $\Pi^0_1(\mathcal{N})$.

Lemma 9.7.5. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and $\varphi(G)$ be a Σ^0_1 formula.

1. If $\sigma_i \mathrel{?{\vdash}} \varphi(G)$, then there is an extension of p forcing $\varphi(G_i)$;
2. If $\sigma_i \mathrel{?{\nvdash}} \varphi(G)$, then there is an extension of p forcing $\neg\varphi(G_i)$. ★

Proof. Let $\mathcal{V} = \{Z : \exists \rho \subseteq Z\ \varphi(\sigma_i \cup \rho)\}$.

Suppose first $\sigma_i \mathrel{?{\vdash}} \varphi(G)$. Then $\mathcal{U}^{\mathcal{M}}_D \cap \mathcal{V}$ is large, so by Lemma 9.6.24, $\langle \mathcal{U}^{\mathcal{M}}_D \rangle \subseteq \mathcal{V}$. Since part i of p is valid, then $A_i \cap X \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle \subseteq \mathcal{V}$. Unfolding the definition of $\mathcal{V}$, there is some $\rho \subseteq A_i \cap X$ such that $\varphi(\sigma_i \cup \rho)$ holds. Letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, (τ_0, τ_1, Y) is an extension forcing $\varphi(G_i)$.

Suppose now $\sigma_i \mathrel{?{\nvdash}} \varphi(G)$. Then $\mathcal{U}^{\mathcal{M}}_D \cap \mathcal{V}$ is not large, so by Exercise 9.6.13, there is a finite set $F \subseteq D$ such that $\mathcal{U}^{\mathcal{M}}_F \cap \mathcal{V}$ is not large. For every k, let $\mathcal{C}_k$ be the $\Pi^0_1(\mathcal{M})$ class of all sets $Z_0 \oplus \dots \oplus Z_{k-1}$ such that $Z_0 \cup \dots \cup Z_{k-1} = \mathbb{N}$ and for every $j < k$, $Z_i \notin \mathcal{U}^{\mathcal{M}}_F \cap \mathcal{V}$. By assumption, $\mathcal{C}_k \neq \emptyset$ for some $k \in \mathbb{N}$, so since $\mathcal{M}$ is a Scott ideal, there is such a set $Z_0 \oplus \dots \oplus Z_{k-1}$ in $\mathcal{C}_k \cap \mathcal{M}$. By partition regularity of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$, there is some $j < k$ such that $X \cap Z_j \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle$. In particular, $Z_j \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle \subseteq \mathcal{U}^{\mathcal{M}}_F$ so $Z_j \notin \mathcal{V}$. Letting $Y = X \cap Z_j$, $q = (\sigma_0, \sigma_1, Y)$ is an extension such that for every $\rho \subseteq Y$, $\neg\varphi(\sigma_i \cup \rho)$ holds. It follows that q forces $\neg\varphi(G_i)$. ■

Syntactic forcing relation. We now turn to second-jump control. The forcing relation for Σ^0_1, Π^0_1 and Σ^0_2 formulas is the usual one. It will be convenient to work with the following syntactic forcing relation for Π^0_2 formulas.

> **Definition 9.7.6.** Let $p = (\sigma_0, \sigma_1, X)$ be a condition, $i < 2$ be a part and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. Let $p \Vdash \varphi(G_i)$ hold if for every $\rho \subseteq X$ and every $x \in \mathbb{N}$, $\sigma_i \cup \rho \mathrel{?{\vdash}} \psi(G, x)$.[33] ◇

33: Assuming the forcing question for Σ^0_1 formulas meets its specification, this forcing relation says that for every x and every future extension of the stem, there will be an extension forcing $\psi(G_i, x)$. Thus, this forcing question states, for each x, the density below p of the set of conditions forcing $\psi(G_i, x)$. Since the forcing question for Σ^0_1 formulas meets its specification on valid parts, then this syntactic forcing relation implies the true forcing relation one the parts which remain valid in the future.

One easily proves that this syntactic forcing relation is closed under condition extension. The following lemma states that, for every sufficiently generic filter $\mathcal{F}$ with valid part i, if $p \Vdash \varphi(G_i)$ for some $p \in \mathcal{F}$, then p forces $\varphi(G_i)$.

Lemma 9.7.7. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and $\varphi(G) \equiv$

$\forall x\psi(G,x)$ be a Π^0_2 formula. If $p \Vdash \varphi(G_i)$, then for every $x \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\psi(G_i, x)$. ★

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G_i)$, then in particular, for $\rho = \emptyset$, $\sigma_i \operatorname{?\vdash} \psi(G,x)$. By Lemma 9.7.5, there is an extension of p forcing $\psi(G_i, x)$. ■

Disjunctive forcing question for Σ^0_2-formulas. The notion of forcing admits a Σ^0_2-preserving disjunctive forcing question for Σ^0_2 formulas, but which satisfies its specification only if *both parts* of the condition are valid.

Definition 9.7.8. Given a condition $p = (\sigma_0, \sigma_1, X)$ and a pair of Σ^0_2 formulas $\varphi_0(G)$ and $\varphi_1(G)$, with $\varphi_i(G) \equiv \exists x\psi_i(G,x)$, define $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ to hold if for every 2-partition $Z_0 \cup Z_1 = X$, there is some $i < 2$, some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \operatorname{?\vdash} \psi_i(G,x)$.[34] ◊

34: As usual, the formula ψ_i being Π^0_1, we use here the forcing question for Π^0_1 formulas obtained by taking the negation of the forcing question for Σ^0_1 formulas.

By compactness, this forcing question holds iff there is a level $\ell \in \mathbb{N}$ such that for every 2-partition $Z_0 \cup Z_1 = X\upharpoonright_\ell$, there is some $i < 2$, some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \operatorname{?\vdash} \psi_i(G,x)$. The formula $\sigma_i \cup \rho \operatorname{?\vdash} \psi_i(G,x)$ is $\Sigma^0_1(\mathcal{N})$ uniformly in σ_i, ρ and ψ_i, thus the overall forcing question is $\Sigma^0_1(\mathcal{N})$ uniformly in p, φ_0 and φ_1.

Lemma 9.7.9. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with both valid parts and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_1 formulas.

1. If $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension of p forcing $\varphi(G_i)$ for some $i < 2$;
2. If $p \operatorname{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension q of p with $q \Vdash \neg\varphi(G_i)$ for some $i < 2$. ★

Proof. Say $\varphi_i(G) \equiv \exists x\psi_i(G,x)$.

Suppose first $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$. Then, letting $Z_0 = X \cap A_0$ and $Z_1 = X \cap A_1$, there is some $i < 2$, some $x \in \mathbb{N}$ and some $\rho \subseteq X \cap A_i$ such that $\sigma_i \cup \rho \operatorname{?\vdash} \psi_i(G,x)$. In particular, letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, $q = (\tau_0, \tau_1, Y)$ is an extension such that both parts are valid. By Lemma 9.7.5, there is an extension of q forcing $\psi_i(G_i, x)$, hence forcing $\varphi(G_i)$.

Suppose now $p \operatorname{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$. Let $\mathscr{C}$ be the $\Pi^0_1(\mathcal{N})$ class of all Z such that, letting $Z_0 = Z$ and $Z_1 = \overline{Z}$, for every $i < 2$, every $x \in \mathbb{N}$, and every $\rho \subseteq X \cap Z_i$, $\sigma_i \cup \rho \operatorname{?\nvdash} \psi_i(G,x)$. Since $\mathcal{N}$ is a Scott ideal, there is such a set $Z \in \mathscr{C} \cap \mathcal{N}$. By partition regularity of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$, there is some $i < 2$ such that $X \cap Z_i \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle$. The condition $q = (\sigma_0, \sigma_1, X \cap Z_i)$ is an extension of p such that $q \Vdash \neg\varphi_i(G_i)$. ■

Degenerate forcing question. In most cases, for sufficiently Cohen generic or sufficiently random sets A, both parts of every conditions will be valid. Unfortunately, in some degenerate cases, there might be some condition $p = (\sigma_0, \sigma_1, X)$ with only one valid part, say part 0, and the disjunctive forcing question may not work because it would yield an extension deciding the formula on part 1. In this case, for every extension of p, part 1 will stay invalid, and part 0 will be valid. We will therefore make a degenerate construction in the valid part.

If some part of a condition is not valid, then it is witnessed by a large $\Sigma^0_1(\mathcal{M})$ superclass of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$ in the following sense.

Definition 9.7.10. A *witness of invalidity* of part i of a condition $p = (\sigma_0, \sigma_1, X)$ is a $\Sigma^0_1(\mathcal{M})$ large class $\mathcal{V} \supseteq \langle \mathcal{U}^{\mathcal{M}}_D \rangle$ such that $X \cap A_i \notin \mathcal{V}$. ◇

If part i of p is not valid, then by definition, $X \cap A_i \notin \langle \mathcal{U}^{\mathcal{M}}_D \rangle$, so by Lemma 9.6.24, there is some $\Sigma^0_1(\mathcal{M})$ class $\mathcal{V}$ such that $X \cap A_i \notin \mathcal{V}$. Thus, every invalid part admits a witness of invalidity. One can exploit this witness to design a non-disjunctive forcing question for Σ^0_2 formulas on the valid part with the good definitional properties.

Definition 9.7.11. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with witness of invalidity $\mathcal{V}$ on part $1-i$, and let $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Define $p \operatorname{?\vdash}^{\mathcal{V}} \varphi(G_i)$ to hold if for every 2-partition $Z_0 \sqcup Z_1 = X$ such that $Z_{1-i} \notin \mathcal{V}$, there is some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \operatorname{?\vdash} \psi_i(G, x)$. ◇

Again, by compactness, this degenerate forcing question is $\Sigma^0_1(\mathcal{N})$. The following lemma shows that this forcing question meets its specification.

Lemma 9.7.12. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with witness of invalidity $\mathcal{V}$ on part $1-i$, and let $\varphi(G)$ be a Σ^0_2 formula.

1. If $p \operatorname{?\vdash}^{\mathcal{V}} \varphi(G_i)$, then there is an extension of p forcing $\varphi(G_i)$.
2. If $p \operatorname{?\nvdash}^{\mathcal{V}} \varphi(G_i)$, then there is an extension $q \leq p$ such that $q \Vdash \neg\varphi(G_i)$.
★

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$.

Suppose first $p \operatorname{?\vdash}^{\mathcal{V}} \varphi(G_i)$. In particular, for $Z_0 = A_0 \cap X$ and $Z_1 = A_1 \cap X$, there is some $x \in \mathbb{N}$ and some $\rho \subseteq A_i \cap X$ such that $\sigma_i \cup \rho \operatorname{?\vdash} \psi_i(G, x)$. Letting $\tau_i = \sigma_i \cup \rho$, $\tau_{1-i} = \sigma_{1-i}$ and $Y = X \setminus \{0, \dots, |\rho|\}$, $q = (\tau_0, \tau_1, Y)$ is an extension such that part $1-i$ is invalid, hence part i is valid. By Lemma 9.7.5, there is an extension of q forcing $\psi_i(G_i, x)$, hence forcing $\varphi(G_i)$.

Suppose now $p \operatorname{?\nvdash}^{\mathcal{V}} \varphi(G_i)$. Let $\mathcal{C}$ be the $\Pi^0_1(\mathcal{N})$ class of all Z such that, letting $Z_0 = Z$ and $Z_1 = \overline{Z}$, then $Z_{1-i} \notin \mathcal{V}$ and for every $x \in \mathbb{N}$, and every $\rho \subseteq X \cap Z_i$, $\sigma_i \cup \rho \operatorname{?\nvdash} \psi_i(G, x)$. Since $\mathcal{N}$ is a Scott ideal, there is such a set $Z \in \mathcal{C} \cap \mathcal{N}$. By partition regularity of $\langle \mathcal{U}^{\mathcal{M}}_D \rangle$, since $X \cap Z_{1-i} \notin \mathcal{V} \supseteq \langle \mathcal{U}^{\mathcal{M}}_D \rangle$, then $X \cap Z_i \in \langle \mathcal{U}^{\mathcal{M}}_D \rangle$. The condition $q = (\sigma_0, \sigma_1, X \cap Z_i)$ is an extension of p such that $q \Vdash \neg\varphi_i(G_i)$. ■

We are now ready to prove Theorem 9.7.1.

Suppose first there is a condition p with some invalid part $1-i$. Let $\mathcal{F}$ be a sufficiently generic filter containing p and let $G_i = G_{\mathcal{F},i}$. Then part i is valid in $\mathcal{F}$. By Lemma 9.7.7, the syntactic forcing relation for Π^0_2 formulas implies the true forcing relation on part i. By Lemma 9.7.12 and by adapting Theorem 9.3.5, for every Turing functional Φ_e, there is some condition $q \in \mathcal{F}$ forcing $\Phi_e^{G_i'} \neq C$, so C is not $\Delta^0_2(G_i)$.

Suppose now that for every condition, both parts are valid. Let $\mathcal{F}$ be a sufficiently generic filter, and let $G_i = G_{\mathcal{F},i}$ for $i < 2$. By Lemma 9.7.7, the syntactic forcing relation for Π^0_2 formulas implies the true forcing relation on both parts. By Lemma 9.7.9 and by adapting Theorem 9.3.5, for every pair of Turing functionals Φ_{e_0}, Φ_{e_1}, there is some condition $q \in \mathcal{F}$ forcing $\Phi_{e_0}^{G_0'} \neq C \vee \Phi_{e_1}^{G_1'} \neq C$. By a pairing argument, there is some $i < 2$ such that C is not $\Delta^0_2(G_i)$. This completes the proof of Theorem 9.7.1. ■

Exercise 9.7.13 (Monin and Patey [31]). Let $f : \mathbb{N} \to \mathbb{N}$ be $\emptyset'$-hyperimmune. Adapt the proof of Theorem 9.7.1 and Theorem 3.6.4 to show that for every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that f is H'-hyperimmune. ★

Jump compactness avoidance 10

Jump compactness avoidance combines the complexity of two orthogonal problematics, namely, second-jump control and compactness avoidance. As one shall expect, from a purely abstract viewpoint, it can be reduced to the design of a forcing question for Σ^0_2 formulas with the appropriate merging properties. However, in real world applications, such as variants of Mathias forcing in reverse mathematics, both techniques do not necessarily combine well, adding an extra layer of complexity.

Prerequisites: Chapters 2 to 5 and 9

10.1 Context and motivation

Jump PA avoidance plays a particularly important role in reverse mathematics, due to its connections with the cohesiveness principle. Recall from Section 3.4 that an infinite set $C \subseteq \mathbb{N}$ is *cohesive* for a sequence of sets $\vec{R} = R_0, R_1, \dots$ if for every $n \in \mathbb{N}$, $C \subseteq^* R_n$ or $C \subseteq^* \overline{R}_n$, where $\subseteq^*$ means "included up to finite changes". The *cohesiveness principle* is the problem COH whose instances are infinite sequences of sets, and whose solutions are infinite cohesive sets.

As mentioned in Chapter 9, COH should be considered as a statement about jump computation, as it is computably equivalent[1] to the statement "For every Δ^0_2 infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, there is a Δ^0_2-approximation of an infinite path." There exists a uniformly computable sequence of sets[2] such that the degrees of its cohesive sets are exactly those whose jump is PA over $\emptyset'$. Such an instance is *maximal*, in the sense that every solution to this instance compute a solution to every other computable instance. Moreover, for every set P of PA degree over $\emptyset'$, there exists an ω-model $\mathcal{M}$ of COH such that for every $X \in \mathcal{M}$, $X' \leq_T P$. Therefore, separating a problem from COH over ω-models can be reduced without loss of generality to jump PA avoidance.

1: This equivalence also holds over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, but not RCA_0 alone. Indeed, RCA_0 + COH is Π^1_1-conservative over RCA_0 (Exercise 7.3.14), while by Fiori-Carones et al. [62, Proposition 4.4], the other statement implies $\mathsf{B}\Sigma^0_2$ over RCA_0.

2: Actually, it suffices to consider the sequence of all primitive recursive sets.

Definition 10.1.1. A problem P admits *jump PA avoidance*[3] if for every pair of sets Z and $D \leq_T Z$ such that Z' is not of PA degree over D', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)'$ is not of PA degree over D'.[4] ◇

3: As usual, the unrelativized formulation with $Z = D = \emptyset$ is far more natural, but does not behave well with artificial problems.

4: One can also define the notion of strong jump PA avoidance, by considering arbitrary instances of P instead of Z-computable ones.

The cohesiveness principle can be considered as a sequential version of the pigeonhole principle. An instance is a countable sequences of instances of RT^1_2, that is, a countable sequence of sets $R_0, R_1, \dots$, and a solution is a single set which is, up to finite changes, a solution to every R_n. One can define a similar statement capturing the degrees whose jump are DNC over $\emptyset'$, in terms of the *thin set theorem*. The thin set theorem for n-tuples (TS^n) is a statement introduced by Friedman, whose instances are colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite sets $H \subseteq \mathbb{N}$ such that $f[H]^n \neq \mathbb{N}$. Such sets are called *f-thin*.

Exercise 10.1.2 (Patey [88]). Given a uniformly computable sequence $\vec{g} = g_0, g_1, \dots$ of functions of type $\mathbb{N} \to \mathbb{N}$, an infinite set $C \subseteq \mathbb{N}$ is *thin $\vec{g}$-cohesive* if for every $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $C \setminus [0, k]$ is g_n-thin.

1. Let $\vec{f} = f_0, f_1, \ldots$ be the sequence of all primitive recursive functions of type $\mathbb{N} \to \mathbb{N}$. Show that for every infinite thin $\vec{f}$-cohesive set C, C' is of DNC degree over $\emptyset'$.
2. Let $\vec{g} = g_0, g_1, \ldots$ be a uniformly computable sequence of functions of type $\mathbb{N} \to \mathbb{N}$ and D be a set whose jump is of DNC degree over $\emptyset'$. Show that D computes an infinite thin $\vec{g}$-cohesive set. ⋆

The degrees whose jump are DNC over $\emptyset'$ received less attention than their PA counterpart, but can be used to prove separations over another well-known statement: the rainbow Ramsey theorem for pairs. A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is k-*bounded* if for every $c \in \mathbb{N}$, $f^{-1}(c)$ has size at most k. A set $H \subseteq \mathbb{N}$ is an f-*rainbow* if f is injective on $[H]^n$, that is, each color is used at most once. The *rainbow Ramsey theorem* for n-tuples and k-bounded colorings (RRT^n_k) is the problem whose instances are k-bounded colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, and whose solutions are infinite f-rainbows.

Exercise 10.1.3 (Miller). Construct a computable 2-bounded coloring $f : [\mathbb{N}]^2 \to \mathbb{N}$ such that for every $\emptyset'$-c.e. set $W^{\emptyset'}_e$, if $\operatorname{card} W^{\emptyset'}_e \geq 2e+2$, then $W^{\emptyset'}_e$ is not extendible into an infinite f-rainbow. Deduce that every infinite f-rainbow is of DNC degree over $\emptyset'$.[5] ⋆

5: This uses the characterization of DNC degrees in terms of effectively immune functions. See Section 6.2 for more details. Miller actually proved a reversal: for every computable k-bounded coloring $f : [\mathbb{N}]^2 \to \mathbb{N}$, every DNC function over $\emptyset'$ computes an infinite f-rainbow.

It follows that if a problem P admits jump DNC avoidance in the following sense, then there is an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ which is not a model of RRT^2_2.

Definition 10.1.4. A problem P admits *jump DNC avoidance* if for every pair of sets Z and $D \leq_T Z$ such that Z' is not of DNC degree over D', every Z-computable instance X of P admits a solution Y such that $(Y \oplus Z)'$ is not of DNC degree over D'. ◇

10.2 Jump PA avoidance

As explained, the pure theory of jump compactness avoidance is a simple adaptation of the techniques of compactness avoidance to Σ^0_2 formulas. In this section, we give two examples with Cohen genericity and tree forcing for the sake of concreteness, and then state the general abstract theorem, leaving its proof as an exercise.

Theorem 10.2.1
For every sufficiently Cohen generic set G, G' is not of PA degree over $\emptyset'$.

Proof. Consider Cohen forcing, that is, the set $2^{<\mathbb{N}}$ of binary strings, partially ordered by the prefix relation. We defined in Section 9.3 a forcing question for Σ^0_2 formulas.

Definition 10.2.2. Let σ be a Cohen condition, and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Define $\sigma \operatorname{?\vdash} \varphi(G)$ to hold if there exists some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that τ strongly forces $\psi(G, x)$, that is, for every $\rho \geq \tau$, $\psi(\rho, x)$ holds. ◇

This forcing question satisfies a strong version of its specifications, that is, if $\sigma \operatorname{?\vdash} \varphi(G)$ does not hold, then σ itself already forces $\neg\varphi(G)$. It follows that, given two Σ^0_2-formulas $\varphi_0(G)$ and $\varphi_1(G)$, if none of $\sigma \operatorname{?\vdash} \varphi_i(G)$ holds, then σ forces $\neg\varphi_0(G) \wedge \neg\varphi_1(G)$. This property is exploited in the following lemma:

Lemma 10.2.3. For every condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index $e \in \mathbb{N}$, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$.[6] ★

6: Recall that a degree is PA iff it computes a $\{0,1\}$-valued DNC function. This equivalence also holds relative to any oracle.

Proof. Consider the following set:

$$U = \{(x,v) \in \mathbb{N} \times 2 : \sigma \mathrel{?\vdash} \Phi_e^{G'}(x)\downarrow = v\}$$

Since the forcing question is Σ_2^0-preserving, the set U is Σ_2^0. There are three cases:

- Case 1: $(x, \Phi_x^{\emptyset'}(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x^{\emptyset'}(x)\downarrow$. By Property (1) of the forcing question, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G'}(x)\downarrow = \Phi_x^{\emptyset'}(x)$.
- Case 2: there is some $x \in \mathbb{N}$ such that $(x,0),(x,1) \notin U$. Then σ already forces $\neg(\Phi_e^{G'}(x)\downarrow = 0)$, $\neg(\Phi_e^{G'}(x)\downarrow = 1)$, so σ forces $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_2^0 graph of a $\{0,1\}$-valued DNC function over $\emptyset'$. This contradicts the fact that $\mathbf{0}'$ is not PA over $\emptyset'$. ■

We are now ready to prove Theorem 10.2.1. Given $e \in \mathbb{N}$, let $\mathcal{D}_e$ be the set of all conditions $\sigma \in 2^{<\mathbb{N}}$ forcing $\Phi_e^{G'}$ not to be a $\{0,1\}$-valued DNC function over $\emptyset'$. It follows from Lemma 10.2.3 that every $\mathcal{D}_e$ is dense, hence every sufficiently generic filter $\mathcal{F}$ is $\{\mathcal{D}_e : e \in \mathbb{N}\}$-generic, so $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$. This completes the proof of Theorem 10.2.1. ■

If a problem P admits a low basis, then it admits jump PA avoidance. Thus, by the low basis theorem for Π_1^0 classes (Theorem 4.4.6), there exists a PA degree which is low, hence whose jump is not PA over $\emptyset'$. More generally, as explained in Section 9.2, it is preferable to use an effective first-jump construction rather than a second-jump one when available, as the former usually involves a simpler machinery.

Although WKL admits a low basis, it is sometimes necessary to use a forcing construction with a second-jump control, when trying for example to preserve a first-jump and second-jump property simultaneously, as it was the case for Theorem 9.4.1. We now prove that WKL can simultaneously avoid a cone, and have a jump of non-PA degree over $\emptyset'$.

Theorem 10.2.4
Let C be a non-computable set. For every non-empty Π_1^0 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists a member $G \in \mathcal{P}$ such that $C \not\leq_T G$ and G' is not of PA degree over $\emptyset'$.

Proof. The proof is an adaptation of Theorem 9.4.1, using the same notion of forcing and the same forcing question. More precisely, we use a restriction of the Jockusch-Soare forcing to infinite *primitive recursive* binary trees, partially ordered by the inclusion relation. By Lemma 9.4.2, every Π_1^0 class in $2^{\mathbb{N}}$ can be represented as the class of paths of a primitive recursive binary tree.

The forcing question for Σ_1^0-formulas is the same as in Exercise 3.3.7 and Theorem 9.4.1. We recall it for the sake of completeness.

Definition 10.2.5. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_1 formula $\varphi(G)$, define $T \mathrel{?\vdash} \varphi(G)$ to hold if there is some level $\ell \in \mathbb{N}$ such that $\varphi(\sigma)$ holds for every node σ at level ℓ in T. ◊

This forcing question is Σ^0_1-preserving and admits strong properties: if $T \mathrel{?\vdash} \varphi(G)$, then σ already forces $\varphi(G)$. On the other hand, if $T \mathrel{?\nvdash} \varphi(G)$, then one must restrict T to an infinite primitive recursive sub-tree S in order to force $\neg\varphi(G)$ (see Lemma 9.4.4). By Theorem 3.3.4 for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$.

Definition 10.2.6. Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $T \mathrel{?\vdash} \varphi(G)$ to hold if there is some $x \in \mathbb{N}$ and an extension $S \leq T$ such that $S \mathrel{?\vdash} \psi(G, x)$. ◊

The forcing question for Σ^0_2-formulas is Σ^0_2-preserving, and also satisfies strong properties, but on Π^0_2-formulas rather than Σ^0_2-formulas. By Lemma 9.4.6, if $T \mathrel{?\nvdash} \varphi(G)$, then T already forces $\neg\varphi(G)$. This property, similar to the case of Cohen forcing, is exploited to prove the following lemma:

Lemma 10.2.7. For every condition T and every Turing index $e \in \mathbb{N}$, there is an extension $S \subseteq T$ forcing $\Phi_e^{G'}$ not to be a $\{0, 1\}$-valued DNC function over $\emptyset'$. ★

Proof. Consider the following set:

$$U = \{(x, v) \in \mathbb{N} \times 2 : T \mathrel{?\vdash} \Phi_e^{G'}(x)\downarrow = v\}$$

Since the forcing question is Σ^0_2-preserving, the set U is Σ^0_2. There are three cases:

- Case 1: $(x, \Phi_x^{\emptyset'}(x)) \in U$ for some $x \in \mathbb{N}$ such that $\Phi_x^{\emptyset'}(x)\downarrow$. By Property (1) of the forcing question, there is an extension $S \subseteq T$ forcing $\Phi_e^{G'}(x)\downarrow = \Phi_x^{\emptyset'}(x)$.
- Case 2: there is some $x \in \mathbb{N}$ such that $(x, 0), (x, 1) \notin U$. Then T already forces $\neg(\Phi_e^{G'}(x)\downarrow = 0) \wedge \neg(\Phi_e^{G'}(x)\downarrow = 1)$, so T forces $\Phi_e^{G'}$ not to be a $\{0, 1\}$-valued DNC function over $\emptyset'$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_2 graph of a $\{0, 1\}$-valued DNC function over $\emptyset'$. This contradicts the fact that $\mathbf{0}'$ is not PA over $\emptyset'$. ■

Putting all the pieces together, for every sufficiently generic filter $\mathcal{F}$, $C \not\leq_T G_{\mathcal{F}}$ by Theorem 3.3.4, and $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$ by Lemma 10.2.7. This completes the proof of Theorem 10.2.4. ■

Recall from Section 5.1 that given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*merging* if for every $p \in \mathbb{P}$ and every pair of Γ-formulas $\varphi_0(G), \varphi_1(G)$, if $p \mathrel{?\vdash} \varphi_0(G)$ and $p \mathrel{?\vdash} \varphi_1(G)$ both hold, then there is an extension $q \leq p$ forcing $\varphi_0(G) \wedge \varphi_1(G)$.

Exercise 10.2.8. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_2-preserving Π^0_2-merging forcing question. Adapt the proof of Theorem 5.1.9 to show that for every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is not of PA degree over $\emptyset'$. ★

10.3 Mathias forcing and COH

Solutions to Ramsey-type theorems are usually built using variants of Mathias forcing. As seen in Proposition 9.5.1, Mathias-like notions of forcing tend to produce sets of high degree when the reservoirs are only under computability-theoretic restrictions. Indeed, by considering sufficiently sparse reservoirs, one can ensure that the principal function[7] generic set G eventually dominates every total computable function. By Martin's domination theorem, these sets are of high degree.

7: Recall that the *principal function* of an infinite set $X = \{x_0 < x_1 < \dots\}$ is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto x_n$.

We therefore developed in Section 9.6 a framework of partition regularity, yielding variants of Mathias forcing enjoying many of the combinatorial features of Mathias forcing, but with a good second-jump control.[8] Recall that a class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is *partition regular* if it is non-empty, it is closed under superset, and for every $X \in \mathcal{P}$ and every 2-cover $Y_0 \cup Y_1 \supseteq X$, there is some $i < 2$ such that $Y_i \in \mathcal{P}$. The idea is to work with Mathias conditions (σ, X) such that $X \in \mathcal{P}$, where $\mathcal{P}$ is a partition regular class containing only "non-sparse" infinite sets.

8: The reader must be familiar with Section 9.6 to understand the remainder of this section.

Restricting the reservoirs to a well-chosen partition regular class enabled to prevent the reservoirs from being too sparse, while still allowing the basic operations on reservoirs, such as finite truncation, or finite partitioning. Unfortunately, although this restriction is sufficient to obtain strong jump cone avoidance, there is no hope of obtaining jump PA avoidance using a notion of forcing which allows finite partitioning of the reservoir.

Proposition 10.3.1. Fix a partition regular class $\mathcal{P} \subseteq 2^{\mathbb{N}}$. Let $\mathbb{P}$ be the restriction of computable Mathias forcing where reservoirs belong to $\mathcal{P}$. For every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is of PA degree over $\emptyset'$. ★

Proof. Fix a uniformly computable sequence of sets $R_0, R_1, \dots$ such that for every infinite $\vec{R}$-cohesive set C, C' is of PA degree over $\emptyset'$. We claim that for every sufficiently generic filter $\mathcal{F}$, $G_{\mathcal{F}}$ is $\vec{R}$-cohesive. Indeed, given a condition (σ, X) and some n, either $X \cap R_n$, or $X \cap \overline{R}_n$ belongs to $\mathcal{P}$, so either $(\sigma, X \cap R_n)$ or $(\sigma, X \cap \overline{R}_n)$ is a valid extension. Any sufficiently generic filter $\mathcal{F}$ containing the former (latter) extension satisfies $G_{\mathcal{F}} \subseteq^* R_n$ ($G_{\mathcal{F}} \subseteq^* \overline{R}_n$). ■

The previous proposition can be considered as a sanity check, but does not help designing an appropriate notion of forcing. In order to better understand the problem, let us consider the forcing question for Σ^0_2-formulas for the most basic variant of Mathias forcing with a good second-jump control. For this, we need to reintroduce some pieces of notation from Section 9.6.

Letting $W^Z_0, W^Z_1, \dots$ be the list of all Z-c.e. sets of strings, this induce a list $\mathcal{U}^Z_0, \mathcal{U}^Z_1, \dots$ of all $\Sigma^0_1(Z)$ classes of sets, upward-closed by inclusion, as follows: $\mathcal{U}^Z_e = \{X : (\exists \rho \in W^Z_e)\rho \subseteq X\}$. Fix a countable Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$, coded by a set $M = \bigoplus_n Z_n$. Any set $X \in \mathcal{M}$ is represented by an integer $a \in \mathbb{N}$ such that $X = Z_a$. We then say that a is an *M-code* of X. One will consider exclusively partition regular classes of the form $\mathcal{U}^{\mathcal{M}}_C = \bigcap_{(e,i) \in C} \mathcal{U}^{Z_i}_e$, for some set of indices $C \subseteq \mathbb{N}^2$.

Thinking of a partition regular class as a "reservoir of reservoirs", the smaller the partition regular class is, the more positive information it imposes on the reservoirs. The idea is therefore to fix a maximal set of indices $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}^{\mathcal{M}}_C$ is partition regular. Such a class is then called *$\mathcal{M}$-minimal*. Consider the notion of forcing whose conditions are pairs (σ, X), where $X \in \mathcal{U}_C^{\mathcal{M}}$ and $X \in \mathcal{M}$, and whose extension is usual Mathias extension. The forcing question for Σ^0_2-formulas is defined as follows:

Definition 10.3.2. Given a condition (σ, X) and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $(\sigma, X) ?\vdash \varphi(G)$ to hold if there is some finite $\rho \subseteq X$ and some $x \in \mathbb{N}$ such that the following class is not large[9]

$$\mathcal{U}_C^{\mathcal{M}} \cap \{Z : \exists \eta \subseteq Z\ \neg\psi(\sigma \cup \rho \cup \eta, x)\}$$

9: Recall that a class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is *large* if it is upward-closed, and for every $k \in \mathbb{N}$ and every k-cover $Y_0 \cup \cdots \cup Y_{k-1} = \mathbb{N}$, there is some $i < k$ such that $Y_i \in \mathcal{A}$. By Proposition 9.6.10, an upward-closed class $\mathcal{A}$ is large iff it contains a partition regular subclass. An arbitrary union of partition regular classes being partition regular, $\mathcal{A}$ contains a maximal partition regular subclass, written $\mathcal{L}(\mathcal{A})$.

This forcing question is $\Sigma^0_1(M' \oplus C)$ and Π^0_2-merging, which is almost sufficient to apply Exercise 10.2.8. However, even in the case where the Scott set $\mathcal{M}$ is coded by a set of low degree, the natural algorithm to build an $\mathcal{M}$-minimal class $\mathcal{U}_C^{\mathcal{M}}$ produces a Δ^0_3 set of indices C (see Proposition 9.6.19), yielding a Σ^0_3 forcing question for Σ^0_2-formulas. In the case of jump cone avoidance, we circumvented this problem by considering a weaker notion of minimality, called *$\mathcal{M}$-cohesiveness*. By Proposition 9.6.25, PA degrees over M' are sufficient (and necessary[10]) to compute a set $C \subseteq \mathbb{N}^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive, which is sufficient to obtain a diagonalization lemma by the cone avoidance basis theorem.

10: Le Houérou, Levy Patey and Mimouni [83, Lemma 4.15] gave a direct proof of the necessity of PA degrees over M', but there is a less direct argument: if there were an $\mathcal{M}$-cohesive class $\mathcal{U}_C^{\mathcal{M}}$ with $C \oplus M'$ of non-PA degree over $\emptyset'$, then one would be able to construct an infinite cohesive set whose jump is not of PA degree over $\emptyset'$, yielding a contradiction.

In the case of jump PA avoidance, however, having a Π^0_2-merging forcing question for Σ^0_2-formulas which is Σ^0_1 relative to a PA degree over $\emptyset'$ is not sufficient to apply Exercise 10.2.8. One must therefore give up the notions of $\mathcal{M}$-minimality and $\mathcal{M}$-cohesiveness, and work with evolving partition regular classes. Consider therefore a new notion of forcing, whose conditions are of the form (σ, X, C), where

1. (σ, X) is a Mathias condition;
2. $\mathcal{U}_C^{\mathcal{M}}$ is a partition regular subclass of $\mathcal{L}_X$;[11]
3. $X \in \mathcal{M}$ and $M' \oplus C$ is not of PA degree over $\emptyset'$.

11: Recall that

$$\mathcal{L}_X = \{Z : Z \cap X \text{ is infinite }\}$$

If one only asked X to belong to $\mathcal{U}_C^{\mathcal{M}}$, then by considering a partition regular subclass $\mathcal{U}_D^{\mathcal{M}} \subseteq \mathcal{U}_C^{\mathcal{M}}$, X might no belong to $\mathcal{U}_D^{\mathcal{M}}$, so (σ, X, D) would not be a valid extension. Requiring that $\mathcal{U}_C^{\mathcal{M}}$ is a partition regular subclass of $\mathcal{L}_X$ is a way to strongly ensure that X will belong to all partition regular subclasses of $\mathcal{U}_C^{\mathcal{M}}$.

A condition (τ, Y, D) *extends* (σ, X, C) if (τ, Y) Mathias extends (σ, X) and $D \supseteq C$. The latter constraint ensures that $\mathcal{U}_D^{\mathcal{M}} \subseteq \mathcal{U}_C^{\mathcal{M}}$, so the partition regular class becomes more and more restrictive during the construction. The new forcing question for Σ^0_2-formulas can be defined as follows:

Definition 10.3.3. Given a condition (σ, X, C) and a Σ^0_2 formula $\varphi(G) \equiv \exists x \psi(G, x)$, define $(\sigma, X, C) ?\vdash \varphi(G)$ to hold if the following class is not large[12]

$$\mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq X} \{Z : \exists \eta \subseteq Z\ \neg\psi(\sigma \cup \rho \cup \eta, x)\}$$

12: This forcing question coincides with Definition 10.3.2 in the case $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive by Lemma 9.6.23. However, in the more general case of an arbitrary partition regular class, one must use the latter formulation.

This new forcing question is again $\Sigma^0_1(M' \oplus C)$, but letting M be of low degree, one can ensure that $M' \oplus C \equiv_T \emptyset'$, hence that the forcing question is Σ^0_2-preserving. This improved complexity is at one cost: the new forcing question is not Π^0_2-merging. Indeed, suppose $(\sigma, X, C) ?\nvdash \varphi(G)$, then letting $D \supseteq C$ be a set of indices such that

$$\mathcal{U}_D^{\mathcal{M}} = \mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq X} \{Z : \exists \eta \subseteq Z\ \neg\psi(\sigma \cup \rho \cup \eta, x)\}$$

the condition (σ, X, D) is an extension of (σ, X, C) forcing $\neg\varphi(G)$. However, suppose that $\varphi_0(G) \equiv \exists x \psi_0(G, x)$ and $\varphi_1(G) \equiv \exists x \psi_1(G, x)$ be two Σ^0_2-formulas, if $(\sigma, X, C) ?\nvdash \varphi_i(G)$ for both $i < 2$, then letting $D_i \supseteq C$ be the corresponding set of indices for each $i < 2$, it might be that $\mathcal{U}_{D_0}^{\mathcal{M}}$ and $\mathcal{U}_{D_1}^{\mathcal{M}}$ are both partition regular, but $\mathcal{U}_{D_0 \cup D_1}^{\mathcal{M}} = \mathcal{U}_{D_0}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}}$ is not, and therefore one cannot choose $(\sigma, X, D_0 \cup D_1)$ as the desired extension. Again, by Proposition 10.3.1, this notion of forcing cannot admit a forcing question with the right properties, as it produces cohesive sets. One must therefore modify the notion of forcing.

The solution consists of keeping both partition regular classes $\mathcal{U}_{D_0}^{\mathcal{M}}$ and $\mathcal{U}_{D_1}^{\mathcal{M}}$ even if they are incompatible, and commit to preserve the positive information from both classes. Concretely, $\mathcal{U}_{D}^{\mathcal{M}} = \mathcal{U}_{D_0}^{\mathcal{M}} \times \mathcal{U}_{D_1}^{\mathcal{M}}$ is a class over $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is partition regular in the following sense: for every $(X_0, X_1) \in \mathcal{U}_{D}^{\mathcal{M}}$, for every $Z_0 \cup Z_1 \supseteq X_0$ and $R_0 \cup R_1 \supseteq X_1$, there is some $i, j < 2$ such that $(Z_i, R_j) \in \mathcal{P}$. We shall therefore obtain a generalized condition[13] of the form (σ, X_0, X_1, D), where X_0, X_1 are two reservoirs and $\mathcal{U}_{D}^{\mathcal{M}}$ is a partition regular class over $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is a sub-class of

$$\mathcal{L}_{X_0, X_1} = \{(Z_0, Z_1) : X_0 \cap Z_0 \text{ and } X_1 \cap Z_1 \text{ are both infinite}\}$$

Because the forcing question will be used multiple times, the dimension of the product space will increase over conditions extensions. Moreover, we shall manipulate partition regular classes over product spaces which cannot be expressed as the cartesian product of partition regular classes over $2^{\mathbb{N}}$. We therefore need to develop the framework of product partition regularity.

13: Generalizing Mathias conditions to multiple reservoirs is a way to get rid of the issue of Proposition 10.3.1. Indeed, if (σ, X_0, X_1, D) is a condition, and R is a set, then maybe neither $(\sigma, X_0 \cap R, X_1 \cap R, D)$ nor $(\sigma, X_0 \cap \overline{R}, X_1 \cap \overline{R}, D)$ will be a valid extension, so this notion of forcing does not produce in general cohesive sets.

10.4 Product largeness

The theory of product partition regularity is a fairly straightforward generalization of standard partition regularity and will therefore not receive as a detailed development as in Section 9.6. In particular, many proofs will be left as exercise. In what follows, fix a finite set I, which will serve as the index set[14] of the product space. We shall therefore work with sub-classes of $I \to 2^{\mathbb{N}}$.[15] Elements of the set I will be denoted ν or μ, which for now can be thought of as integers, but later will be better represented as strings.

14: From now on, we shall use *index set* to denote the set of indices in the product space. This should not be confused with the set $C \subseteq \mathbb{N}^2$ of indices representing the class $\mathcal{U}_C^{\mathcal{M}}$.

15: The reason we do not use $I = \{0, \dots, n-1\}$ and work with products of the form $2^{\mathbb{N}} \times \cdots \times 2^{\mathbb{N}}$ will become apparent in the next section, where we will use a hierarchy of index sets forming a tree structure.

One could define partition regularity for product classes, yielding a well-behaving generalization of partition regularity over $2^{\mathbb{N}}$. However, in the next sections, all the necessary combinatorics can be formulated in terms of largeness rather than partition regularity. We shall therefore solely introduce largeness for product classes, to reduce the number of concepts.

Definition 10.4.1. A class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ is *large*[16] if

1. For all $\langle X_\nu : \nu \in I \rangle \in \mathcal{A}$ and $Y_\nu \supseteq X_\nu$, then $\langle Y_\nu : \nu \in I \rangle \in \mathcal{A}$.[17]
2. For every $k \in \mathbb{N}$ and every k-cover $Y_0 \cup \cdots \cup Y_{k-1} = \mathbb{N}$, there is some $j : I \to k$ such that $\langle Y_{j(\nu)} : \nu \in I \rangle \in \mathcal{A}$. ◇

16: When I is a singleton, this corresponds to standard largeness over $2^{\mathbb{N}}$.

17: We use the notation $\langle X_\nu : \nu \in I \rangle$ to represent an element of $I \to 2^{\mathbb{N}}$. Any such element can be coded by an element of $2^{\mathbb{N}}$.

The following fundamental lemma generalizes Exercise 9.6.13 and plays an important role in the effective theory of large classes:

Lemma 10.4.2 (Monin and Patey [78]). Suppose $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence of large classes. Then $\bigcap_s \mathcal{A}_s$ is large. ★

Proof. If $\langle X_\nu : \nu \in I \rangle \in \bigcap_s \mathcal{A}_s$ and $Y_\nu \supseteq X_\nu$ for every $\nu \in I$, then for every s, since $\mathcal{A}_s$ is large, $\langle Y_\nu : \nu \in Y \rangle \in \mathcal{A}_s$, so $\langle Y_\nu : \nu \in Y \rangle \in \bigcap_s \mathcal{A}_s$. Let

$Y_0 \cup \dots \cup Y_k = \mathbb{N}$ for some $k \in \mathbb{N}$. For every $s \in \mathbb{N}$, by largeness of $\mathcal{A}_s$, there is some $j : I \to k$ such that $\langle Y_{j(\nu)} : \nu \in I \rangle \in \mathcal{A}_s$. By the infinite pigeonhole principle, there is some $j : I \to k$ such that $\langle Y_{j(\nu)} : \nu \in I \rangle \in \mathcal{A}_s$ for infinitely many s. Since $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence, $\langle Y_{j(\nu)} : \nu \in I \rangle \in \bigcap_s \mathcal{A}_s$. ■

Recall that for every infinite set $X \in 2^{\mathbb{N}}$, the class $\mathcal{L}_X = \{Y : X \cap Y \text{ is infinite }\}$ is partition regular. We generalize the definition to product classes.

Definition 10.4.3. Given $\langle X_\nu : \nu \in I \rangle$, let

$$\mathcal{L}_{\langle X_\nu : \nu \in I \rangle} = \{\langle Y_\nu : \nu \in I \rangle : \forall \nu \in I, Y_\nu \cap X_\nu \text{ is infinite}\}$$

The following easy exercise simply states that the definition is invariant under finite modifications of the sets.

Exercise 10.4.4 (Monin and Patey [78]). Let $\langle X_\nu : \nu \in I \rangle$ and $\langle Y_\nu : \nu \in I \rangle$ be such that $X_\nu =^* Y_\nu$[18] for every $\nu \in I$. Then $\mathcal{L}_{\langle X_\nu : \nu \in I \rangle} = \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$. ★

18: The notation $X =^* Y$ means that X and Y are equal *up to finite changes*.

In general, $\mathcal{L}_X \cap \mathcal{L}_Y \supsetneq \mathcal{L}_{X \cap Y}$ for infinite sets X, Y. For instance, if X and Y are the sets of all odd and even numbers, respectively, then $\mathbb{N} \in \mathcal{L}_X \cap \mathcal{L}_Y$ but $\mathcal{L}_{X \cap Y} = \emptyset$. On the other hand, if $\mathcal{L}_X \cap \mathcal{L}_Y$ is large, then so is $\mathcal{L}_{X \cap Y}$. The following lemma generalizes this property.

Lemma 10.4.5 (Monin and Patey [78]). Let $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ be a large class and $\langle X_\nu : \nu \in I \rangle, \langle Y_\nu : \nu \in I \rangle$ be two tuples. If $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \cap \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$ is large, then so is $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu \cap Y_\nu : \nu \in I \rangle}$. ★

Proof. First, note that $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu \cap Y_\nu : \nu \in I \rangle}$ is upward-closed for inclusion. Let $Z_0 \cup \dots \cup Z_{k-1} = \mathbb{N}$. By refining the covering, we can assume that for every $t < k$ and $\nu \in I$, Z_t is both X_ν and Y_ν-homogeneous. Since $\mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \cap \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$ is large, there is some $j : I \to k$ such that $\langle Z_{j(\nu)} : \nu \in I \rangle \in \mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \cap \mathcal{L}_{\langle Y_\nu : \nu \in I \rangle}$. We claim that $Z_{j(\nu)} \subseteq X_\nu \cap Y_\nu$ for every $\nu \in I$. Indeed, since $\langle Z_{j(\nu)} : \nu \in I \rangle \in \mathcal{L}_{\langle X_\nu : \nu \in I \rangle}$, then $Z_{j(\nu)} \cap X_\nu$ is infinite, so by X_ν-homogeneity of $Z_{j(\nu)}$, $Z_{j(\nu)} \subseteq X_\nu$. Similarly, $Z_{j(\nu)} \subseteq Y_\nu$. Thus $\langle Z_{j(\nu)} : \nu \in I \rangle \in \mathcal{A} \cap \mathcal{L}_{\langle X_\nu \cap Y_\nu : \nu \in I \rangle}$. ■

Recall from Section 9.6 that every large class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ admits a maximal partition regular sub-class $\mathcal{L}(\mathcal{A})$, which admits a formulation purely in terms of largeness thanks to Exercise 9.6.12. We give a similar definition for product classes.

Proposition 10.4.6 (Monin and Patey [78]). Let $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ be a non-trivial large class. The class

$$\mathcal{L}(\mathcal{A}) = \{\langle X_\nu : \nu \in I \rangle \in \mathcal{A} : \mathcal{A} \cap \mathcal{L}_{\langle X_\nu : \nu \in I \rangle} \text{ is large }\}$$

is a large sub-class of $\mathcal{A}$. ★

Proof. First, $\mathcal{L}(\mathcal{A})$ is by definition a sub-class of $\mathcal{A}$. Moreover, it is upward-closed for inclusion. Suppose for the contradiction that $\mathcal{L}(\mathcal{A})$ is not large. Then there is some $k \in \mathbb{N}$ and some k-cover $X_0 \cup \dots \cup X_{k-1} = \mathbb{N}$ such that for every $j : I \to k$, $\langle X_{j(\nu)} : \nu \in I \rangle \notin \mathcal{L}(\mathcal{A})$. Unfolding the definition, for every $j : I \to k$, $\mathcal{A} \cap \mathcal{L}_{\langle X_{j(\nu)}:\nu\in I\rangle}$ is not large. Thus for every $j : I \to k$, there is some $k_j \in \mathbb{N}$ and some k_j-cover $Y_0 \cup \dots \cup Y_{k_j-1} = \mathbb{N}$ such that for every $i : I \to k_j$, $\langle Y_{i(\nu)} : \nu \in I\rangle \notin \mathcal{A}$. Let $Z_0 \cup \dots Z_{\ell-1} = \mathbb{N}$ be the common refinement of all these covers. Then, for every $r : I \to \ell$, $\langle Z_{r(\nu)} : \nu \in I\rangle \notin \mathcal{A} \cap \mathcal{L}_{\langle Z_{r(\nu)}:\nu\in I\rangle}$. However, since $\mathcal{A}$ is large, there is some $r : I \to \ell$ such that $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{A}$, and since $\mathcal{A}$ is non-trivial, $Z_{r(\nu)}$ is infinite for every $\nu \in I$, so $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{L}_{\langle Z_{r(\nu)}:\nu\in I\rangle}$. It follows that $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{A} \cap \mathcal{L}_{\langle Z_{r(\nu)}:\nu\in I\rangle}$. Contradiction. ■

Exercise 10.4.7.

1. Define the notion of partition regularity of sub-classes of $I \to 2^{\mathbb{N}}$.
2. Show that if $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ is large, then $\mathcal{L}(\mathcal{A})$ is the maximal partition regular subclass of $\mathcal{A}$. ★

10.4.1 Effective classes

Let $W_0^{Z,I}, W_1^{Z,I}, \dots$ be a list of all Z-c.e. subsets of $I \to 2^{<\mathbb{N}}$. As above, this induces a list $\mathcal{U}_0^{Z,I}, \mathcal{U}_1^{Z,I}, \dots$ of all $\Sigma^0_1(Z)$ sub-classes of $I \to 2^{\mathbb{N}}$, upward-closed by inclusion. Fix a countable Scott ideal $\mathcal{M} = \{Z_0, Z_1, \dots\}$ coded by a set $M = \bigoplus_n Z_n$. Given a set $C \subseteq \mathbb{N}^2$, we write $\mathcal{U}_C^{\mathcal{M},I}$ for $\bigcap_{(e,i)\in C} \mathcal{U}_e^{Z_i,I}$.

Lemma 10.4.8. Let $C \subseteq \mathbb{N}^2$ be a set. The statement "$\mathcal{U}_C^{\mathcal{M},I}$ is large" is $\Pi^0_1(C \oplus M')$ uniformly in C, M and I. ★

Proof. Let us first show that the statement "$\mathcal{U}_e^{Z,I}$ is large" is $\Pi^0_2(Z)$ uniformly in e, Z and I. Indeed, by compactness, $\mathcal{U}_e^{Z,I}$ is large iff for every $k \in \mathbb{N}$, there is some $\ell \in \mathbb{N}$ such that for every k-cover $Y_0 \cup \dots \cup Y_{k-1} = \{0, \dots, \ell\}$, there is some $j : I \to k$ and some $\rho \in W_e^I$ such that for each $\nu \in I$, $\rho(\nu) \subseteq Y_{j(\nu)}$. This statement is $\Pi^0_2(Z)$ uniformly in e and Z. Then, by Lemma 10.4.2, $\mathcal{U}_C^{\mathcal{M},I}$ is large iff for every finite set $F \subseteq C$, $\mathcal{U}_F^{\mathcal{M},I}$ is large. The resulting statement is therefore $\Pi^0_1(C \oplus M')$. ■

We shall work exclusively with non-trivial classes of the form $\mathcal{U}_C^{\mathcal{M},I}$ where $\mathcal{M}$ is a Scott ideal coded by a set of low degree, and $C \subseteq \mathbb{N}^2$ is Δ^0_2. The following exercise shows that such classes are Π^0_2.

Exercise 10.4.9. Let $\mathcal{M}$ be a Scott ideal, coded by a set M of low degree. Let $C \subseteq \mathbb{N}^2$ be Σ^0_2. Show that $\mathcal{U}_C^{\mathcal{M},I}$ is Π^0_2. ★

10.4.2 Projections

We developed so far a theory of product largeness for a fixed set of indices I. The main theorem of this chapter will invoke the pigeonhole principle over I to obtain a sub-set $J \subseteq I$ over which the large class admits better properties. We must therefore define a proper notion of projection of a class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ over a sub-set $J \subseteq I$.

19: There exist multiple candidate notions of projection. For instance, one could have asked the class to be non-empty instead of large. However, this definition enjoys better combinatorial properties.

Definition 10.4.10. Given a class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ and a subset $J \subseteq I$, let $\pi_J(\mathcal{A})$ be the class of all $\langle X_\nu : \nu \in J\rangle$ such that the following class is large:[19]

$$\{\langle X_\nu : \nu \in I \setminus J\rangle : \langle X_\nu : \nu \in I\rangle \in \mathcal{A}\}$$

It is not clear at first sight that this definition of projection is not too strong, that is, asking the residual class to be large instead of non-empty might yield a small projection. Thankfully, the following lemma states that a large number of elements satisfies this property.

Lemma 10.4.11 (Monin and Patey [78]). Let $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ be a large class, and $J \subseteq I$ be a subset. Then $\pi_J(\mathcal{A})$ is large. ⋆

Proof. The class $\pi_J(\mathcal{A})$ is upward-closed by upward-closure of $\mathcal{A}$. Let $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$ for some $k \in \mathbb{N}$. Suppose for the contradiction that for every $j : J \to k$, $\langle Y_{j(\nu)} : \nu \in J\rangle \notin \pi_J(\mathcal{A})$. Unfolding the definition, for every $j : J \to k$, the following class is not large:

$$\{\langle X_\nu : \nu \in I \setminus J\rangle : \langle X_\nu : \nu \in I \setminus J\rangle \cdot \langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{A}\}$$

Let $Z_0 \cup \dots \cup Z_{\ell-1} = \mathbb{N}$ be the common refinement of all the covers witnessing that these classes are not large, and of $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$. Since $\mathcal{A}$ is large, there is some $r : I \to \ell$ such that $\langle Z_{r(\nu)} : \nu \in I\rangle \in \mathcal{A}$. Since the cover refines $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$, there is a function $j : J \to k$ such that for every $\nu \in J$, $Y_{j(\nu)} \supseteq Z_{r(\nu)}$. Let $i : I \setminus J \to \ell$ be the restriction of r to $I \setminus J$. Then by upward-closure of $\mathcal{A}$, $\langle Z_{i(\nu)} : \nu \in I \setminus J\rangle \cup \langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{A}$, which contradicts the fact that $Z_0 \cup \dots \cup Z_{\ell-1} = \mathbb{N}$ refines the witness of non-largeness for j. ■

The following lemma states the existence of a commutative diagram between large classes and their projections. It will be very useful to consider each projection independently, and obtain a decreasing sequence of large sub-classes of $I \to 2^{\mathbb{N}}$.

Lemma 10.4.12 (Monin and Patey [78]). Let $\mathcal{U}_C^{\mathcal{M},I} \subseteq I \to 2^{\mathbb{N}}$ be a large class for some Δ^0_2 set $C \subseteq \mathbb{N}^2$, $J \subseteq I$ be a subset of indices and $\mathcal{A} \subseteq \pi_J(\mathcal{U}_C^{\mathcal{M},I})$ be a Π^0_2 large class. Then there is a Δ^0_2 set $D \supseteq C$ such that $\mathcal{U}_D^{\mathcal{M},I} \subseteq \mathcal{U}_C^{\mathcal{M},I}$ is large, and $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) = \mathcal{A}$. ⋆

Proof. Say $\mathcal{A} = \mathcal{U}_E^{\mathcal{M},J}$ for some Δ^0_2 set $E \subseteq \mathbb{N}^2$. There exists an increasing computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$ and every oracle Z, $\mathcal{U}_{f(e)}^{Z,I} = \{\langle X_\nu : \nu \in I\rangle : \langle X_\nu : \nu \in J\rangle \in \mathcal{U}_e^{Z,J}\}$. Let $D = C \cup \{(f(e), i) : (e, i) \in E\}$. Then D is Δ^0_2 and $\mathcal{U}_D^{\mathcal{M},I}$ is the class of all $\langle X_\nu : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$ such that $\langle X_\nu : \nu \in J\rangle \in \mathcal{A}$. Since $D \supseteq C$, $\mathcal{U}_D^{\mathcal{M},I} \subseteq \mathcal{U}_C^{\mathcal{M},I}$.

20: This claim is precisely the reason we defined projection in terms of largeness rather than non-emptiness.

We claim that $\mathcal{U}_D^{\mathcal{M},I}$ is large.[20] Note that it is upward-closed, as both $\mathcal{U}_C^{\mathcal{M},I}$ and $\mathcal{A}$ are. Let $k \in \mathbb{N}$ and $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$. Since $\mathcal{A} \subseteq J \to 2^{\mathbb{N}}$ is large, there is some $j : J \to k$ such that $\langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{A}$. Moreover, since $\mathcal{A} \subseteq \pi_J(\mathcal{U}_C^{\mathcal{M},I})$, the class

$$\{\langle X_\nu : \nu \in I \setminus J\rangle : \langle X_\nu : \nu \in J \setminus I\rangle \cup \langle Y_{j(\nu)} : \nu \in J\rangle \in \mathcal{U}_C^{\mathcal{M},I}\rangle$$

is large. Therefore, there is some $i : I \setminus J \to k$ such that $\langle Y_{i(\nu)} : \nu \in I \setminus J\rangle$ belongs to this class. Letting $r : I \to k$ be the common extension of i and j, $\langle Y_{r(\nu)} : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$. Thus, $\langle Y_{r(\nu)} : \nu \in I\rangle \in \mathcal{U}_D^{\mathcal{M},I}$. This proves our claim.

We claim that $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) = \mathcal{A}$. By definition, given $\langle Y_\nu : \nu \in J\rangle \in \mathcal{A}$, the class $\mathcal{B} = \{\langle Y_\nu : \nu \in I \setminus J\rangle : \langle Y_\nu : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}\}$ is large since $\mathcal{A} \subseteq \pi_J(\mathcal{U}_C^{\mathcal{M},I})$. By construction of $\mathcal{U}_D^{\mathcal{M},I}$, $\mathcal{B} = \{\langle Y_\nu : \nu \in I \setminus J\rangle : \langle Y_\nu : \nu \in I\rangle \in \mathcal{U}_D^{\mathcal{M},I}\}$, so $\langle Y_\nu : \nu \in J\rangle \in \pi_J(\mathcal{U}_D^{\mathcal{M},I})$. It follows that $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) \supseteq \mathcal{A}$. Suppose now that $\langle Y_\nu : \nu \in J\rangle \in \pi_J(\mathcal{U}_D^{\mathcal{M},I})$. Then the class $\mathcal{D} = \{\langle Y_\nu : \nu \in I \setminus J\rangle : \langle Y_\nu : \nu \in I\rangle \in \mathcal{U}_D^{\mathcal{M},I}\}$ is large, and in particular non-empty. By definition of $\mathcal{U}_D^{\mathcal{M},I}$, $\langle Y_\nu : \nu \in J\rangle \in \mathcal{A}$. Thus $\pi_J(\mathcal{U}_D^{\mathcal{M},I}) \subseteq \mathcal{A}$. ■

Exercise 10.4.13. Let $I = \{0, 1\}$, $J = \{0\}$, let Odd and Even be the sets of odd and even numbers, respectively. Let $\mathcal{B} = (\mathcal{L}_{\mathsf{Odd}} \times 2^{\mathbb{N}}) \cup (\mathcal{L}_{\mathsf{Even}} \times \{\mathbb{N}\})$. Let $\hat{\pi}_J(\mathcal{B})$ be the set of all $X \in 2^{\mathbb{N}}$ such that $(X, Y) \in \mathcal{B}$ for some set Y.[21]

1. Show that $\mathcal{B}$ is large.
2. What is $\pi_J(\mathcal{B})$? What is $\hat{\pi}_J(\mathcal{B})$?
3. Show that $\mathcal{L}_{\mathsf{Even}}$ is a Π^0_2 sub-class of $\hat{\pi}_J(\mathcal{B})$, but there is no large sub-class $\mathcal{D} \subseteq \mathcal{B}$ such that $\hat{\pi}_J(\mathcal{D}) = \mathcal{L}_{\mathsf{Even}}$. ★

21: In other words, $\hat{\pi}_J(\mathcal{B})$ is the alternative notion of projection. The goal of this exercise is to show that such version does not satisfy Lemma 10.4.12.

10.4.3 Index sets

So far, we only manipulated large classes over product spaces for a fixed index set I, and reduced the dimension of a space using projection. One of the main interest of product spaces is to force multiple positive information on the reservoirs by considering the cartesian product of two large classes. Given two index sets I and K, there exists a natural one-to-one correspondence between the following two classes:[22]

$$K \to (I \to 2^{\mathbb{N}}) \quad \text{and} \quad K \times I \to 2^{\mathbb{N}}$$

22: The translation from the second class to the first class is known in computer science as *curryfication*.

We therefore identify the two classes, and given a class $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$, we consider $K \to \mathcal{A}$ as a sub-class of $K \times I \to 2^{\mathbb{N}}$.

Definition 10.4.14. Given two index sets I and J, we write $J \le I$ if there is an index set K such that $J = K \times I$. Given two classes $\mathcal{A} \subseteq I \to 2^{\mathbb{N}}$ and $\mathcal{B} \subseteq J \to 2^{\mathbb{N}}$, we write $\mathcal{B} \le \mathcal{A}$ if $J = K \times I$ for some K and $\mathcal{B} \subseteq K \to \mathcal{A}$. ◇

If $J \le I$ as witnessed by an index set K, we call *canonical surjection* the function $f : J \to I$ defined for every $(\mu, \nu) \in J \times I$ by $f(\mu, \nu) = \nu$.

Exercise 10.4.15. Let $I_0 \ge I_1 \ge I_2$ be three index sets and $\mathcal{A}_i \subseteq I_i \to 2^{\mathbb{N}}$ be classes for each $i < 3$. Show that if $\mathcal{A}_3 \le \mathcal{A}_2$ and $\mathcal{A}_2 \le \mathcal{A}_1$, then $\mathcal{A}_3 \le \mathcal{A}_1$. ★

10.5 Product Mathias forcing

Let us now exemplify the concepts introduced in this chapter by designing a variant of Mathias forcing whose generic sets have a jump of non-PA degree over $\emptyset'$. The main theorem of this chapter will be an elaboration of this notion of forcing, with many subtleties due to the disjunctive nature of the pigeonhole principle.

Fix a countable Scott ideal $\mathcal{M}$, coded by a set M of low degree. Consider the notion of forcing[23] whose conditions[24] are tuples $(\sigma, \langle X_\nu : \nu \in I\rangle, C)$, where

23: This notion of forcing may seem quite complex at first sight, but it is arguably the natural refinement of Mathias forcing with a good second-jump control which produces non-cohesive solutions.

24: One could have merged the sets $\langle X : \nu \in I\rangle$ into a single set $X = \bigcup_{\nu \in I} X_\nu$, and worked with tuples (σ, X, I, C), such that $\mathcal{U}_C^{\mathcal{M},I}$ is a large sub-class of $\mathcal{L}_{\langle X : \nu \in I\rangle}$. The use of multiple reservoirs will however be needed for our later refinement of Mathias forcing.

1. I is a finite index set;
2. $(\sigma, \bigcup_{\nu \in I} X_\nu)$ is a Mathias condition;
3. $\mathcal{U}_C^{\mathcal{M},I}$ is a large sub-class of $\mathcal{L}_{\langle X_\nu : \nu \in I \rangle}$;
4. $\langle X_\nu : \nu \in I \rangle \in \mathcal{M}$ and C is Δ_2^0.

A condition $(\tau, \langle Y_\mu : \mu \in J \rangle, D)$ *extends* $(\sigma, \langle X_\nu : \nu \in I \rangle, C)$ if $(\tau, \bigcup_{\mu \in J} Y_\mu)$ Mathias extends $(\sigma, \bigcup_{\nu \in I} X_\nu)$, $J \leq I$ with canonical surjection $f : J \to I$, $\mathcal{U}_D^{\mathcal{M},J} \leq \mathcal{U}_C^{\mathcal{M},I}$, and for every $\mu \in J$, $Y_\mu \subseteq X_{f(\mu)}$.

Every filter $\mathcal{F}$ for this notion of forcing induces a set $G_\mathcal{F} = \bigcup\{\sigma : (\sigma, \langle X_\nu : \nu \in I \rangle, C) \in \mathcal{F}\}$. The following extension lemma states that not only for every sufficiently generic filter $\mathcal{F}$, the set $G_\mathcal{F}$ is infinite, but if $\mathcal{F}$ contains a condition $(\sigma, \langle X_\nu : \nu \in I \rangle, C)$, then $G_\mathcal{F} \cap X_\nu$ is infinite for every $\nu \in I$.

Lemma 10.5.1. Let $(\sigma, \langle X_\nu : \nu \in I \rangle, C)$ be a condition and $x \in X_\nu$ for some $\nu \in I$. Then $(\sigma \cup \{x\}, \langle X_\nu \setminus [0, x] : \nu \in I \rangle, C)$ is a valid extension. ★

Proof. Immediate by Exercise 10.4.4. ■

As one expects, the use of multiple reservoirs prevents $G_\mathcal{F}$ to be cohesive as a set. The following lemma states that for every computable instance $\vec{R}$ of COH with no computable solution, and every sufficiently generic filter $\mathcal{F}$, the set $G_\mathcal{F}$ is not $\vec{R}$-cohesive.

Lemma 10.5.2. Let $\vec{R} = R_0, R_1, \dots$ be a uniformly computable sequence of sets with no computable infinite $\vec{R}$-cohesive set. For every condition $(\sigma, \langle X_\nu : \nu \in I \rangle, C)$, and every $\mu \in I$, there is an extension $(\sigma, \langle Y_{(i,\nu)} : (i, \nu) \in 2 \times I \rangle, D)$ and some $n \in \mathbb{N}$ such that $Y_{(0,\mu)} \subseteq R_n$ and $Y_{(1,\mu)} \subseteq \overline{R}_n$. ★

Proof. Pick any $\mu \in I$ and let $\mathcal{A} = \pi_{\{\mu\}}(\mathcal{U}_C^{\mathcal{M},I})$. Note that $\mathcal{A}$ is a Π_2^0 sub-class of $\mathcal{L}_{X_\mu}$. By Exercise 9.6.27, there is some $n \in \mathbb{N}$ such that $\mathcal{A} \cap \mathcal{L}_{R_n}$ and $\mathcal{A} \cap \mathcal{L}_{\overline{R}_n}$ are both large. By Lemma 10.4.5, $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{L}_{R_n \cap X_\mu}$ and $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{L}_{\overline{R}_n \cap X_\mu}$ are both large. By Lemma 10.4.12, there are two Δ_2^0 sets $D_0, D_1 \supseteq C$ such that $\mathcal{U}_{D_i}^{\mathcal{M},I} \subseteq \mathcal{U}_C^{\mathcal{M},I}$ is large and $\pi_{\{\mu\}}(\mathcal{U}_{D_i}^{M,I}) = \mathcal{A}_i$ for each $i < 2$. Let $J = 2 \times I$, $D \subseteq \mathbb{N}^2$ be such that $\mathcal{U}_D^{\mathcal{M},J} = \mathcal{U}_{D_0}^{\mathcal{M},I} \times \mathcal{U}_{D_1}^{\mathcal{M},I}$. Then $\mathcal{U}_D^{\mathcal{M},J} \leq \mathcal{U}_C^{\mathcal{M},I}$. Let $Y_{(0,\mu)} = X_\mu \cap R_n$, $Y_{(1,\mu)} = X_\mu \cap \overline{R}_n$, and $Y_{(i,\nu)} = X_\nu$ otherwise. Then the condition $(\sigma, \langle Y_\nu : \nu \in J \rangle, D)$ is the desired extension. ■

Having a notion of forcing producing non-cohesive generic sets is a sanity check, but it might be the case that the generic set *computes* a cohesive set for a computable instance of COH. We shall prove later that this does not happen, by designing a Π_2^0-merging and Σ_2^0-preserving forcing question for Σ_2^0-formulas.

Forcing question for Σ_1^0-formulas. We now design a forcing question for Σ_1^0-formulas. It essentially corresponds to the forcing question for computable Mathias forcing.[25]

25: Contrary to the proof of Theorem 9.7.1, the reservoirs belong to $\mathcal{M}$, so the forcing question can directly involve the reservoirs rather than using an over-approximation in terms of largeness. The forcing question therefore has a good definitional complexity and is Π_1^0-extremal.

Definition 10.5.3. Given a Mathias condition (σ, X) and a Σ_1^0 formula $\varphi(G)$, define $(\sigma, X) ?\vdash \varphi(G)$ to hold there exists some $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. ◇

Note that this relation is $\Sigma_1^0(X)$. The proof of validity of the forcing question for Σ_1^0-formulas is straightforward and is left as an exercise.

Exercise 10.5.4. Let $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ be a condition and $\varphi(G)$ be a Σ^0_1 formula. Prove that

1. if $(\sigma, \bigcup_\nu X_\nu) \mathrel{?\vdash} \varphi(G)$, then there is an extension of p forcing $\varphi(G)$;
2. if $(\sigma, \bigcup_\nu X_\nu) \mathrel{?\nvdash} \varphi(G)$, then there is an extension of p forcing $\neg\varphi(G)$. ⋆

Syntactic forcing relation. As in the proof of Theorem 9.7.1, it will be convenient to define a syntactic forcing relation for Π^0_2-formulas.

Definition 10.5.5. Let $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ be a condition and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. Let $p \Vdash \varphi(G)$ hold if for every $\rho \subseteq \bigcup_{\nu \in I} X_\nu$ and every $x \in \mathbb{N}$,[26] [27]

$$\mathcal{U}^{\mathcal{M},I}_C \subseteq \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \mathrel{?\vdash} \psi(G, x)\}$$

26: One would be tempted to only require that the intersection of the left and right-hand side of the inclusion is large. However, since $\mathcal{U}^{\mathcal{M},I}_C$ may decrease over condition extension, this forcing relation would not be closed under extension. Asking for inclusion is a way to strongly enforce the largeness of the intersection, for every further restriction of $\mathcal{U}^{\mathcal{M},I}_C$.

27: Technically, we should have used

$$(\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu \setminus [0, \max \rho])$$

to ensure that the minimum of the reservoirs is larger than the stems, but we drop this restriction for simplicity of the notation.

Since the size of the index set may increase over condition extension, it is not completely clear that this syntactic forcing relation is closed under extension. The following lemma shows that it is the case.

Lemma 10.5.6. Let p be a condition and $\varphi(G)$ be a Π^0_2-formula such that $p \Vdash \varphi(G)$. For every extension $q \le p$, $q \Vdash \varphi(G)$. ⋆

Proof. Say $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$, $q = (\tau, \langle Y_\mu : \mu \in J\rangle, D)$, and $\varphi(G) \equiv \forall x \psi(G, x)$. Let K be such that $J = K \times I$, and let $f : J \to I$ be the canonical surjection. Fix some $x \in \mathbb{N}$ and some $\rho \subseteq \bigcup_{\mu \in J} Y_\mu$. Since $(\tau, \bigcup_{\mu \in J} Y_\mu)$ Mathias extends $(\sigma, \bigcup_{\nu \in I} X_\nu)$, there is some $\eta \subseteq \bigcup_{\nu \in I} X_\nu$ such that $\tau \cup \rho = \sigma \cup \eta$. Since $p \Vdash \varphi(G)$, then

$$\mathcal{U}^{\mathcal{M},I}_C \subseteq \{\langle R_\nu : \nu \in I\rangle : (\sigma \cup \eta, \bigcup_{\nu \in I} R_\nu) \mathrel{?\vdash} \psi(G, x)\}$$

We claim that

$$\mathcal{U}^{\mathcal{M},J}_D \subseteq \{\langle Z_\mu : \mu \in J\rangle : (\tau \cup \rho, \bigcup_{\mu \in J} Z_\mu) \mathrel{?\vdash} \psi(G, x)\}$$

Fix some $\langle Z_\mu : \mu \in J\rangle \in \mathcal{U}^{\mathcal{M},J}_D$. Since $\mathcal{U}^{\mathcal{M},J}_D \le \mathcal{U}^{\mathcal{M},I}_C$, $\mathcal{U}^{\mathcal{M},J}_D \subseteq K \to \mathcal{U}^{\mathcal{M},I}_C$. It follows that there is some $\langle R_\nu : \nu \in I\rangle \in \mathcal{U}^{\mathcal{M},I}_C$ such that $\bigcup_{\mu \in J} Z_\mu \supseteq \bigcup_{\nu \in I} R_\nu$. Since $(\sigma \cup \eta, \bigcup_{\nu \in I} R_\nu) \mathrel{?\vdash} \psi(G, x)$, then $(\tau \cup \rho, \bigcup_{\mu \in J} Z_\mu) \mathrel{?\vdash} \psi(G, x)$. ■

Together with Lemma 10.5.6, the following lemma states that, for every sufficiently generic filter $\mathcal{F}$, if $p \Vdash \varphi(G)$ for some $p \in \mathcal{F}$, then p forces $\varphi(G)$.

Lemma 10.5.7. Let $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ be a condition and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. If $p \Vdash \varphi(G)$, then for every $x \in \mathbb{N}$, there is an extension $q \le p$ forcing $\psi(G, x)$. ⋆

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G)$, then in particular, for $\rho = \emptyset$,

$$\mathcal{U}^{\mathcal{M},I}_C \subseteq \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \mathrel{?\vdash} \psi(G, x)\}$$

Since $\langle X_\nu : \nu \in I\rangle \in \mathcal{U}^{\mathcal{M},I}_C$, then $(\sigma, \bigcup_{\nu \in I} X_\nu) \mathrel{?\vdash} \psi(G, x)$. By Exercise 10.5.4, there is an extension of p forcing $\psi(G, x)$. ■

Forcing question for Σ^0_2-formulas. We now have all the necessary tools to define a forcing question for Σ^0_2-formulas with good definitional and combinatorial properties.

Definition 10.5.8. Let $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ be a condition and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Let $p \operatorname{?\vdash} \varphi(G)$ hold if the following class is not large:

$$\mathcal{U}^{\mathcal{M},I}_C \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq \bigcup_{\nu \in I} X_\nu} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \operatorname{?\vdash} \psi(G, x)\}$$

By Lemma 10.4.8, the forcing question is $\Sigma^0_1(C \oplus M')$, hence Σ^0_2 since M is low and C Δ^0_2. It follows that the forcing question is Σ^0_2-preserving. We now prove that it meets its specifications.

Lemma 10.5.9. Let p be a condition and $\varphi(G)$ a Σ^0_2-formula.

1. If $p \operatorname{?\vdash} \varphi(G)$, then there is an extension of p forcing $\varphi(G)$.
2. If $p \operatorname{?\nvdash} \varphi(G)$, then there is an extension q of p with $q \Vdash \neg\varphi(G)$. ★

Proof. Say $p = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ and $\varphi(G) \equiv \exists x \psi(G, x)$.

Suppose first $p \operatorname{?\vdash} \varphi(G)$. Then there is some finite set $F \subseteq C$, some $\ell \in \mathbb{N}$ and some $x_0, \dots, x_{\ell-1} \in \mathbb{N}$ and $\rho_0, \dots, \rho_{\ell-1} \subseteq \bigcup_{\nu \in I} X_\nu$ such that

$$\mathcal{A} = \mathcal{U}^{\mathcal{M},I}_F \cap \bigcap_{s<\ell} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho_s, \bigcup_{\nu \in I} Y_\nu) \operatorname{?\vdash} \psi(G, x_s)\}$$

is not large. Given $k \in \mathbb{N}$, let $\mathcal{C}_k$ be the $\Pi^0_1(\mathcal{M})$ class of all $Y_0 \oplus \dots \oplus Y_{k-1} \in 2^\mathbb{N}$ such that $Y_0 \cup \dots \cup Y_{k-1} = \mathbb{N}$ and for every $j : I \to k$, $\langle Y_{j(\nu)} : \nu \in I\rangle \notin \mathcal{A}$. There is some $k \in \mathbb{N}$ such that $\mathcal{C}_k \neq \emptyset$. Since $\mathcal{M}$ is a Scott ideal, there is some $Y_0 \oplus \dots \oplus Y_{k-1} \in \mathcal{C}_k \cap \mathcal{M}$. By Proposition 10.4.6, there is some $j : I \to k$ such that $\mathcal{U}^{\mathcal{M},I}_C \cap \mathcal{L}_{\langle Y_{j(\nu)} : \nu \in I\rangle}$ is large. Since $\langle Y_{j(\nu)} : \nu \in I\rangle \notin \mathcal{A}$, there is some $s < \ell$ such that $(\sigma \cup \rho_s, \bigcup_{\nu \in I} Y_{j(\nu)}) \operatorname{?\vdash} \psi(G, x_s)$. By definition of a condition, $\mathcal{U}^{\mathcal{M},I}_C \subseteq \mathcal{L}_{\langle X_\nu : \nu \in I\rangle}$, so by Lemma 10.4.5, $\mathcal{U}^{\mathcal{M},I}_C \cap \mathcal{L}_{\langle X_\nu \cap Y_{j(\nu)} : \nu \in I\rangle}$ is large. For every $\nu \in I$, let $Z_\nu = X_\nu \cap Y_{j(\nu)}$. Let $D \supseteq C$ be a Δ^0_2 set such that $\mathcal{U}^{\mathcal{M},I}_D = \mathcal{U}^{\mathcal{M},I}_C \cap \mathcal{L}_{\langle Z_\nu : \nu \in I\rangle}$. Then $q = (\sigma \cup \rho_s, \langle Z_\nu : \nu \in I\rangle, D)$ is an extension of p such that $(\sigma \cup \rho_s, \bigcup_{\nu \in I} Y_{j(\nu)}) \operatorname{?\vdash} \psi(G, x_s)$. By Exercise 10.5.4, there is an extension of q forcing $\psi(G, x_s)$, hence forcing $\varphi(G)$.

Suppose first $p \operatorname{?\nvdash} \varphi(G)$. Let $D \supseteq C$ be a Δ^0_2 set such that

$$\mathcal{U}^{\mathcal{M},I}_D = \mathcal{U}^{\mathcal{M},I}_C \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq \bigcup_{\nu \in I} X_\nu} \{\langle Y_\nu : \nu \in I\rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \operatorname{?\vdash} \psi(G, x)\}$$

Then $q = (\sigma, \langle X_\nu : \nu \in I\rangle, C)$ is an extension of p such that $q \Vdash \neg\varphi(G)$. ■

Our last lemma states that the forcing question for Σ^0_2-formulas is Π^0_2-merging. It follows from Exercise 10.2.8 that for every sufficiently generic filter $\mathcal{F}$, $G'_\mathcal{F}$ is not of PA degree over $\emptyset'$.

Lemma 10.5.10. Let p be a condition and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_2-formulas. If $p \operatorname{?\nvdash} \varphi_0(G)$ and $p \operatorname{?\nvdash} \varphi_1(G)$, then there is an extension q of p with $q \Vdash \neg\varphi_0(G)$ and $q \Vdash \neg\varphi_1(G)$. ★

Proof. Say $p = (\sigma, \langle X_\nu : \nu \in I \rangle, C)$ and $\varphi_i(G) \equiv \exists x \psi_i(G, x)$ for each $i < 2$. For each $i < 2$, let $D_i \supseteq C$ be a Δ^0_2 set such that

$$\mathcal{U}^{\mathcal{M},I}_{D_i} = \mathcal{U}^{\mathcal{M},I}_{C} \cap \bigcap_{x \in \mathbb{N}, \rho \subseteq \bigcup_{\nu \in I} X_\nu} \{\langle Y_\nu : \nu \in I \rangle : (\sigma \cup \rho, \bigcup_{\nu \in I} Y_\nu) \mathrel{?\vdash} \psi_i(G, x)\}$$

Let $D \subseteq \mathbb{N}^2$ be a Δ^0_2 set such that $\mathcal{U}^{\mathcal{M},2\times I}_{D} = \mathcal{U}^{\mathcal{M},I}_{D_0} \times \mathcal{U}^{\mathcal{M},I}_{D_1}$. For each $(i, \nu) \in 2 \times I$, let $Y_{(i,\nu)} = X_\nu$. Then $q = (\sigma, \langle Y_{(i,\nu)} : (i, \nu) \in 2 \times I \rangle, D)$ is the desired extension of p. ∎

Exercise 10.5.11. Fix a uniformly computable sequence $\vec{g} = g_0, g_1, \dots$ of functions of type $\mathbb{N} \to \mathbb{N}$. Use product Mathias forcing to show that there exists an infinite thin $\vec{g}$-cohesive[28] set $C \subseteq \mathbb{N}$ such that C' is not of PA degree over $\emptyset'$. ★

28: Recall that an infinite set $C \subseteq \mathbb{N}$ is *thin $\vec{g}$-cohesive* if for every $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $C \setminus [0, k]$ is g_n-thin.

10.6 Pigeonhole principle

As explained in Section 3.4, Ramsey's theorem for pairs can be decomposed into the cohesiveness principle (COH) and the pigeonhole principle for Δ^0_2 instances ($(\mathsf{RT}^1_2)'$). It is natural to wonder whether this decomposition is strict, that is, whether COH implies $(\mathsf{RT}^1_2)'$ or $(\mathsf{RT}^1_2)'$ implies COH over RCA_0. The former question can easily be answered negatively by a first-jump control argument (see Hirschfeldt et al. [47]), while the former was a long-standing open question. It was first answered negatively by Chong, Slaman and Yang [29] using non-standard models.[29] More recently, Monin and Patey [78] proved that $(\mathsf{RT}^1_2)'$ does not imply COH over ω-models, by proving that $(\mathsf{RT}^1_2)'$ admits jump PA avoidance using a variant of the product Mathias forcing.

29: Chong, Slaman and Yang [29] constructed a non-standard model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + (\mathsf{RT}^1_2)'$ in which every set is of low degree (from the viewpoint of the model). Such a model cannot be standard, as Downey et al. [28] constructed a Δ^0_2 set with no infinite subset of it or its complement of low degree.

Theorem 10.6.1 (Monin and Patey [78])
Let $A \subseteq \mathbb{N}$ be a Δ^0_2 set. There exists an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that H' is not of PA degree over $\emptyset'$.[30]

30: The statement relativizes as follows: For every set Z such that Z' is not of PA degree over $\emptyset'$, and every $\Delta^0_2(Z)$ set A, there exists an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that $(H \oplus Z)'$ is not of PA degree over $\emptyset'$.

The natural attempt would be to adapt product Mathias forcing to construct solutions to $(\mathsf{RT}^1_2)'$, the same way Mathias forcing was adapted in the proof of Theorem 3.4.6. Fix a Δ^0_2 set A and a countable Scott ideal $\mathcal{M}$, coded by a set M of low degree. Let $A_0 = A$ and $A_1 = \overline{A}$, and consider the notion of forcing $(\mathbb{Q}, \leq)$ whose conditions are tuples of the form $(\sigma_0, \sigma_1, \langle X_\nu : \nu \in I \rangle, C)$, where $(\sigma_i, \langle X_\nu : \nu \in I \rangle, C)$ is a product Mathias forcing condition for each $i < 2$, and $\sigma_i \subseteq A_i$. Condition extension is defined accordingly. One must really think of such notion of a condition as two product Mathias conditions sharing the reservoirs and notions of largeness. Any filter $\mathcal{F}$ induces two sets $G_{\mathcal{F},0}$ and $G_{\mathcal{F},1}$, defined by $G_{\mathcal{F},i} = \bigcup \{\sigma_i : (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I \rangle, C) \in \mathcal{F}\}$.

Syntactic forcing relation. The syntactic forcing relation for Π^0_2-formulas is a straightforward adaptation of Definition 10.5.5. The only difference comes from the structural constraint of homogeneity, which requires ρ to be included in A_i.

Definition 10.6.2. Let $p = (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I \rangle, C)$ be a condition, $i < 2$ be a part and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. Let $p \Vdash \varphi(G_i)$ hold if for every $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$ and every $x \in \mathbb{N}$,

$$\mathcal{U}_C^{\mathcal{M},I} \subseteq \{\langle Y_\nu : \nu \in I\rangle : (\sigma_i \cup \rho, \bigcup_{\nu \in I} Y_\nu) ?\vdash \psi(G, x)\}$$

The proof of stability of the syntactic forcing relation under condition extension is left as an exercise.

Exercise 10.6.3. Adapt the proof of Lemma 10.5.6 to show that if p is a condition and $\varphi(G)$ is a Π^0_2-formula such that $p \Vdash \varphi(G_i)$ for some $i < 2$, then for every extension $q \leq p$, $q \Vdash \varphi(G_i)$. ★

Contrary to product Mathias forcing, this syntactic forcing relation does not entail the semantic one in general, because the stem must be a subset of A_i. One must therefore introduce a notion of validity as in Theorem 9.7.1.

Definition 10.6.4. We say that part i of $(\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ is *valid* if $\langle X_\nu \cap A_i : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$. Part i of a filter $\mathcal{F}$ is *valid* if part i is valid for every condition in $\mathcal{F}$.[31] ◊

31: One could have strengthened the definition of validity by requiring that $\mathcal{U}_C^{\mathcal{M},I} \cap \mathcal{L}_{\langle X_\nu \cap A_i : \nu \in I\rangle}$ is large. Indeed, Lemma 10.6.13 already proves the existence of a valid part in the stronger sense.

A new problem arises in the realm of product spaces: if $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is large, there is not necessarily some $i < 2$ such that $(A_i, A_i) \in \mathcal{A}$. It follows that every condition does not necessarily have a valid side. We shall leave this issue for now. The notion of validity is designed so that the following lemma holds.

Lemma 10.6.5 (Monin and Patey [78]). Let $p = (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ be a condition with valid part i and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_2 formula. If $p \Vdash \varphi(G_i)$, then for every $x \in \mathbb{N}$, there is an extension $q \leq p$ forcing $\psi(G_i, x)$. ★

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G_i)$, then in particular, for $\rho = \emptyset$,

$$\mathcal{U}_C^{\mathcal{M},I} \subseteq \{\langle Y_\nu : \nu \in I\rangle : (\sigma_i \cup \rho, \bigcup_{\nu \in I} Y_\nu) ?\vdash \psi(G, x)\}$$

By validity of part i of p, $\langle X_\nu \cap A_i : \nu \in I\rangle \in \mathcal{U}_C^{\mathcal{M},I}$, so $(\sigma_i, A_i \cap \bigcup_{\nu \in I} X_\nu) ?\vdash \psi(G, x)$. Let $\mu \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$ be such that $\psi(\sigma_i \cup \mu, x)$ holds. Let $\tau_i = \sigma_i \cup \mu$, $\tau_{1-i} = \sigma_{1-i}$, and for each $\nu \in I$, let $Y_\nu = X_\nu \setminus \{0, \dots, \max \mu\}$. Then $(\tau_0, \tau_1, \langle Y_\nu : \nu \in I\rangle, C)$ is an extension forcing $\psi(G_i, x)$. ■

Together with Exercise 10.6.3, the previous lemma implies that, for every sufficiently generic filter $\mathcal{F}$ with valid part i, if $p \Vdash \varphi(G_i)$ for some $p \in \mathcal{F}$, then p forces $\varphi(G_i)$.[32]

32: This statement might be vacuous as the existence of a sufficiently generic filter with a valid part is not clear.

Exercise 10.6.6 (Monin and Patey [78]). Let $p, q \in \mathbb{Q}$ be two conditions such that $q \leq p$. Show that if part i of q is valid, then so is part i of p. ★

The following exercise implies that for every sufficiently generic filter $\mathcal{F}$ with valid part i, $G_{\mathcal{F},i}$ is infinite.

Exercise 10.6.7 (Monin and Patey [78]). Let $p = (\sigma_0, \sigma_1, \langle X_\nu : \nu \in I\rangle, C)$ be a condition. Show that if part i of p is valid, then there is an extension $q = (\tau_0, \tau_1, \langle Y_\nu : \nu \in I\rangle, D)$ such that $\operatorname{card} \tau_i > \operatorname{card} \sigma_i$.[33] ★

33: Note that the extension has the same index set as the condition. This will be useful in combination with Lemma 10.6.14.

Index sets. As mentioned, if $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is large, there is not necessarily some $i < 2$ such that $(A_i, A_i) \in \mathcal{A}$. On the other hand, if $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, by the pigeonhole principle, there is some $i < 2$ and some $a < b < 3$ such that $(A_i, A_i) \in \pi_{\{a,b\}}(\mathcal{A})$. We shall therefore work with a more complex notion of condition over a larger index set, representing multiple $\mathbb{Q}$-conditions by projections. To do this, we shall define an infinite sequence of big index sets $\mathcal{I}_0 \geq \mathcal{I}_1 \geq \dots$ where $\mathcal{I}_n$ contains only finite sequences of length n, satisfying some appropriate Ramsey property on its index subsets.

Example 10.6.8. Say $\mathcal{I}_1 = \{0, 1, 2\}$ and let $I \lhd \mathcal{I}_1$ if $I \subseteq \mathcal{I}_1$ and $\operatorname{card} I = 2$. By the pigeonhole principle, for every 2-partition of $\mathcal{I}_1$, there is some monochromatic $I \lhd \mathcal{I}_1$.

We now generalize the previous example for argument for every n. Let $u_0, u_1, \dots$ be inductively defined by $u_0 = 1$ and $u_{n+1} = \binom{2u_n+1}{2} u_n$.

Definition 10.6.9. Given $n \in \mathbb{N}$, the *meta n-index set* $\mathcal{I}_n$ is defined inductively defined as follows: $\mathcal{I}_0 = \{\epsilon\}$, and

$$\mathcal{I}_{n+1} = (2u_n + 1) \times \mathcal{I}_n = \{x \cdot \nu : x \leq 2u_n \wedge \nu \in I_n\}$$

Technically, meta index sets are nothing but index sets. However, they differ by their role, as they should be thought of families of index sets $\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\}$, for some relation $\lhd$ that we define now:

Definition 10.6.10. Let $\lhd$ be the smallest relation satisfying $\{\epsilon\} \lhd \mathcal{I}_0$, and if $I \lhd \mathcal{I}_n$ and $x < y \leq 2u_n$, then $(x \cdot I \cup y \cdot I) \lhd \mathcal{I}_{n+1}$.[34] ◊

34: The notation $x \cdot I$ means $\{x \cdot \nu : \nu \in I\}$.

Note that if $I \lhd \mathcal{I}_n$, then $I \subseteq \mathcal{I}_n$. Moreover, if $J \lhd \mathcal{I}_{n+1}$, then there is some $I \lhd \mathcal{I}_n$ such that $J \leq I$. An easy counting argument yields the following lemma.

Lemma 10.6.11 (Monin and Patey [78]). For every $n \in \mathbb{N}$, $\operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = u_n$. ★

Proof. By induction over n. For $n = 0$, there is exactly one $I \subseteq \mathcal{I}_0$ such that $I \lhd \mathcal{I}_0$, namely, $\{\epsilon\}$, and $u_0 = 1$. Suppose $\operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = u_n$. Then $\operatorname{card}\{J \subseteq \mathcal{I}_{n+1} : J \lhd \mathcal{I}_{n+1}\} = \binom{2u_n+1}{2} \operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = \binom{2u_n+1}{2} u_n = u_{n+1}$. ■

The following lemma states that the meta index sets satisfy some desired Ramsey property. It will play an essential role in proving that every meta-condition contains a branch with a valid side.

Lemma 10.6.12 (Monin and Patey [78]). For every $n \in \mathbb{N}$ and every 2-cover $B_0 \cup B_1 = \mathcal{I}_n$, there is some $I \lhd \mathcal{I}_n$ and some $i < 2$ such that $I \subseteq B_i$. ★

Proof. By induction on n. The case $n = 0$ is trivial. Assume it holds for n. Let $B_0 \cup B_1 = \mathcal{I}_{n+1}$. For every $x \leq 2u_n$ and $i < 2$, let $B_{x,i} = \{\nu : x \cdot \nu \in B_i\}$. Note that for each $x \leq 2u_n$, $B_{x,0} \cup B_{x,1} = \mathcal{I}_n$, so by induction hypothesis, there is some $I_x \lhd \mathcal{I}_n$ and $i_x < 2$ such that $I_x \subseteq B_{x,i_x}$. By Lemma 10.6.11, $\operatorname{card}\{I \subseteq \mathcal{I}_n : I \lhd \mathcal{I}_n\} = u_n$, so by the pigeonhole principle, there is some $x < y \leq 2u_n$, some $I \lhd \mathcal{I}_n$ and $i < 2$ such that $I = I_x = I_y$ and $i = i_x = i_y$. Letting $J = x \cdot I \cup y \cdot I$, we have $J \lhd \mathcal{I}_{n+1}$ and $J \subseteq B_i$. ■

Meta-conditions. We now define a more complex notion of forcing $(\mathbb{P}, \leq)$, whose conditions are of the form $(\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ for some $n \in \mathbb{N}$, where

1. $\sigma_i^I \subseteq A_i$ for each $i < 2$ and $I \lhd \mathcal{I}_n$;
2. $(\sigma_i^I, \bigcup_{\nu \in I} X_\nu)$ is a Mathias condition for each $i < 2$ and $I \lhd \mathcal{I}_n$;
3. $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \subseteq \mathcal{I}_n \to 2^{\mathbb{N}}$ is a large sub-class of $\mathcal{L}_{\langle X_\nu : \nu \in \mathcal{I}_n\rangle}$;
4. $\langle X_\nu : \nu \in \mathcal{I}_n\rangle \in \mathcal{M}$ and C is Δ_2^0.

We write $\mathbb{P}_n$ for the set of meta-conditions indexed by $\mathcal{I}_n$, and $\mathbb{Q}_n$ for the set of conditions indexed by some $I \lhd \mathcal{I}_n$. One should really think of a meta-condition $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ as u_n-many parallel $\mathbb{Q}$-conditions $c^{[I]} = (\sigma_0^I, \sigma_1^I, \langle X_\nu : \nu \in I\rangle, C^I)$ for each $I \lhd \mathcal{I}_n$, where $C^I \subseteq \mathbb{N}^2$ is such that $\mathcal{U}_{C^I}^{\mathcal{M}, I} = \pi_I(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$. We shall refer to $c^{[I]}$ as *branches* of c. The notion of meta-condition has been design so that it satisfies the following validity lemma:

Lemma 10.6.13 (Monin and Patey [78]). For every meta-condition $c \in \mathbb{P}_n$, there is some $I \lhd \mathcal{I}_n$ such that $c^{[I]}$ admits a valid part. ★

Proof. Say $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$. Since $A_0 \cup A_1 = \mathbb{N}$ and by Proposition 10.4.6, $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$ is large, there is some $j : \mathcal{I}_n \to 2$ such that $\langle A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle \in \mathcal{L}(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$. Thus, $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu : \nu \in \mathcal{I}_n\rangle} \cap \mathcal{L}_{\langle A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle}$ is large, so by Lemma 10.4.5, $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle}$ is large.

Let $B_i = \{\nu \in \mathcal{I}_n : j(\nu) = i\}$ for each $i < 2$. Since $B_0 \cup B_1 = \mathcal{I}_n$, then by Lemma 10.6.12, there is some $I \lhd \mathcal{I}_n$ and some $i < 2$ such that $I \subseteq B_i$. Since $\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap A_{j(\nu)} : \nu \in \mathcal{I}_n\rangle}$ is large, then $\langle X_\nu \cap A_{j(\nu)} : \nu \in I\rangle \in \pi_I(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$. As $I \subseteq B_i$, $\langle X_\nu \cap A_i : \nu \in I\rangle = \langle X_\nu \cap A_{j(\nu)} : \nu \in I\rangle \in \pi_I(\mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n})$, so part i of the $\mathbb{Q}$-condition $c^{[I]}$ is valid. ■

A meta-condition $d = (\langle \tau_0^J, \tau_1^J : J \lhd \mathcal{I}_m\rangle, \langle Y_\mu : \mu \in \mathcal{I}_m\rangle, D)$ *extends* $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ if $m \geq n$, and for every $J \lhd \mathcal{I}_m$, letting $I \lhd \mathcal{I}_n$ be the unique index set such that $J \leq I$, $d^{[J]} \leq c^{[I]}$ as $\mathbb{Q}$-conditions. The following commutative diagram will be very useful to propagate lemmas from $(\mathbb{Q}, \leq)$ forcing to $(\mathbb{P}, \leq)$ forcing.

Lemma 10.6.14 (Monin and Patey [78]). Fix a meta-condition $c \in \mathbb{P}_n$ and $I \lhd \mathcal{I}_n$. For every $\mathbb{Q}_n$-condition $q \leq c^{[I]}$, there is a meta-condition $d \leq c$ in $\mathbb{P}_n$ such that $d^{[I]} = q$.[35] ★

35: One must be a bit careful when using this lemma: it only states the existence of a commutative diagram for a fixed n.

Proof. Say $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n\rangle, \langle X_\nu : \nu \in \mathcal{I}_n\rangle, C)$ and $q = (\tau_0^I, \tau_1^I, \langle Y_\nu : \nu \in I\rangle, D^I)$. By Lemma 10.4.12, there is a Δ_2^0 set $D \supseteq C$ such that $\mathcal{U}_D^{\mathcal{M}, \mathcal{I}_n} \subseteq \mathcal{U}_C^{\mathcal{M}, \mathcal{I}_n}$ is a large class and $\pi_I(\mathcal{U}_D^{\mathcal{M}, \mathcal{I}_n}) = \mathcal{U}_{D^I}^{\mathcal{M}, I}$. For every $J \lhd \mathcal{I}_n$ with $J \neq \mathcal{I}$ and $i < 2$, let $\tau_i^J = \sigma_i^J$. For every $\nu \in \mathcal{I}_n \setminus I$, let $Y_\nu = X_\nu$. The meta-condition $d = (\langle \tau_0^I, \tau_1^I : I \lhd \mathcal{I}_n\rangle, \langle Y_\nu : \nu \in \mathcal{I}_n\rangle, D)$ is an extension of c such that $d^{[I]} = q$. ■

Forcing question for Σ_2^0-formulas. A meta-condition representing multiple $\mathbb{Q}$-conditions, requirements must be forced on every branch of the meta-condition.

Definition 10.6.15. Given a requirement $\mathcal{R}(G)$, a part $i < 2$ and a meta-condition $c \in \mathbb{P}_n$, let $\mathcal{R}(c, i)$ be the set of all $I \lhd \mathcal{I}_n$ such that $c^{[I]}$ does not force $\mathcal{R}(G_i)$.[36] ◇

36: This definition and the following explanation is slightly approximative in the sense given to "forcing". In our setting, a positive answer to the forcing question yields an extension strongly forcing the Σ^0_2 formula, while the witness of a negative answer syntactically forces its negation. As seen, the syntactical forcing relation implies the semantical one only on valid parts. A requirement being often a disjunction between wrong computation and partiality, the formal sense given to "forcing" actually depends on the side of the disjunction. We will therefore give a more formal sense in the case of jump PA avoidance in Definition 10.6.20.

One could define a non-disjunctive Σ^0_2-preserving forcing question for Σ^0_2-formulas on $\mathbb{Q}$-conditions which would meet its specifications, and witness the answer by an extension with the same index set. For a single Σ^0_2-formula, one could then use Lemma 10.6.14 to define a finite decreasing sequence of meta-conditions $c = c_0 \geq c_1 \geq \cdots \geq c_k$ such that $\mathcal{R}(c_{s+1}, i) \subsetneq \mathcal{R}(c_s, i)$, eventually yielding $\mathcal{R}(c_k, i) = \emptyset$ for each $i < 2$, thus forcing the requirement on every part of every branch.

However, in order to obtain jump PA avoidance, one must design a Π^0_2-merging forcing question. The forcing question for Σ^0_2-formulas on $\mathbb{Q}$-conditions is Π^0_2-merging, but the witnessed extension is obtained by considering the cartesian product of multiple large classes, hence increasing the index set. Trying to adapt Lemma 10.6.14 to increasing index sets would yield an extension d with more branches. Then $\mathcal{R}(d, i)$ might be larger than $\mathcal{R}(c, i)$, which would not yield a progress towards forcing the requirements on all the branches.

We shall therefore directly design a forcing question for Σ^0_2-formulas on meta-conditions c, parameterized by the set $\mathcal{R}(c, i)$, with the following property: either there exists an extension d with the same index set forcing $\mathcal{R}(G_i)$ on some branch $I \in \mathcal{R}(c, i)$, yielding $\mathcal{R}(d, i) \subseteq \mathcal{R}(c, i) \setminus \{I\}$, or there exists an extension $d \in \mathbb{P}_m$ with a larger index set, but forcing $\mathcal{R}(G_i)$ on every branch $J \lhd \mathcal{I}_m$ such that $J \leq I$ for some $I \in \mathcal{R}(c, i)$, so $\mathcal{R}(d, i) = \emptyset$.[37]

37: The idea was already present in the proof of Liu's theorem [12], who designed a forcing question for Σ^0_1-formulas with the same features. It is also present in Theorem 5.3.3.

Definition 10.6.16. Let $c = (\langle \sigma^I_0, \sigma^I_1 : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C)$ be a meta-condition, $H \subseteq \{I \lhd \mathcal{I}_n\}$, $i < 2$ and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_2 formula. Let $c \mathrel{?\vdash}_H \varphi(G_i)$ hold if the following class is not large:

$$\mathcal{U}^{\mathcal{M}, \mathcal{I}_n}_C \cap \bigcap_{\substack{I \in H, x \in \mathbb{N}, \\ \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu}} \{\langle Z_\mu : \mu \in \mathcal{I}_n \rangle : (\sigma_i \cup \rho, \bigcup_{\nu \in I} Z_\nu) \mathrel{?\nvdash} \psi(G, x)\}$$

Note that the relation in Σ^0_2 uniformly in H, i and $\varphi(G)$. The following lemma states that the forcing question meets its specifications and the witnessed extension has the same index set.

Lemma 10.6.17 (Monin and Patey [78]). Let $c \in \mathbb{P}_n$ be a meta-condition, $H \subseteq \{I \lhd \mathcal{I}_n\}$, $i < 2$, and $\varphi(G)$ be a Σ^0_2 formula.

1. If $c \mathrel{?\vdash}_H \varphi(G_i)$, then there is an extension $d \leq c$ in $\mathbb{P}_n$ and some $I \in H$ such that $d^{[I]}$ strongly forces[38] $\varphi(G_i)$.
2. If $c \mathrel{?\nvdash}_H \varphi(G_i)$, then there is an extension $d \leq c$ in $\mathbb{P}_n$ such that for every $I \in H$, $d^{[I]} \Vdash \neg\varphi(G_i)$. ★

38: Recall that given a notion of forcing $(\mathbb{P}, \leq)$, a condition p *strongly forces* a formula $\varphi(G)$ if the formula holds for *every* filter containing p.

Proof. Say $\varphi(G) \equiv \exists x \psi(G, x)$ and $c = (\langle \sigma^I_0, \sigma^I_1 : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C)$. For every $I \in H$, $x \in \mathbb{N}$ and $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$, let

$$\mathcal{A}_{I,x,\rho} = \{\langle Z_\mu : \mu \in \mathcal{I}_n \rangle : (\sigma^I_i \cup \rho, \bigcup_{\nu \in I} Z_\nu) \mathrel{?\nvdash} \psi(G, x)\}$$

Suppose first $c \mathrel{?\vdash}_H \varphi(G_i)$. Then there is some finite set $F \subseteq C$ and some $t \in$

$\mathbb{N}$ such that the following class is not large:

$$\mathcal{B} = \mathcal{U}_F^{\mathcal{M},\mathcal{I}_n} \bigcap_{I \in H, x<t, \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu \upharpoonright t} \mathcal{A}_{I,x,\rho}$$

Since $\mathcal{B}$ is $\Sigma^0_1(\mathcal{M})$ and $\mathcal{M}$ is a Scott ideal, there is some $k \in \mathbb{N}$ and a k-cover $Z_0 \cup \dots \cup Z_{k-1} = \mathbb{N}$ in $\mathcal{M}$ such that for every $j : \mathcal{I}_n \to k$, $\langle Z_{j(\nu)} : \nu \in I \rangle \notin \mathcal{B}$. By Proposition 10.4.6, $\mathcal{L}(\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n})$ is large, so there is some $j : \mathcal{I}_n \to k$ such that $\langle Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle \in \mathcal{L}(\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n})$. In particular, $\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu : \nu \in \mathcal{I}_n \rangle} \cap \mathcal{L}_{\langle Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle}$ is large, so by Lemma 10.4.5, so is $\mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle}$. In particular, $\langle X_\nu \cap Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle \in \mathcal{U}_F^{\mathcal{M},\mathcal{I}_n}$, so there is some $I \in H$, some $x < t$ and some $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu \upharpoonright t$ such that $\langle X_\nu \cap Z_{j(\nu)} \rangle \notin \mathcal{A}_{I,x,\rho}$. Unfolding the definition of $\mathcal{A}_{I,x,\rho}$, $(\sigma_i^I \cup \rho, \bigcup_{\nu \in I} Z_{j(\nu)}) \operatorname{?\nvdash} \psi(G, x)$, so $(\sigma_i^I \cup \rho, \bigcup_{\nu \in I} Z_{j(\nu)})$ strongly forces $\psi(G, x)$, hence strongly forces $\varphi(G)$. Let $D \subseteq C$ be a Δ^0_2 set such that $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_n} = \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \mathcal{L}_{\langle X_\nu \cap Z_{j(\nu)} : \nu \in \mathcal{I}_n \rangle}$. For every $\nu \in \mathcal{I}_n$, let $Y_\nu = (X_\nu \cap Z_{j(\nu)}) \setminus \{0, \dots, t\}$. Let $\tau_i^I = \sigma_I^i \cup \rho$ and $\tau_{1-i}^I = \sigma_{1-i}^I$. For every $J \lhd \mathcal{I}_n$ with $J \neq I$, let $\tau_0^J = \sigma_0^J$ and $\tau_1^J = \sigma_1^J$. The meta-condition $d = (\langle \tau_0^J, \tau_1^J : J \lhd \mathcal{I}_n \rangle, \langle Y_\nu : \nu \in \mathcal{I}_n \rangle, D)$ is an extension of c such that $d^{[I]}$ strongly forces $\varphi(G_i)$.

Suppose now $c \operatorname{?\nvdash}_H \varphi(G_i)$. Let $D \supseteq C$ be a Δ^0_2 set such that

$$\mathcal{U}_D^{\mathcal{M},\mathcal{I}_n} = \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \bigcap_{I \in H, x \in \mathbb{N}, \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu} \mathcal{A}_{I,x,\rho}$$

The meta-condition $d = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, D)$ is an extension of c such that $d^{[I]} \Vdash \neg\varphi(G_i)$ for every $I \in H$. ■

Recall from Section 5.2 that given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *weakly* Γ*-merging*[39] if for every $p \in \mathbb{P}$, there is some $k \in \mathbb{N}$ such that for every k-tuple of Γ-formulas $\varphi_0(G), \dots, \varphi_{k-1}(G)$, if $p \operatorname{?\vdash} \varphi_i(G)$ for each $i < k$, then there is an extension $q \leq p$ and two indices $i < j < k$ such that q forces $\varphi_i(G) \wedge \varphi_j(G)$. Thanks to Liu's notion of valuation (see Section 5.2), if a notion of forcing admits a Σ^0_2-preserving and weakly Π^0_2-merging forcing question for Σ^0_2-formulas, then every sufficiently generic filter yields a set whose jump is not of PA degree over $\emptyset'$.

39: Note that in the definition of a weakly Γ-merging forcing question, the parameter k might depend on the condition p.

This notion of weak Π^0_2-merging forcing question does not apply directly on meta-conditions due to the branching and disjunctive nature of meta-conditions, but the same combinatorial argument holds, with the necessary adaptation. In particular, the following lemma informally states that the forcing question on meta-conditions for Σ^0_2-formulas is weakly Π^0_2-merging.

Lemma 10.6.18 (Monin and Patey [78]). Let $c \in \mathbb{P}_n$ be a meta-condition, $H \subseteq \{I \lhd \mathcal{I}_n\}$, $i < 2$ and $\varphi_0(G), \dots, \varphi_{2u_n}(G)$ be $2u_n + 1$ many Σ^0_2 formulas. Suppose that for every $s \leq 2u_n$, $c \operatorname{?\nvdash}_H \varphi_s(G_i)$. Then there is some extension $d \in \mathbb{P}_{n+1}$ such that for every $I \in H$ and every $J \lhd \mathcal{I}_{n+1}$ such that $J \leq I$, there are some $a < b \leq 2u_n$ such that

$$d^{[J]} \Vdash \neg\varphi_a(G_i) \quad \text{and} \quad d^{[J]} \Vdash \neg\varphi_b(G_i)$$

Proof. Say $c = (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C)$ and $\varphi_s(G) \equiv \exists x \psi_s(G, x)$ for each $s \leq 2u_n$. For every $s \leq 2u_n$, the following class is large:

$$\mathcal{A}_s = \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n} \cap \bigcap_{\substack{I \in H, x \in \mathbb{N}, \\ \rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu}} \{\langle Z_\mu : \mu \in \mathcal{I}_n \rangle : (\sigma_i^I \cup \rho, \bigcup_{\nu \in I} Z_\nu) \, {?\nVdash} \, \psi_s(G, x)\}$$

Let $D \subseteq \mathbb{N}^2$ be a Δ_2^0 set such that $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}} = \prod_{j \leq 2u_n} \mathcal{A}_s$. In particular, $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}}$ is large. For every $(j, \nu) \in \mathcal{I}_{n+1}$, let $Y_{(j,\nu)} = X_\nu$. For every $J \lhd \mathcal{I}_{n+1}$, let $\tau_0^J = \sigma_0^I$ and $\tau_1^J = \sigma_1^I$, where $I \lhd \mathcal{I}_n$ is the unique index set such that $J \leq I$. Note that $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}} \subseteq \mathcal{L}_{\langle Y_\mu : \mu \in \mathcal{I}_{n+1} \rangle}$ and $\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}} \leq \mathcal{U}_C^{\mathcal{M},\mathcal{I}_n}$. The meta-condition $d = (\langle \tau_0^J, \tau_1^J : J \lhd \mathcal{I}_{n+1} \rangle, \langle Y_\mu : \mu \in \mathcal{I}_{n+1} \rangle, D)$ is an extension of c.

Fix $I \in H$ and $J \lhd \mathcal{I}_{n+1}$ such that $J \leq I$. Let $a < b \leq 2u_n$ be such that $J = \{a, b\} \times I$. We claim that $d^{[J]} \Vdash \neg \varphi_a(G_i)$ and $d^{[J]} \Vdash \neg \varphi_b(G_i)$. We prove the former, the latter being symmetric. Fix some $x \in \mathbb{N}$ and $\rho \subseteq A_i \cap \bigcup_{\mu \in J} Y_\mu$. In particular, $\rho \subseteq A_i \cap \bigcup_{\nu \in I} X_\nu$. Fix $\langle Z_\mu : \mu \in J \rangle \in \pi_J(\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}})$. In particular,

$$\langle Z_{(a,\nu)} : \nu \in I \rangle \in \mathcal{A}_a \subseteq \{\langle Z_\mu : \mu \in \mathcal{I}_n \rangle : (\sigma_i^I \cup \rho, \bigcup_{\nu \in I} Z_\nu) \, {?\nVdash} \, \psi_a(G, x)\}$$

so $(\sigma_i^I \cup \rho, \bigcup_{\nu \in I} Z_{(a,\nu)}) \, {?\nVdash} \, \psi_a(G, x)$. As $\sigma_i^I = \tau_i^J$ and $\bigcup_{\nu \in I} Z_{(a,\nu)} \subseteq \bigcup_{\mu \in J} Z_\mu$, then $(\tau_i^J \cup \rho, \bigcup_{\mu \in J} Z_\mu) \, {?\nVdash} \, \psi_a(G, x)$. Thus, for every $x \in \mathbb{N}$ and $\rho \subseteq A_i \cap \bigcup_{\mu \in J} Y_\mu$, $\pi_J(\mathcal{U}_D^{\mathcal{M},\mathcal{I}_{n+1}}) \subseteq \{\langle Z_\mu : \mu \in J \rangle : (\tau_i^J \cup \rho, \bigcup_{\mu \in J} Z_\mu) \, {?\nVdash} \, \psi_a(G, x)\}$, so $d^{[J]} \Vdash \neg \varphi_a(G_i)$. ■

Diagonalization. We now use the forcing question for Σ_2^0-formulas to prove the appropriate diagonalization lemmas in the context of jump PA avoidance. Because of the weakly Π_2^0-merging nature of the forcing question for meta-conditions, one needs to use the valuation machinery introduced by Liu [12].

Recall from Section 5.2 that a *valuation* is a partial $\{0, 1\}$-valued function $h \subseteq \mathbb{N} \to 2$. A valuation is finite if it has finite support, that is, $\operatorname{dom} h$ is finite. A valuation h is Z*-correct* if for every $n \in \operatorname{dom} h$, $\Phi_n^Z(n) \downarrow \neq h(n)$. Two valuations f and h are *compatible* if for every $n \in \operatorname{dom} f \cap \operatorname{dom} h$, $f(n) = h(n)$. The following lemma is a relativization of Lemma 5.2.3.

Lemma 10.6.19 (Liu [12]). Fix a set Z. Let U be a Z-c.e. set of finite valuations. Either U contains a Z-correct[40] valuation, or for every $k \in \mathbb{N}$, there are k pairwise incompatible finite valuations outside of U. ★

40: Note that the appropriate relativization of Lemma 5.2.3 requires to relativize the notion of correctness, as it is a computability-theoretic property.

For every $e \in \mathbb{N}$, let $\mathcal{R}_e(G)$ be the requirement "either $\Phi_e^{G'}$ is partial, or $\Phi_e^{G'}(x) \downarrow = \Phi_x^{\emptyset'}(x)$ for some $x \in \mathbb{N}$." As mentioned in a note next to Definition 10.6.15, we overload the forcing relation for the requirement $\mathcal{R}_e(G)$.

> **Definition 10.6.20.** Given a $\mathbb{Q}$-condition p, some index $e \in \mathbb{N}$ and a part $i < 2$, we say that p *forces* $\mathcal{R}_e(G_i)$ if
>
> 1. either p strongly forces "$\Phi_e^{G_i'}$ is incompatible with h" for a $\emptyset'$-correct valuation h,
> 2. or $p \Vdash$ "$\Phi_e^{G_i'}$ is compatible with h_s" for two incompatible valuations h_0, h_1.[41] ◊

41: The statement "$\Phi_e^{G'}$ is incompatible with h" is $\Sigma_2^0(G)$, as it is equivalent to $\exists x \Phi_e^{G'}(x) \downarrow \neq h(x)$.

According to Definition 10.6.15, given a meta-condition $c \in \mathbb{P}_n$ we write $\mathcal{R}_e(c, i)$ for the set of index sets $I \lhd \mathcal{I}_n$ such that $c^{[I]}$ does not force $\mathcal{R}_e(G_i)$.

Lemma 10.6.21 (Monin and Patey [78]). For every meta-condition c, every part $i < 2$ and index $e \in \mathbb{N}$ such that $\mathcal{R}_e(c, i) \neq \emptyset$, there is an extension $d \leq c$ such that $\operatorname{card} \mathcal{R}_e(d, i) < \operatorname{card} \mathcal{R}_e(c, i)$. ★

Proof. Let $H = \mathcal{R}_e(c, i)$, and let U be the set of all valuations h such that $c \mathrel{?{\vdash}_H}$ "$\Phi_e^{G'_i}$ is incompatible with h". Note that the set U is $\emptyset'$-c.e., so by Lemma 10.6.19, we have two cases. Case 1: $h \in U$ for some $\emptyset'$-correct valuation h. Then, by Lemma 10.6.17, there is an extension $d \leq c$ in $\mathbb{P}_n$ and some $I \in H$ such that $d^{[I]}$ strongly forces $\Phi_e^{G'_i}$ to be incompatible with h. In particular, $\mathcal{R}_e(d, i) \subsetneq \mathcal{R}_e(c, i)$, hence $\operatorname{card} \mathcal{R}_e(d, i) < \operatorname{card} \mathcal{R}_e(c, i)$. Case 2: $h_0, \dots, h_{2u_n} \notin U$ for $2u_n + 1$ pairwise incompatible valuations. By Lemma 10.6.18, there is an extension $d \leq c$ in $\mathbb{P}_{n+1}$ such that for every $I \in H$ and every $J \lhd \mathcal{I}_{n+1}$ such that $J \leq I$, there are some $a < b \leq 2u_n$ such that $d^{[J]} \Vdash$ "$\Phi_e^{G'_i}$ is compatible with h_a" and $d^{[J]} \Vdash$ "$\Phi_e^{G'_i}$ is compatible with h_b", hence $d^{[J]}$ forces $\mathcal{R}_e(G_i)$. It follows that $\mathcal{R}_e(d, i) = \emptyset$, so $\operatorname{card} \mathcal{R}_e(d, i) < \operatorname{card} \mathcal{R}_e(c, i)$. ■

We say that a meta-condition $c \in \mathbb{P}_n$ *forces* $\mathcal{R}_e(G)$ if $c^{[I]}$ forces $\mathcal{R}_e(G_i)$ for every $I \lhd \mathcal{I}_n$ and $i < 2$.

Lemma 10.6.22 (Monin and Patey [78]). For every meta-condition c and $e \in \mathbb{N}$, there is an extension $d \leq c$ forcing $\mathcal{R}_e(G)$. ★

Proof. Apply iteratively Lemma 10.6.21 to obtain a meta-condition $d_0 \leq c$ such that $\mathcal{R}_e(d_0, 0) = \emptyset$. Then, apply again iteratively Lemma 10.6.21 to obtain a meta-condition $d_1 \leq d_0$ such that $\mathcal{R}_e(d_1, 1) = \emptyset$. ■

Tree structure. The partial order of meta-conditions being countable, every $\mathbb{P}$-filter can be identified with an infinite decreasing sequence of meta-conditions $c_0 \geq c_1 \geq \dots$ Each meta-conditions represents multiple $\mathbb{Q}$-conditions, each of which admits two parts. By Lemma 10.6.13, every meta-condition admits a branch with a valid part, and by Exercise 10.6.6, the valid parts a upward-closed under the extension relation. The valid parts of $\mathbb{Q}$-conditions along a decreasing sequence of meta-conditions therefore naturally form a tree structure, motivating the following definition.

Definition 10.6.23. A *path* through a $\mathbb{P}$-filter $\mathcal{F}$ is a pair $\langle P, i \rangle$ where $i < 2$, such that

1. for every $n \in \mathbb{N}$, $P(n) \lhd \mathcal{I}_n$ such that $P(n+1) \leq P(n)$;
2. for every $c \in \mathcal{F} \cap \mathbb{P}_n$, part i of $c^{[P(n)]}$ is valid. ◊

By Lemma 10.6.13 and Exercise 10.6.6, every $\mathbb{P}$-filter admits a path. For every $\mathbb{P}$-filter $\mathcal{F}$ and every path $\langle P, i \rangle$, let

$$G_{\mathcal{F},P,i} = \bigcup \{\sigma_i^{P(n)} : (\langle \sigma_0^I, \sigma_1^I : I \lhd \mathcal{I}_n \rangle, \langle X_\nu : \nu \in \mathcal{I}_n \rangle, C) \in \mathcal{F}\}$$

If $\mathcal{F}$ is a sufficiently generic $\mathbb{P}$-filter and $\langle P, i \rangle$ is a path through $\mathcal{F}$, then $\mathcal{F}_P = \{c^{[P(n)]} : c \in \mathcal{F} \cap \mathbb{P}_n, n \in \mathbb{N}\}$ might not be a sufficiently generic $\mathbb{Q}$-filter. Thankfully, if a $\mathbb{Q}$-condition p strongly forces a Σ^0_1, a Π^0_2 or a Σ^0_2-formula, then the property holds for *every* $\mathbb{Q}$-filter containing p, with no consideration of genericity. The following lemma states that the syntactic forcing relation for Π^0_2-formulas holds along paths of every sufficiently generic $\mathbb{P}$-filter.

Lemma 10.6.24 (Monin and Patey [78]). Let $\mathcal{F}$ be a sufficiently generic $\mathbb{P}$-filter, and let $\langle P, i\rangle$ be a path through $\mathcal{F}$. Let $\varphi(G)$ be a Π^0_2-formula and $c \in \mathcal{F}$. If $c^{[P(n)]} \Vdash \varphi(G_i)$, then $\varphi(G_{\mathcal{F},P,i})$ holds. ⋆

Proof. Fix some $x \in \mathbb{N}$ and say $\varphi(G) \equiv \forall x \psi(G, x)$. Let $\mathcal{D}_x$ be the set of meta-conditions $d \leq c$ such that $d^{[I]}$ forces $\psi(G_i, x)$ for every branch $I \leq P(n)$ such that part i of $d^{[I]}$ is valid. By Exercise 10.6.3, Lemma 10.6.5 and Lemma 10.6.14, the set $\mathcal{D}_x$ is dense below c, so by genericity of $\mathcal{F}$, there is some $d \in \mathcal{D}_x \cap \mathcal{F}$. Say $d \in \mathcal{P}_m$. Since $P(m) \leq P(n)$ and part i of $d^{[I]}$ is valid, $d^{[P(m)]}$ forces $\psi(G_i, x)$, so $\psi(G_{\mathcal{F},P,i}, x)$ holds. Thus $\varphi(G_{\mathcal{F},P,i})$ holds.■

We are now ready to prove Theorem 10.6.1.

Proof of Theorem 10.6.1. Let $\mathcal{F}$ be a sufficiently generic $\mathbb{P}$-filter, and let $\langle P, i\rangle$ be a path through $\mathcal{F}$. By definition of a meta-condition, $G_{\mathcal{F},P,i} \subseteq A_i$. By Exercise 10.6.7 and Lemma 10.6.14, $G_{\mathcal{F},P,i}$ is infinite. By Lemma 10.6.22, for every $e \in \mathbb{N}$, the set of meta-conditions forcing $\mathcal{R}_e(G)$ is dense, hence there is some $d_e \in \mathbb{P} \cap \mathcal{F}$ such that d_e forces $\mathcal{R}_e(G)$. By Lemma 10.6.24, it follows that $\mathcal{R}_e(G_{\mathcal{F},P,i})$ holds for every $e \in \mathbb{N}$, so $G'_{\mathcal{F},P,i}$ is not of PA degree over $\emptyset'$. This completes the proof of Theorem 10.6.1. ■

10.7 Jump DNC avoidance

As mentioned in the introduction, jump DNC avoidance did not receive as much attention as jump PA avoidance since the DNC counterpart to COH did not occur naturally in reverse mathematics.

Exercise 10.7.1. Adapt the proof of Theorem 10.2.1 to show that for every sufficiently Cohen generic set G, G' is not of DNC degree over $\emptyset'$. ⋆

Exercise 10.7.2. Adapt the proof of Theorem 10.2.4 to show that given a non-computable set C and a non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there exists a member $G \in \mathcal{P}$ such that $C \not\leq_T G$ and G' is not of DNC degree over $\emptyset'$. ⋆

Recall from Section 5.8 that given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is *countably* Γ*-merging* if for every $p \in \mathbb{P}$ and every countable sequence of Γ-formulas $(\varphi_s(G))_{s\in\mathbb{N}}$, if $p \mathrel{?\vdash} \varphi_s(G)$ for each $s \in \mathbb{N}$, then there is an extension $q \leq p$ forcing $\forall s \varphi_s(G)$.

Exercise 10.7.3. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ^0_2-preserving, countably Π^0_2-merging forcing question. Adapt the proof of Theorem 5.8.4 to show that for every sufficiently generic filter $\mathcal{F}$, $G'_{\mathcal{F}}$ is not of DNC degree over $\emptyset'$.⋆

Both in the cases of Cohen forcing and WKL, we actually exploited a stronger feature of the forcing question for Σ^0_2-formulas. A forcing question for Σ^0_n-formulas is Π^0_n*-extremal* if for every Σ^0_n-formula φ and every condition $p \in \mathbb{P}$, if $p \mathrel{?\nvdash} \varphi(G)$, then p forces $\neg\varphi(G)$.

Exercise 10.7.4. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Π^0_n-extremal forcing question. Show that the forcing question is countably Π^0_n-merging. ⋆

The status of the pigeonhole principle with respect to DNC degrees is slightly different than PA degrees. First of all, contrary to PA degrees (see Theorem 5.4.3), for every set X, there exists an instance of RT^1_2 such that every solution is of DNC degree over X. Such instance is constructed thanks to the notion of effective immunity. Recall from Section 6.2 that given a function $h : \mathbb{N} \to \mathbb{N}$, an infinite set A is *h-immune* if for every c.e. set W_e such that $W_e \subseteq A$, then $\operatorname{card} W_e \leq h(e)$. An infinite set is *effectively immune* if it is h-immune for some computable function $h : \mathbb{N} \to \mathbb{N}$.

Proposition 10.7.5 (Hirschfeldt et al. [47]). For every set X, there is an X'-computable effectively bi-X-immune[42] set A. ⋆

42: The relativization of effective immunity has two parameters: a set A is Y-effectively X-immune if there is an Y-computable function $h : \mathbb{N} \to \mathbb{N}$ such that for every X-c.e. set W^X_e with $W^X_e \subseteq A$, then $\operatorname{card} W^X_e \leq h(e)$.

Proof. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by $h(e) = 3e + 2$. We build an h-X-immune set A by stages using an X'-computable construction. At stage e, assume $A \upharpoonright_e$ is defined, and $A(n)$ is defined for at most $2e$ other n's. Decide X'-computably whether W^X_e has at least $3e + 2$ many elements. If so, then there are at least two elements $n_0, n_1 \in W^X_e$ for which A has not yet been decided. Let $A(n_0) = 0$ and $A(n_1) = 1$. In any case, if $A(e)$ is not defined yet, let $A(e)$ be any value among 0 and 1. This completes the construction. ■

In particular, letting $X = \emptyset'$, there exists a Δ^0_3 instance of RT^1_2 such that every solution computes a DNC function over $\emptyset'$. This implies that RT^1_2 does not admit strong DNC avoidance, and *a fortiori* does not admit strong jump DNC avoidance.

Exercise 10.7.6. Use Proposition 5.7.2 to prove the existence, for every set X, of an X'-computable set A such that every infinite subset of A or of $\overline{A}$ is of DNC degree over X. ⋆

Of course, the pigeonhole principle being computably true, every Δ^0_2 instance of RT^1_2 admits a Δ^0_2 solution, hence a solution which is not of DNC degree over $\emptyset'$. The following question remains open:

Question 10.7.7. Is there a Δ^0_2 instance of RT^1_2 such that for every solution H, H' is of DNC degree over $\emptyset'$? ⋆

One would naturally want to adapt the proof of Theorem 10.6.1 and work with ω-product largeness to obtain a countably Π^0_2-merging forcing question for Σ^0_2-formulas. However, ω-product spaces do not behave as nicely as finite product spaces, leaving the question open.

Higher jump cone avoidance 11

Prerequisites: Chapters 2, 3 and 9

The conceptual gap from second to iterated jump control is not as significant as from first to second jump control. Indeed, the main difficulty comes from dealing with non-continuous functionals, which already occurs at the Σ^0_2 level. There is therefore often a natural generalization from second to all the levels of the arithmetic hierarchy.

New difficulties arise when trying to control the jump at transfinite levels. The arithmetic hierarchy extends to the hyperarithmetic hierarchy through iterations along computable ordinals. While the arithmetic hierarchy is indexed by integers, which are left unchanged when considering relativization to a generic set, the hyperarithmetic hierarchy is indexed by computable ordinals, which is a relative notion: the generic set might compute more ordinals, and therefore might have more levels in its relative hyperarithmetic hierarchy.

11.1 Context and motivation

The study of iterated jump control at the arithmetic and hyperarithmetic levels has two different motivations, both coming from reverse mathematics.

Arithmetic jump control. At the arithmetic level, arithmetic jump control is an essential tool in the study of Ramsey-type hierarchies. Consider for instance the rainbow Ramsey theorem, which is a particular case of the canonical Ramsey theorem of Erdős and Rado.

> **Definition 11.1.1.** A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is *k-bounded* if each color appears at most k times, that is, $|f^{-1}(c)| \leq k$ for every $c \in \mathbb{N}$. A set $H \subseteq \mathbb{N}$ is an *f-rainbow* if f is injective on $[H]^n$. The *rainbow Ramsey theorem* for n-tuples and k-bounds (RRT^n_k) states that every k-bounded coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ admits an infinite f-rainbow. ◊

As for Ramsey's theorem, the rainbow Ramsey theorem forms a hierarchy of statements based on the size n of the tuples. However, while RT^n_2 collapses and is equivalent to ACA_0 for $n \geq 3$, Wang [15] proved that RRT^n_2 is strictly weaker than ACA_0 for every $n \geq 1$. Whether or not the rainbow Ramsey hierarchy is strict remains open.

Csima and Mileti [80] proved that every computable instance of RRT^n_2 admits a Π^0_n solution, while there exists a computable instance of RRT^n_2 with no Σ^0_n solution. The most promising approach to separate RRT^n_2 from RRT^{n+1}_2 is using the natural invariant lying at the Δ^0_n level of the arithmetic hierarchy, namely, low$_n$ness. By Cholak, Jockusch and Slaman [27] and Wang [89], every computable instance of RRT^n_2 admits a low$_n$ solution for $n \in \{2, 3\}$. The general case is likely to be solved using arithmetic jump control.

Hyperarithmetic jump control. The duality between computability and definability is omnipresent in reverse mathematics. The base theory, RCA_0, captures "computable mathematics", and its ω-models admit a nice characterization in terms of Turing ideals. The systems WKL_0 and ACA_0 also admit computability-theoretic formulations, in terms of existence of PA degrees and of the halting set, respectively. On the other hand, the two highest systems of the Big Five, namely, ATR_0 and Π^1_1-CA_0, are better explained in terms of higher recursion theory, stating the existence of every transfinite iterations of the halting set, and the existence of Kleene's $\mathcal{O}$, respectively. Given the importance of arithmetic jump control in the study of the lower systems of reverse mathematics, one can reasonably guess that hyperarithmetic jump control will play some role in the study of principles at the level of ATR_0 and Π^1_1-CA_0.

11.2 First examples

As mentioned, there exists a natural generalization from second jump to arithmetic jump control, using inductive definitions. We illustrate this using Cohen forcing.

Theorem 11.2.1 (Feferman [90])
Fix $n \geq 1$ and let C be a non-Δ^0_n set. For every sufficiently Cohen generic filter $\mathcal{F}$, C is not $\Delta^0_n(G_\mathcal{F})$.

Proof. In order to prove our theorem, we need to define a Σ^0_n-preserving forcing question for Σ^0_n-formulas.

Definition 11.2.2. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_n formula for $n \geq 1$.

1. For $n = 1$, let $\sigma \operatorname{?\vdash} \varphi(G)$ hold if there is some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\psi(\tau, x)$ holds.
2. For $n > 1$, let $\sigma \operatorname{?\vdash} \varphi(G)$ hold if there is some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \operatorname{?\vdash} \psi(G, x)$.[1] ◇

1: Here, ψ is a Π^0_{n-1}-formula. The notation $\tau \operatorname{?\vdash} \psi(G, x)$ is therefore a shorthand for $\tau \operatorname{?\nvdash} \neg\psi(G, x)$, that is, the forcing question for Π^0_{n-1}-formulas induced by taking the negation of the forcing question for Σ^0_{n-1}-formulas.

A simple induction on the structure of the formulas shows that given a Σ^0_n-formula $\varphi(G)$, the relation $\sigma \operatorname{?\vdash} \varphi(G)$ is Σ^0_n uniformly in its parameters. The following lemma shows that the definition of the forcing question meets a strong version of its specifications.

Lemma 11.2.3. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition and $\varphi(G)$ be a Σ^0_n formula for $n \geq 1$.

1. If $\sigma \operatorname{?\vdash} \varphi(G)$, then there is an extension $\tau \geq \sigma$ forcing $\varphi(G)$.
2. If $\sigma \operatorname{?\nvdash} \varphi(G)$, then σ forces $\neg\varphi(G)$.[2] ★

2: This property states that the forcing question for Σ^0_n-formulas is Π^0_n-extremal (see Definition 7.6.5). It follows that sufficiently Cohen generic sets preserve many computational properties.

Proof. We prove simultaneously both items inductively on the structure of the formula $\varphi(G)$. Say $\varphi(G) \equiv \exists\psi(G, x)$ where $\psi(G, x)$ is Π^0_{n-1}.

Base case: $n = 1$.[3] If $\sigma \operatorname{?\vdash} \varphi(G)$, then, letting $\tau \geq \sigma$ and $x \in \mathbb{N}$ witness the definition, for every filter $\mathcal{F}$ containing τ, $G_\mathcal{F} \geq \tau$, hence $\psi(G_\mathcal{F}, x)$ holds, so $\varphi(G_\mathcal{F})$ holds. It follows that τ is an extension of σ forcing $\varphi(G)$. Conversely, if σ does not force $\neg\varphi(G)$, then there is a filter $\mathcal{F}$ containing σ such that $\varphi(G_\mathcal{F})$ holds. Then, by the use property, there is a finite $\tau \prec G_\mathcal{F}$ and some $x \in \mathbb{N}$ such that $\psi(\tau, x)$ holds. Since $\sigma \prec G_\mathcal{F}$, by taking τ long enough, one has $\sigma \prec \tau$, thus $\sigma \operatorname{?\vdash} \varphi(G)$.

3: The base case is a solution to Exercise 3.3.6.

Inductive case: $n > 1$. If $\sigma \operatorname{?\vdash} \varphi(G)$, then there is some $x \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \operatorname{?\vdash} \psi(G, x)$. By induction hypothesis, there is some $\rho \geq \tau$ forcing $\psi(G, x)$. In particular, ρ is an extension of σ forcing $\varphi(G)$. If $\sigma \operatorname{?\nvdash} \varphi(G)$, then for every $x \in \mathbb{N}$ and every $\tau \geq \sigma$, $\tau \operatorname{?\nvdash} \psi(G, x)$. By induction hypothesis, for every $x \in \mathbb{N}$ and every $\tau \geq \sigma$, there is some $\rho \geq \tau$ forcing $\neg\psi(G, x)$. In other words, for every $x \in \mathbb{N}$, the set of all ρ forcing $\neg\psi(G, x)$ is dense below σ. Thus, for every sufficiently generic filter $\mathcal{F}$ containing σ and for every $x \in \mathbb{N}$, there is some $\rho \in \mathcal{F}$ forcing $\neg\psi(G, x)$, hence $\forall x \neg\psi(G_{\mathcal{F}}, x)$ holds. In other words, σ forces $\neg\varphi(G)$. ■

The following diagonalization lemma is a straightforward generalization of Lemma 3.2.2.

Lemma 11.2.4. For every Cohen condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index e, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G^{(n-1)}} \neq C$. ⋆

Proof. Consider the following set[4]

$$U = \{(x, v) \in \mathbb{N} \times 2 : \sigma \mathrel{?\vdash} \Phi_e^{G^{(n-1)}}(x)\downarrow = v\}$$

Since the forcing question is Σ_n^0-preserving, the set U is Σ_n^0. There are three cases:

- Case 1: $(x, 1 - C(x)) \in U$ for some $x \in \mathbb{N}$. By Lemma 11.2.3(1), there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G^{(n-1)}}(x)\downarrow = 1 - C(x)$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. By Lemma 11.2.3(2), there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{G^{(n-1)}}(x)\uparrow$ or $\Phi_e^{G^{(n-1)}}(x)\downarrow \neq C(x)$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ_n^0 graph of the characteristic function of C, hence C is Δ_n^0. This contradicts our hypothesis. ■

4: By Post's theorem, the following property is Σ_n^0, although the translation is not straightforward:

$$\Phi_e^{G^{(n-1)}}(x)\downarrow = v$$

We are now ready to prove Theorem 11.2.1. Let $\mathcal{F}$ be a sufficiently generic filter for Cohen forcing, and let $G_{\mathcal{F}} = \bigcup \mathcal{F}$. By genericity of $\mathcal{F}$, $G_{\mathcal{F}}$ is an infinite binary sequence, and by Lemma 11.2.4, $C \not\leq_T G_{\mathcal{F}}^{(n-1)}$, in other words C is not $\Delta_n^0(G)$. This completes the proof of Theorem 11.2.1. ■

Exercise 11.2.5. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Σ_n^0-preserving forcing question. Show that for every non-Δ_n^0 set C and every sufficiently generic filter $\mathcal{F}$, C is not $\Delta_n^0(G_{\mathcal{F}})$. ⋆

Exercise 11.2.6 (Wang [82]). Let $(\mathbb{P}, \leq)$ be the primitive recursive Jockusch-Soare forcing, that is, $\mathbb{P}$ is the set of all infinite primitive recursive binary trees $T \subseteq 2^{<\mathbb{N}}$, partially ordered by inclusion.

1. Adapt the proof of Theorem 9.4.1 to design a Σ_n^0-preserving forcing question for Σ_n^0-formulas.
2. Deduce that for every non-Δ_n^0 set C and every sufficiently generic $\mathbb{P}$-filter $\mathcal{F}$, C is not $\Delta_n^0(G_{\mathcal{F}})$. ⋆

11.3 Pigeonhole principle

Although the conceptual gap from second-jump to higher jump control is much smaller than from first to second-jump control, the generalization sometimes requires some non-trivial adaptation. The pigeonhole principle is a good example of a statement with a reasonably simple first-jump control (Theorem 3.4.6), with a second-jump control requiring the development of a whole new machinery (Theorem 9.7.1), and whose generalization to higher jump control still contains some subtleties.[5]

5: In order to understand this section, it is mandatory to be completely familiar with the material of Chapter 9.

Theorem 11.3.1 (Monin and Patey [31])
Fix $n \geq 1$ and let C be a non-Δ^0_n set. For every set A, there is an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that C is not $\Delta^0_n(H)$.

Proof. The case $n = 1$ is Theorem 3.4.6 and the case $n = 2$ is Theorem 9.7.1. We therefore assume that $n \geq 3$, although one could prove all cases simultaneously with more case analysis within the definitions and the proof. Fix C and A. As in the previous cases, we shall construct two sets $G_0 \subseteq A$ and $G_1 \subseteq \overline{A}$ using a disjunctive notion of forcing. For simplicity, let $A_0 = A$ and $A_1 = \overline{A}$.

Hierarchy of Scott ideals. By multiple applications of the low basis theorem (Theorem 4.4.6) and Theorem 4.3.2, there exists a sequence of sets $M_0, \dots, M_{n-2}$ such that for every $s < n-1$,

1. M_s is of low degree over $\emptyset^{(s)}$;
2. M_s is a code for a Scott ideal $\mathcal{M}_s$ containing $\emptyset^{(s)}$.

By the cone avoidance basis theorem (Theorem 3.2.6) relativized to $\emptyset^{(n-1)}$ and Theorem 4.3.2, there is a code M_{n-1} for a Scott ideal $\mathcal{M}_{n-1}$ containing $\emptyset^{(n-1)}$ such that $C \not\leq_T M_{n-1}$. Note that for every $s < n-1$, $M'_s \in \mathcal{M}_{s+1}$.

Hierarchy of partition regular classes. We construct a sequence $D_0, \dots, D_{n-2}$ such that for every $s < n-1$,

1. $\mathcal{U}^{\mathcal{M}_s}_{D_s}$ is an $\mathcal{M}_s$-cohesive large class;
2. $\mathcal{U}^{\mathcal{M}_{s+1}}_{D_{s+1}} \subseteq \langle \mathcal{U}^{\mathcal{M}_s}_{D_s} \rangle$ if $s < n-2$.

First, by Proposition 9.6.25, $\mathcal{M}_1$ contains a set $D_0 \subseteq \mathbb{N}^2$ such that $\mathcal{U}^{\mathcal{M}_0}_{D_0}$ is an $\mathcal{M}_0$-cohesive class. Suppose D_s is defined and belongs to $\mathcal{M}_{s+1}$, with $s < n-2$. By Proposition 9.6.19, there is an $(M'_s \oplus D_s)'$-computable set $E_s \supseteq D_s$ such that $\mathcal{U}^{\mathcal{M}_s}_{E_s}$ is $\mathcal{M}_s$-minimal.[6] In particular, E_s is M'_{s+1}-computable, so $E_s \in \mathcal{M}_{s+2}$. Furthermore, since $M_s \in \mathcal{M}_{s+1}$ and M_{s+1} is a Scott code, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$, $f(e)$ is an M_{s+1}-code and e is an M_s-code of the same set. Let $F_{s+1} = \{(a, f(e)) : (a, e) \in E_s\}$. Then $\mathcal{U}^{\mathcal{M}_{s+1}}_{F_{s+1}} = \mathcal{U}^{\mathcal{M}_s}_{E_s}$ and $F_{s+1} \in \mathcal{M}_{s+2}$. By Proposition 9.6.25, $\mathcal{M}_{s+2}$ contains a set $D_{s+1} \supseteq F_{s+1}$ such that $\mathcal{U}^{\mathcal{M}_{s+1}}_{D_{s+1}}$ is $\mathcal{M}_{s+1}$-cohesive. In particular,

6: Note that $\mathcal{U}^{\mathcal{M}_s}_{E_s} = \langle \mathcal{U}^{\mathcal{M}_s}_{D_s} \rangle$ by Lemma 9.6.24 and by $\mathcal{M}_s$-cohesiveness of the class $\mathcal{U}^{\mathcal{M}_s}_{D_s}$.

$$\mathcal{U}^{\mathcal{M}_{s+1}}_{D_{s+1}} \subseteq \mathcal{U}^{\mathcal{M}_{s+1}}_{F_{s+1}} = \mathcal{U}^{\mathcal{M}_s}_{E_s} = \langle \mathcal{U}^{\mathcal{M}_s}_{D_s} \rangle$$

Notion of forcing. The notion of forcing is a variant of Mathias forcing whose conditions are triples (σ_0, σ_1, X), where[7]

7: This notion of forcing is very similar to the one of Theorem 9.7.1, with $\mathcal{M}_{n-1}$ playing the role of the ideal $\mathcal{N}$.

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_i \subseteq A_i$; $X \in \langle \mathcal{U}^{\mathcal{M}_{n-2}}_{D_{n-2}} \rangle$;
3. $X \in \mathcal{M}_{n-1}$.

The interpretation $[\sigma_0, \sigma_1, X]$ of a condition (σ_0, σ_1, X), the notion of extension, the definition of a valid part of a condition are exactly the same as in Theorem 9.7.1. The following lemma also holds, with the same proof as Lemma 9.7.3. Therefore, for every sufficiently generic filter $\mathcal{F}$ with valid part i, $G_{\mathcal{F},i}$ is infinite and belongs to $\langle \mathcal{U}^{\mathcal{M}_{n-2}}_{D_{n-2}} \rangle$.

Lemma 11.3.2. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and let $\mathcal{V} \supseteq \langle \mathcal{U}_{D_{n-2}}^{\mathcal{M}_{n-2}} \rangle$ be a large $\Sigma_1^0(\mathcal{M}_{n-2})$ class. There is an extension (τ_0, τ_1, Y) of p such that $[\tau_i] \subseteq \mathcal{V}$. ★

Forcing question at lower levels. In the proof of Theorem 9.7.1, we defined a non-disjunctive $\Pi_2^0(\mathcal{N})$ forcing question for Σ_1^0-formulas and a disjunctive $\Sigma_1^0(\mathcal{N})$ forcing question for Σ_2^0-formulas. The generalization to Theorem 11.3.1 goes as follows: the non-disjunctive forcing question will be extended to every Σ_s^0-formula, for $s \in \{1, \dots, n-1\}$, yielding a $\Pi_1^0(\mathcal{M}_s)$ forcing question for Σ_s^0-formulas, and one will keep the same disjunctive $\Sigma_1^0(\mathcal{M}_{n-1})$ forcing question for Σ_n^0-formulas.

Definition 11.3.3. Given a string $\sigma \in 2^{<\mathbb{N}}$ and a Σ_1^0 formula $\varphi(G)$, define $\sigma \mathrel{?\vdash} \varphi(G)$ to hold if the following class is large:[8]

$$\mathcal{U}_{D_0}^{\mathcal{M}_0} \cap \{Z : \exists \rho \subseteq Z\ \varphi(\sigma \cup \rho)\}$$

Given a string $\sigma \in 2^{<\mathbb{N}}$ and a Σ_s^0-formula $\varphi(G) \equiv \exists x \psi(G, x)$ for $s \in \{2, \dots, n-1\}$, define $\sigma \mathrel{?\vdash} \varphi(G)$ to hold if the following class is large:[9]

$$\mathcal{U}_{D_{s-1}}^{\mathcal{M}_{s-1}} \cap \{Z : \exists \rho \subseteq Z\ \exists x\ \sigma \cup \rho \mathrel{?\vdash} \psi(G, x)\}$$

8: Note that for Σ_s^0-formulas, we consider largeness with respect to $\mathcal{U}_{D_{s-1}}^{\mathcal{M}_{s-1}}$. The advantage is that it yields a better definitional complexity than using $\mathcal{U}_{D_{n-1}}^{\mathcal{M}_{n-1}}$, but it requires to have some compatibility between $\mathcal{U}_{D_{s-1}}^{\mathcal{M}_{s-1}}$ and $\mathcal{U}_{D_{n-1}}^{\mathcal{M}_{n-1}}$. This was the purpose of the construction of $D_0, \dots, D_{n-2}$.

9: As usual, ψ is Π_{s-1}^0, so $\sigma \cup \rho \mathrel{?\vdash} \psi(G, x)$ is a shorthand for $\sigma \cup \rho \mathrel{?\nvdash} \neg\psi(G, x)$.

By induction over the complexity of the formulas and using Lemma 9.6.15, one can prove that for Σ_s^0-formulas, the relation $\sigma \mathrel{?\vdash} \varphi(G)$ is $\Pi_1^0(D_{s-1} \oplus M'_{s-1})$ uniformly in σ and φ. Since $M'_{s-1}, D_{s-1} \in \mathcal{M}_s$, the relation is $\Pi_1^0(\mathcal{M}_s)$. Before proving the validity of Definition 11.3.3, one first needs to focus on the forcing relation for Π_s^0-formulas, for $s \in \{2, \dots, n\}$. Recall that in the proof of Theorem 9.7.1, we defined a custom syntactic forcing relation for Π_2^0-formulas, implying the semantic forcing relation only on the valid parts. It becomes more convenient to define a syntactic relation at every level, both for Σ_s^0 and Π_s^0-formulas.

Definition 11.3.4. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$ be a part. We define the relation $\Vdash$ for Σ_s^0 and Π_s^0-formulas for $s \in \{1, \dots, n\}$ inductively as follows. For a Δ_0^0-formula $\psi(G, x)$,

1. $p \Vdash \exists x \psi(G_i, x)$ if $\psi(\sigma_i, x)$ holds for some $i < 2$;
2. $p \Vdash \forall x \neg\psi(G_i, x)$ if $(\forall \rho \subseteq X)(\forall x)\neg\psi(\sigma_i \cup \rho, x)$.

For a Π_{s-1}^0-formula $\psi(G, x)$ with $s \in \{2, \dots, n\}$

1. $p \Vdash \exists x \psi(G_i, x)$ if $p \Vdash \psi(G_i, x)$ for some $x \in \mathbb{N}$;
2. $p \Vdash \forall x \neg\psi(G_i, x)$ if $(\forall \rho \subseteq X)(\forall x)\sigma_i \cup \rho \mathrel{?\vdash} \neg\psi(G_i, x)$. ◇

The first property that one expects of a forcing relation is that it is stable under condition extension. This is left as an exercise.

Exercise 11.3.5. Let p and q be two conditions, and $i < 2$. Show that for every $s \in \{1, \dots, n\}$ and every Σ_s^0 and Π_s^0-formula $\varphi(G)$, if $p \Vdash \varphi(G_i)$ and $q \leq p$, then $q \Vdash \varphi(G_i)$.[10] ★

10: Note that the closure under extension of the syntactic question also holds if the side is not valid.

There is an interplay between the syntactic forcing relation and the forcing questions. Indeed, the proof that the syntactic forcing relation for Π_s^0-formulas implies the semantic ones uses the validity of the forcing question for lower levels, while the proof of validity of the forcing question involves the syntactic forcing relation at the same level. We therefore start with the proof of validity of Definition 11.3.3, which is a straightforward generalization of Lemma 9.7.5 and is left as an exercise.

Exercise 11.3.6. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and $\varphi(G)$ be a Σ^0_s-formula for $s \in \{1, \dots, n-1\}$. Prove that

1. if $\sigma_i \mathbin{?\vdash} \varphi(G)$, then there is an extension q of p such that $q \Vdash \varphi(G_i)$;
2. if $\sigma_i \mathbin{?\nvdash} \varphi(G)$, then there is an extension q of p such that $q \Vdash \neg\varphi(G_i)$. ⋆

The following trivial lemma shows that if a Π^0_s-formula is syntactically forced on a valid part, then progress can be made on forcing the Π^0_s-formula.

Lemma 11.3.7. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with valid part i and $\varphi(G) \equiv \forall x \psi(G, x)$ be a Π^0_s-formula for some $s \in \{2, \dots, n\}$. If $p \Vdash \varphi(G_i)$, then for every $x \in \mathbb{N}$, there is an extension $q \leq p$ such that $q \Vdash \psi(G_i, x)$. ⋆

Proof. Fix $x \in \mathbb{N}$. Since $p \Vdash \varphi(G_i)$, then in particular, for $\rho = \emptyset$, $\sigma_i \mathbin{?\vdash} \psi(G, x)$. By Exercise 11.3.6, there is an extension q of p such that $q \Vdash \psi(G_i, x)$. ■

We are now ready to prove that the syntactic forcing relation implies the semantic one on valid sides.

Lemma 11.3.8. Let p be a condition, $i < 2$ be a side and $\varphi(G)$ be a Σ^0_s or Π^0_s-formula for some $s \in \{1, \dots, n\}$. If $p \Vdash \varphi(G_i)$, then $\varphi(G_{\mathcal{F},i})$ holds for every sufficiently generic filter $\mathcal{F}$ containing p and whose side i is valid.[11] ⋆

11: Recall that a side $i < 2$ is *valid* in a filter $\mathcal{F}$ if the side is valid for every $p \in \mathcal{F}$. Every filter has at least a valid side.

Proof. By induction over the complexity of the formula φ. The case $s = 1$ is easy and $\varphi(G_{\mathcal{F},i})$ even holds for every filter $\mathcal{F}$ containing p, with no regard to genericity or to validity of the side. Suppose $s \geq 2$. If $\varphi(G) \equiv \exists x \psi(G, x)$ for some Π^0_{s-1}-formula ψ, then by definition, there is some $x \in \mathbb{N}$ such that $p \Vdash \psi(G_i, x)$, so by induction hypothesis, $\psi(G_{\mathcal{F},i}, x)$ holds for every sufficiently generic filter $\mathcal{F}$ containing p and whose side i is valid. In particular, $\varphi(G_{\mathcal{F},i})$ holds for every such filter $\mathcal{F}$. If $\varphi(G) \equiv \forall x \neg\psi(G, x)$ for some Π^0_{s-1}-formula ψ, then we claim that for every $x \in \mathbb{N}$, the following class $\mathcal{D}_x$ is dense below p:

$$\mathcal{D}_x = \{q : \text{ side } i \text{ of } q \text{ is not valid } \vee q \Vdash \neg\psi(G_i, x)\}$$

Indeed, fix $x \in \mathbb{N}$ and let $r = (\tau_0, \tau_1, Y)$ be an extension of p. If side i of r is not valid, then $r \in \mathcal{D}_x$, in which case we are done. Otherwise, by Exercise 11.3.5, $r \Vdash \varphi(G_i)$, so, unfolding the definition, for $\rho = \emptyset$, $\tau_i \mathbin{?\vdash} \neg\psi(G_i, x)$, so by Exercise 11.3.6, there is an extension $q \leq r$ such that $q \Vdash \neg\psi(G_i, x)$, in which case $q \in \mathcal{D}_x$. Thus, $\mathcal{D}_x$ is dense below p.

Let $\mathcal{F}$ be a sufficiently generic filter containing p and whose side i is valid. Since $\mathcal{D}_x$ is dense below p for every $x \in \mathbb{N}$, $\mathcal{F} \cap \mathcal{D}_x \neq \emptyset$ for every $x \in \mathbb{N}$. Moreover, since side i is valid in $\mathcal{F}$, then for $q \in \mathcal{F} \cap \mathcal{D}_x$, we have $q \Vdash \neg\psi(G_i, x)$. By induction hypothesis, $\neg\psi(G_{\mathcal{F},i}, x)$ holds, and this for every $x \in \mathbb{N}$, so $\varphi(G_{\mathcal{F},i}, x)$ holds. ■

Forcing question on top level. The design of the forcing question for Σ^0_n formulas is exactly the one of Theorem 9.7.1. It consists of defining two forcing questions: a disjunctive one which works if both sides of the condition are valid, and in case one side is invalid, one designs a degenerate non-disjunctive forcing question exploiting the failure of validity. We define both forcing questions and leave their proofs as exercises.

> **Definition 11.3.9.** Given a condition $p = (\sigma_0, \sigma_1, X)$ and a pair of Σ^0_n formulas $\varphi_0(G)$ and $\varphi_1(G)$, with $\varphi_i(G) \equiv \exists x \psi_i(G, x)$, define $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ to hold if for every 2-partition $Z_0 \cup Z_1 = X$, there is some $i < 2$, some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$. ◇

Exercise 11.3.10. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with both valid parts and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_n-formulas. Prove that

1. if $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension q of p such that $q \Vdash \varphi(G_i)$ for some $i < 2$;
2. if $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is an extension q of p such that $q \Vdash \neg\varphi(G_i)$ for some $i < 2$. ★

A *witness of invalidity* of part i of a condition $p = (\sigma_0, \sigma_1, X)$ is a $\Sigma^0_1(\mathcal{M}_{n-2})$ large class $\mathcal{V} \supseteq \langle \mathcal{U}^{\mathcal{M}_{n-2}}_{D_{n-2}} \rangle$ such that $X \cap A_i \notin \mathcal{V}$.

> **Definition 11.3.11.** Let $p = (\sigma_0, \sigma_1, X)$ be a condition with witness of invalidity $\mathcal{V}$ on part $1 - i$, and let $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_n formula. Define $p \mathrel{?\vdash^{\mathcal{V}}} \varphi(G_i)$ to hold if for every 2-partition $Z_0 \sqcup Z_1 = X$ such that $Z_{1-i} \notin \mathcal{V}$, there is some $x \in \mathbb{N}$ and some $\rho \subseteq Z_i$ such that $\sigma_i \cup \rho \mathrel{?\vdash} \psi_i(G, x)$. ◇

Exercise 11.3.12. Let $p = (\sigma_0, \sigma_1, X)$ be a condition with witness of invalidity $\mathcal{V}$ on part $1 - i$, and let $\varphi(G)$ be a Σ^0_n formula. Prove that

1. if $p \mathrel{?\vdash^{\mathcal{V}}} \varphi(G_i)$, then there is an extension of p forcing $\varphi(G_i)$;
2. if $p \mathrel{?\nvdash^{\mathcal{V}}} \varphi(G_i)$, then there is an extension $q \leq p$ such that $q \Vdash \neg\varphi(G_i)$. ★

By compactness, both forcing questions for Σ^0_n-formulas are $\Sigma^0_1(\mathcal{M}_{n-1})$. We are now ready to prove Theorem 11.3.1.

Suppose first there is a condition p with some invalid part $1 - i$. Let $\mathcal{F}$ be a sufficiently generic filter containing p and let $G_i = G_{\mathcal{F},i}$. Then part i is valid in $\mathcal{F}$. By Lemma 11.3.7, the syntactic forcing relation implies the semantic forcing relation on part i. By Exercise 11.3.12 and by adapting Theorem 9.3.5, for every Turing functional Φ_e, there is some condition $q \in \mathcal{F}$ forcing $\Phi_e^{G_i^{(n-1)}} \neq C$, so C is not $\Delta^0_n(G_i)$.

Suppose now that for every condition, both parts are valid. Let $\mathcal{F}$ be a sufficiently generic filter, and let $G_i = G_{\mathcal{F},i}$ for $i < 2$. By Lemma 11.3.7, the syntactic forcing relation implies the semantic forcing relation on both parts. By Exercise 11.3.10 and by adapting Exercise 11.2.5, for every pair of Turing functionals Φ_{e_0}, Φ_{e_1}, there is some condition $q \in \mathcal{F}$ forcing $\Phi_{e_0}^{G_0^{(n-1)}} \neq C \vee \Phi_{e_1}^{G_1^{(n-1)}} \neq C$. By a pairing argument, there is some $i < 2$ such that C is not $\Delta^0_n(G_i)$. This completes the proof of Theorem 11.3.1. ■

11.4 Computable ordinals

In order to extend iterated jump control to transfinite levels, one first needs to develop a theory of computable ordinals. There are often two approaches to define a mathematical structure : the axiomatic approach (top-down) and the constructive one (bottom-up). For instance, an ordinal can either be defined as the order type of a well-order, or using von Neumann definition, as the set of its smaller ordinals. We shall see that the effective counterparts of these definitions coincide, yielding a robust notion of computable ordinal.[12]

12: We assume the reader has some familiarity with the classical theory of ordinals.

Definition 11.4.1. An ordinal α is *computable* if it is finite or it is the order-type of a computable[13] well-order on $\mathbb{N}$. ◊

13: Actually, one could have replaced "computable" by "polynomial-time computable", "arithmetic", or even "hyperarithmetic", this would have yielded exactly the same class of ordinals, even-though the equivalence is highly non-trivial.

First, note from the above definition that every computable ordinal is witnessed by the program of a computable well-order. There are therefore only countably many ordinals. We first show that one can replace "computable" by "c.e." in the above definition of a computable ordinal.

Lemma 11.4.2. Let $<_R$ be a c.e. total order on $\mathbb{N}$. Then $<_R$ is computable.★

Proof. By totality of $<_R$, $(a, b) \notin <_R$ iff $a = b$ or $(b, a) \in <_R$. Thus, $<_R$ is both c.e. and co-c.e., hence is computable. ■

We shall now prove that the computable ordinals form an initial segment of the ordinals.

Lemma 11.4.3. Let $<_R$ be a c.e. total order on an infinite set $A \subseteq \mathbb{N}$. Then there is a c.e. total order $<_S$ on $\mathbb{N}$ with the same order type as $<_R$. ★

Proof. First, note that A is c.e., since $A = \{a \in \mathbb{N} : \exists b((a, b) \in <_R \vee (b, a) \in <_R)\}$ by totality of $<_R$. Thus, there is a computable bijection $f : \mathbb{N} \to A$. Then, $<_S = \{(f^{-1}(a), f^{-1}(b) : (a, b) \in <_R\}$. ■

Suppose now that α is a computable ordinal, as witnessed by a computable well-order $<_R$ on $\mathbb{N}$, and let $\beta < \alpha$. Then either β is finite, in which case it is computable by definition, or β is the order type of $<_R$ restricted to $\{b \in \mathbb{N} : b <_R a\}$ for some $a \in \mathbb{N}$ with infinitely many predecessors. Then by Lemma 11.4.3 and Lemma 11.4.2, β is the order type of a computable well-order on $\mathbb{N}$, thus is a computable ordinal. Since the computable ordinals form a countable initial segment of the ordinals, then there is a least non-computable ordinal.

Definition 11.4.4. Let ω_1^{ck} denote the least non-computable ordinal.[14] ◊

14: "ck" stands for "Church Kleene", who introduced the concept in [91].

The representation of a computable ordinal using well-orders is not the most effective, in that given a computable well-order $<_R$ on $\mathbb{N}$ and some $a \in \mathbb{N}$, one cannot computably decide wether a is a successor element or a limit. We now give an alternative and more constructive definition of the computable ordinals, which can be seen as an effective counterpart of von Neumann definition.

Definition 11.4.5 (Kleene's O). Let $<_\mathcal{O}$ be the least partial order on $\mathbb{N}$ such that $1 <_\mathcal{O} 2$, satisfying the following closures:[15]

(1) If $a <_\mathcal{O} b$ then $a <_\mathcal{O} 2^b$
(2) For every total function $\Phi_e : \mathbb{N} \to \mathbb{N}$, if $\forall n(\Phi_e(n) <_\mathcal{O} \Phi_e(n+1))$,

15: The choice of 2^b to code the successor of b and $3 \cdot 5^e$ to code for a limit ordinal with cofinal sequence Φ_e is arbitrary. The only requirement is to have a unique notation to be able to deconstruct the inductive definition and distinguish the successor and limit cases. For instance, one could have defined 3^{e+1} instead of $3 \cdot 5^e$.

then for every $n \in \mathbb{N}$, $\Phi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$.

Let $\mathcal{O}$ be the domain of $<_{\mathcal{O}}$.[16] ◇

The above definition might seem quite cryptic, and deserves some explanation. Each element a of $\mathcal{O}$ can be evaluated into a computable ordinal $|a|$, by transfinite induction[17] as follows: First, $|1| = \mathbb{0}$. If $2^a \in \mathcal{O}$, then $|2^a| = |a| + \mathbb{1}$. Last, if $3 \cdot 5^e \in \mathcal{O}$, then $|3 \cdot 5^e| = \sup_n |\phi_e(n)|$. To avoid confusion, we write $\mathbb{0}, \mathbb{1}, \dots$ for the finite ordinals and keep the standard font $0, 1, \dots$ for their codes.[18]

Definition 11.4.6. An ordinal α is *constructible* if $\alpha = |a|$ for some $a \in \mathcal{O}$.◇

The main advantage of constructible ordinals is that one can directly know from a code a whether it codes for $\mathbb{0}$, for a successor ordinal, or is a limit ordinal. In the latter case, one can even effectively find a cofinal sequence of codes.

Exercise 11.4.7. Show that the constructible ordinals are downward-closed.★

Every finite ordinal n admits a unique code in $\mathcal{O}$, namely, the n-fold power of two. The ordinal ω, on the other hand, admits infinitely many codes in $\mathcal{O}$, since there exist countably many computable strictly increasing sequences of finite ordinals. More generally, the limit step introduces infinitely many codes, and one can thus see $\mathcal{O}$ as a tree, which is ω-branching at limit steps. A maximal path[19] through this tree is a linearly ordered subset of $\mathcal{O}$ which is downward-closed, and cofinal in ω_1^{ck}.

Exercise 11.4.8. Show that for every $a \in \mathcal{O}$, the set $\{b \in \mathcal{O} : b <_{\mathcal{O}} a\}$ is uniformly c.e. and linearly ordered.[20] ★

The same way Turing-invariant operators on sets induce operations on the Turing degrees, one can study the effectivity of operations on ordinals by defining functions over their codes. The following exercise shows that ordinal addition is computable.

Exercise 11.4.9. Let $+_{\mathcal{O}} : \mathbb{N}^2 \to \mathbb{N}$ be total computable function defined by $a +_{\mathcal{O}} 1 = a$, $a +_{\mathcal{O}} 2^b = 2^{a +_{\mathcal{O}} b}$, $a +_{\mathcal{O}} 3 \cdot 5^e = 3 \cdot 5^{f(e,a)}$, where $f(e,a)$ is the code of a function[21] such that $\Phi_{f(e,a)}(n) = a +_{\mathcal{O}} \Phi_e(n)$, and $a +_{\mathcal{O}} b = 1$ if b is not in any of those forms. Show that for every $a, b \in \mathcal{O}$, $|a| + |b| = |a +_{\mathcal{O}} b|$.★

Given a non-empty c.e. set of codes of constructible ordinals, its supremum is again constructible, but not uniformly computable. One can however uniformly compute an upper bound:

Lemma 11.4.10 (Sacks [93]). There is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that if $W_e \subseteq \mathcal{O}$, then $f(e) \in \mathcal{O}$ and $\sup_{a \in W_e} |a| \leq |f(e)|$.[22] ★

Proof. One can without loss of generality assume that W_e is infinite, by enumerating all the constructible codes of finite ordinals. For every $e \in \mathbb{N}$, let $f(e) = 3 \cdot 5^a$ where $\Phi_a(n)$ returns the finite ordinal sum (using Exercise 11.4.9) of the n first distinct elements enumerated in W_e, different from 1 (the code of $\mathbb{0}$). One therefore has $\Phi_a(n) <_{\mathcal{O}} \Phi_a(n+1)$ for every $n \in \mathbb{N}$, hence $3 \cdot 5^a \in \mathcal{O}$. Moreover, by construction, $\sup_{a \in W_e} |a| \leq \sup_n |\Phi_a(n)| = |3 \cdot 5^a| = |f(e)|$. ■

16: The sets $<_{\mathcal{O}}$ and $\mathcal{O}$ are both Π^1_1-complete.

17: In order to be allowed to use transfinite induction, one must actually first check that $<_{\mathcal{O}}$ is a well-founded partial ordering. One can define an natural enumeration of $<_{\mathcal{O}}$ by transfinite induction on the ordinals, such that if $a <_{\mathcal{O}} b$ and $b <_{\mathcal{O}} c$, then $a <_{\mathcal{O}} b$ is enumerated at an earlier stage than $b <_{\mathcal{O}} c$. It follows that any infinite decreasing $<_{\mathcal{O}}$-sequence would yield an infinite decreasing sequence of ordinals.

18: One must be careful in distinguishing the constructible code 1 from the ordinal $\mathbb{1}$. Indeed, the code 1 denotes the ordinal $\mathbb{0}$.

19: As noted Chong and Liu [92], not every path can be extended into a maximal path. Indeed, with poor choices at the ω-branching levels, one might obtain only ω^2 for instance.

20: Although $<_{\mathcal{O}}$ is Π^1_1, the restriction of the order to $\{b \in \mathcal{O} : b <_{\mathcal{O}} a\}$ is uniformly c.e. in a.

21: Note that this definition involves Kleene's fixpoint theorem, as the definition of f uses $+_{\mathcal{O}}$. Also note that $a \leq_{\mathcal{O}} a +_{\mathcal{O}} b$ but not necessarily $b \leq_{\mathcal{O}} a +_{\mathcal{O}} b$ because of the limit case.

22: Note that we do not require $<_{\mathcal{O}}$ to be total on W_e. In other words, the inequality holds for ordinals, one does not satisfy $a <_{\mathcal{O}} f(e)$ for every $a \in W_e$.

We shall now prove that the constructible ordinals coincide with the computable ones. Following the intuition, a code for a constructible ordinal carries more information than a computable well-order, in that one can computably transform a code $a \in \mathbb{O}$ into a program for a computable well-order of order type $|a|$, while the reverse translation is not computable.

Theorem 11.4.11 (Kleene, Markwald)
Computable and constructible ordinals coincide.

Proof. Let $a \in \mathbb{O}$ be a code for a constructible ordinal α. If $\alpha < \omega$, then it is computable by definition. If α is infinite, then the relation $<_{\mathbb{O}}$ restricted to $\{b \in \mathbb{O} : b <_{\mathbb{O}} a\}$ is c.e. By Lemma 11.4.3 and Lemma 11.4.2, there is a computable order over $\mathbb{N}$ with the same order type, thus α is computable.

Suppose now that α is a computable ordinal. If $\alpha < \omega$, then the α-fold power of 2 yields a constructible code for α, hence hence α is constructible. If α is infinite, then there is a computable well-order $<_R$ on $\mathbb{N}$ of order type α. Let $f : \mathbb{N} \to \mathbb{N}$ be the function of Lemma 11.4.10, and let $g : \mathbb{N} \to \mathbb{N}$ be the total computable function which on a computes the code e_a of the c.e. set $W_{e_a} = \{g(b) : b <_R a\}$, and outputs $f(e_a)$. One can prove by induction over a that $g(a) \in \mathbb{N}$ and $|g(a)|$ is at least the order type of $<_R$ restricted to the elements below a. Let $W_e = \{g(a) : a \in \mathbb{N}\}$, then $|f(e)| \geq \sup_a |g(a)|$, so $|f(e)|$ is at least the order type of $<_R$.[23] ∎

23: One could be tempted to rather consider $3 \cdot 5^i$ where $\Phi_i(a) = g(a)$. However, although $|g(a)| < |g(a+1)|$, one does not have in general $g(a) <_{\mathbb{O}} g(a+1)$, thus $3 \cdot 5^i$ is not a valid constructible code.

11.5 Hyperarithmetic hierarchy

The arithmetic hierarchy corresponds to the finite levels of the effective counterpart to the Borel hierarchy over $\mathbb{N}$, equipped with the discrete topology.[24] We now generalize the arithmetic hierarchy to transfinite levels, and prove the corresponding generalization of Post theorem, namely, every level of the hierarchy is effectively open relative to the appropriate iteration of the halting set.

24: It seems at first sight that this is just a complicated reformulation of a simple notion. However, the topological considerations are very useful to understand why Post theorem holds for the arithmetic hierarchy, but not for classes over $2^{\mathbb{N}}$. Indeed, since the Borel hierarchy collapses over the discrete topology, every Borel set is open, hence is effectively open relative to an appropriate oracle, while the Borel hierarchy is strict on the Cantor space, hence some Π^0_2 classes are not $\Pi^0_1(A)$ for any oracle A.

Although the arithmetic hierarchy is usually defined in terms of alternations of quantifiers, the generalization to transfinite levels which require to use infinitary effective conjunctions and disjunctions to handle the limit cases. One therefore rather defines the hyperarithmetic hierarchy in terms of codes.

Definition 11.5.1. The hyperarithmetic codes are defined by induction over the computable ordinals[25][26].

1. A Σ^0_1-*code* of a set A is a pair $\langle 0, e\rangle$ such that $W_e = A$.
2. A Π^0_α-*code* of a set A is a pair $\langle 1, e\rangle$, where e is a Σ^0_α-code of the set $\mathbb{N} \setminus A$.
3. A Σ^0_α-*code* of a set $A = \bigcup_n A_n$ is a pair $\langle 2, e\rangle$ where W_e is non-empty, and enumerates $\Pi^0_{\beta_n}$-codes of sets A_n such that $\sup_n(\beta_n + \mathbb{1}) = \alpha$. ◇

25: One could actually define the notion of Σ^0_α-code for arbitrary ordinals. However, an easy induction along the ordinals shows that every Σ^0_α-code is Σ^0_β for some $\beta < \omega_1^{ck}$, hence the hierarchy does not go beyond the computable ordinals.

26: Because Σ^0_α-codes do not distinguish the successor case from the limit case, one cannot uniformly compute a constructible code $a \in \mathbb{O}$ from a $\Sigma^0_{|a|}$-code.

A set A is Σ^0_α (resp. Π^0_α) if it admits a Σ^0_α-code (resp. a Π^0_α-code). A set A is Δ^0_α if it is both Σ^0_α and Π^0_α. An easy induction shows that the finite levels correspond to the arithmetic hierarchy.

Exercise 11.5.2. Show that the Σ^0_α sets are closed under effective countable unions and finite intersections. Moreover, those closure are uniform in Σ^0_α-codes. ★

Exercise 11.5.3. Show that if A is either Σ^0_α or Π^0_α, then A is $\Delta^0_{\alpha+\mathbb{1}}$ uniformly in a Σ^0_α or a Π^0_α-code of A. ★

The following lemma requires a bit more work, thus is fully proven.

Lemma 11.5.4. If A is Δ^0_α and B is $\Sigma^0_1(A)$, then B is Σ^0_α uniformly in a Δ^0_α-code of A and a c.e. index of B.[27] ★

27: A Δ^0_α-code is nothing but a pair of a Σ^0_α-code and a Π^0_α-code.

Proof. Say $B = W^A_e$. Then $B = \{n : \exists\sigma\ (n \in W^\sigma_e \wedge \forall i < |\sigma|\ ((\sigma(i) = 0 \wedge i \notin A) \vee (\sigma(i) = 1 \wedge i \in A))\}$. By induction on α, given $\sigma \in 2^{<\mathbb{N}}$ and $i < 2$, one can uniformly compute a Σ^0_α-code of a set $A_{\sigma,i}$ such that $A_{\sigma,i} = \mathbb{N}$ if $\sigma(i) = A(i)$ and $A_{\sigma,i} = \emptyset$ otherwise. Then $B = \bigcup_\sigma (W^\sigma_e \cap \bigcap_{i<|\sigma|} A_{\sigma,i})$. By Exercise 11.5.2, B is Σ^0_α. ■

The following exercise is proven by a simple induction over codes, and will be useful later.

Exercise 11.5.5. Let $f : \mathbb{N} \to \mathbb{N}$ be a total computable function and A be a Σ^0_α-set. Show that $f[A] = \{f(n) : n \in A\}$ is Σ^0_α uniformly in a Σ^0_α-code of A and a c.e. index of f. ★

We now define transfinite iterations of the Turing jump to state the generalized Post theorem. In the limit case, one naturally wants to join a cofinal sequence of previous iterations. This raises some canonicity issues, as there exist infinitely many cofinal sequences already at the level of ω, and they yield different sets[28]. We will therefore iterate the jump along constructible codes of ordinals.[29]

28: One could for instance define $\emptyset^{(\omega)}$ as $\bigoplus_n \emptyset^{(n)}$, but also as $\bigoplus_n \emptyset^{(2n)}$, among many possibilities.

29: Since constructible codes are integers, it would be confusing to write $\emptyset^{(a)}$ for an $|a|$-iteration of the Turing jump. One therefore traditionally uses the notation H_a, standing for “hyperarithmetic”.

Definition 11.5.6. For every $a \in \mathcal{O}$, let H_a be defined inductively as follows.

1. $H_1 = \emptyset$
2. $H_{2^a} = H'_a$
3. $H_{3\cdot 5^e} = \bigoplus_n H_{\Phi_e(n)}$. ◇

By Spector [94], if a and b are two constructible codes for an ordinal α, then $H_a \equiv_T H_b$. Therefore, this hierarchy defines iterations of the Turing jump over the Turing degrees, and one can write $\mathbf{0}^{(\alpha)}$ for the α-iterate of the Turing jump. The following proposition might be surprising at first, as the transfinite iterations are shifted with respect to the finite levels.

Proposition 11.5.7. For every constructible code $a \in \mathcal{O}$ with $|a| \geq \omega$, H_a is $\Delta^0_{|a|}$ uniformly in a. ★

Proof. By induction along $\mathcal{O}$ starting with $|a| = \omega$.

Suppose first $a = 2^b$ codes of a successor ordinal. Then, by induction hypothesis, H_b is $\Delta^0_{|b|}$ uniformly in b. By Lemma 11.5.4, $H_a = H'_b$ is $\Sigma^0_{|b|}$ uniformly in b, so by Exercise 11.5.3, H_a is $\Delta^0_{|a|}$ uniformly in a.

Suppose now $a = 3 \cdot 5^e$ codes for a limit ordinal. Here, for every n, we have two cases: either $\Phi_e(n)$ is a constructible code of a finite ordinal, in which case Post's theorem yields that $H_{\Phi_e(n)}$ is $\Sigma^0_{|\Phi_e(n)|+\mathbb{1}}$ uniformly in n and e, or $\Phi_e(n)$ is a constructible code of an infinite ordinal. In the latter case, by induction hypothesis, $H_{\Phi_e(n)}$ is $\Delta^0_{|\Phi_e(n)|}$ uniformly in n and e, in which case by Exercise 11.5.3 it is again $\Sigma^0_{|\Phi_e(n)|+\mathbb{1}}$ uniformly in n and e. Note that one can computably decide in which case we are, since being a constructible code of a finite ordinal is decidable. Thus, we can assume in both cases that $H_{\Phi_e(n)}$ is $\Sigma^0_{|\Phi_e(n)|+\mathbb{1}}$ uniformly in n and e.

By Exercise 11.5.5, for each n, the set $B_n = \{\langle m, n\rangle : m \in H_{\Phi_e(n)}\}$ is $\Sigma^0_{|\Phi_e(n)|+\mathbb{1}}$ uniformly in n and e. Then $H_a = \bigcup_n B_n$ is $\Sigma^0_{|\alpha|}$ uniformly in a. By Exercise 11.5.3, $\overline{H}_{\Phi_e(n)}$ is $\Sigma^0_{|\Phi_e(n)|+\mathbb{2}}$ uniformly in n and e. By Exercise 11.5.5, for each n, the set $C_n = \{\langle m, n\rangle : m \in \overline{H}_{\Phi_e(n)}\}$ is $\Sigma^0_{|\Phi_e(n)|+\mathbb{2}}$ uniformly in n and e. Thus, $\overline{H}_a = \bigcup_n C_n$ is $\Sigma^0_{|\alpha|}$ uniformly in a. It follows that H_a is $\Delta^0_{|\alpha|}$ uniformly in a. ■

Corollary 11.5.8
For every constructible code $a \in \mathcal{O}$,

1. *if $|a| < \omega$, then H_a is $\Sigma^0_{|a|}$ uniformly in a;*
2. *if $|a| \geq \omega$, then H_{2^a} is $\Sigma^0_{|a|}$ uniformly in a.*

Proof. The first case holds by Post's theorem. The second case is immediate by Proposition 11.5.7 and Lemma 11.5.4. ■

The bound is actually tight, and one can prove with some extra work that H_{2^a} is $\Sigma^0_{|a|}$-complete when $|a| \geq \omega$. Together with Post's theorem, this yields the following generalized Post theorem:

Theorem 11.5.9 (Monin and Patey [4])
Fix some $a \in \mathcal{O}$.

1. *If $|a| < \omega$, then the set H_a is $\Sigma^0_{|a|}$-complete uniformly in a.*
2. *If $|a| \geq \omega$, then the set H_{2^a} is $\Sigma^0_{|a|}$-complete uniformly in a.*

11.6 Higher recursion theory

Beyond the definition of a robust notion of computable ordinal, and the extension of the arithmetic hierarchy to transfinite levels, there is a whole theory generalizing computability theory along computable ordinals, called *higher recursion theory*. Its development goes far beyond the scope of this book. We however state some of its main concepts and theorems, which will be useful for transfinite jump control. One might refer to Sacks [93], Chong and Yu [92] or to Monin and Patey [4] for an introduction to higher recursion theory.

11.6.1 Hyperarithmetic reduction

Many natural properties on sets induce operations or relations over sets by considering their relativized form. The most basic example is the notion of

Turing machine, whose relativization yields the Turing reduction. One can also relativize the arithmetic hierarchy, yielding the arithmetic reduction by letting X be *arithmetically reducible* to Y if X is $\Sigma^0_n(X)$ for some $n \in \mathbb{N}$. Similarly, one can naturally define the notion of Y-computable ordinal, with ω_1^Y denoting the least non-Y-computable ordinal. The $\Pi^1_1(Y)$ set $\mathcal{O}^Y$ of Y-constructible codes is defined accordingly, with all c.e. operators replaced by Y-c.e. operators.[30] One then defines $\Sigma^0_\alpha(Y)$ classes for $\alpha < \omega_1^Y$ and the sets H_a^Y for $a \in \mathcal{O}^Y$. All the theorems of the previous sections are uniform in Y. In particular, $H^Y_{2^a}$ is uniformly $\Sigma^0_{|a|_Y}$ if $|a|_Y \geq \omega$.

30: If $a \in \mathcal{O}^X \cap \mathcal{O}^Y$, the interpretation $|a|_Y$ of a Y-constructible code might differ from its interpretation $|a|_X$. For convenience, we might assume that for every $a \in \mathcal{O} \cap \mathcal{O}^Y$, $|a| = |a|_Y$.

We shall see that most sets Y satisfy $\omega_1^Y = \omega_1^{ck}$. In other words, it is an "anomaly" to compute non-computable ordinals. However, even if $\omega_1^Y = \omega_1^{ck}$, computable ordinals will have in general more codes in $\mathcal{O}^Y$ than in $\mathcal{O}$.

Definition 11.6.1. A set X is *hyperarithmetically reducible*[31] to a set Y (written $X \leq_h Y$) if it is $\Sigma^0_\alpha(Y)$ for some $\alpha < \omega_1^Y$, or equivalently if there is some $a \in \mathcal{O}^Y$ and $e \in \mathbb{N}$ such that $X = \Phi_e^{H_a^Y}$. ◇

31: It is very important to note that $a \in \mathcal{O}^Y$ and not simply $a \in \mathcal{O}$. Indeed, Y might compute some non-computable ordinals.

The hyperarithmetic reduction is a very robust notion, in that it admits various characterizations of very different nature. A set $X \subseteq \mathbb{N}$ is $\Sigma^1_1(Y)$ if it can be written of the form $\{n \in \mathbb{N} : \exists X \varphi(X, Y, n)\}$, where φ is an arithmetic formula.[32] A set X is $\Pi^1_1(Y)$ if its complement is $\Sigma^1_1(Y)$, and $\Delta^1_1(Y)$ if it is both $\Sigma^1_1(Y)$ and $\Pi^1_1(Y)$. A Y-*modulus* of a set X is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $g : \mathbb{N} \to \mathbb{N}$ dominating[33] f, $g \oplus Y \geq_T X$. Last, a set X is X-*computably encodable* if for every infinite set $A \subseteq \mathbb{N}$, there is an infinite subset $B \subseteq A$ such that $B \oplus Y \geq_T X$. The following theorem shows that all these definitions coincide.

32: By Kleene's normal form theorem, φ can even be taken Π^0_1.

33: A function g *dominates* f if $g(x) \geq f(x)$ for every x. Some authors define it as $g(x) \geq f(x)$ *for all but finitely many* x. This difference does not matter in this context.

Theorem 11.6.2 (Groszek and Slaman [95], Solovay [19], Kleene [96])
Let X and Y be two sets. The following are equivalent:

1. $X \leq_h Y$;
2. X *is* $\Delta^1_1(Y)$;
3. X *admits a* Y*-modulus;*
4. X *is* Y*-computably encodable.*

There exists a whole correspondence[34] between classical computability theory and higher recursion theory. In this correspondence, the Π^1_1 sets play the role of higher c.e. sets, the hyperarithmetic sets are both the higher finite and higher computable sets, and Kleene's $\mathcal{O}$ is the higher halting set.

34: This correspondence is imperfect, in particular because the true higher counterpart of the integers is ω_1^{ck}. It follows that there is a better correspondence between classical computability theory and *metarecursion theory*, a theory which studies the subsets of ω_1^{ck} from a computational viewpoint. See Sacks [93] for an introduction to both theories.

The following theorem is known as the Σ^1_1 *majoration theorem*.

Theorem 11.6.3 (Spector [94])
Let $X \subseteq \mathcal{O}$ be a Σ^1_1 set. Then $\sup_{a \in X} |a| < \omega_1^{ck}$.[35]

35: This theorem is actually uniform in the following sense: one can computably find a constructible code $b \in \mathcal{O}$ such that $\sup_{a\in X} |a| \leq |b|$ from a Σ^1_1-code of X.

Corollary 11.6.4
Let $f : \mathbb{N} \to \mathcal{O}$ be a total Π^1_1-function.[36] *Then $\sup_n |f(n)| < \omega_1^{ck}$.*

36: A function is Π^1_1 if its graph is Π^1_1.

Proof. The graph G_f of f can be written of the form $\{(x, y) : \forall X \Phi_e^X(x, y)\downarrow\}$. Since f is total, $G_f = \{(x, y) : \forall z \exists X (z \neq y \to \Phi_e^X(x, z)\uparrow\}$, which is a Σ^1_1 set, so f is Δ^1_1. In particular, the range of f is a Σ^1_1 subset of $\mathcal{O}$, so by the Σ^1_1 majoration theorem, $\sup_n |f(n)| < \omega_1^{ck}$. ■

11.6.2 Hyperjump operator

As mentioned, Kleene's $\mathcal{O}$ is the higher counterpart of the halting set. The relativization of the halting set induces an operation on the Turing degrees called the Turing jump. Similarly, the map $X \mapsto \mathcal{O}^X$ is compatible with the hyperarithmetic reduction, and therefore induces an operation on the hyperarithmetic degrees, called the *hyperjump*.

Recall that given two sets X, Y, $X \leq_T Y$ iff $X' \leq_m Y'$. The following theorem states its higher counterpart.

Theorem 11.6.5 (Sacks [93])
Fix two sets X, Y. Then $X \leq_h Y$ iff $\mathcal{O}^X \leq_m \mathcal{O}^Y$.

Proof. Suppose first $X \leq_h Y$. Then X is $\Delta^1_1(Y)$ by Theorem 11.6.2, but since $\mathcal{O}^X$ is $\Pi^1_1(X)$, then $\mathcal{O}^X$ is $\Pi^1_1(Y)$.[37] Since $\mathcal{O}^Y$ is $\Pi^1_1(Y)$-complete for the many-one reduction[38], $\mathcal{O}^X \leq_m \mathcal{O}^Y$.

Suppose now $\mathcal{O}^X \leq_m \mathcal{O}^Y$. Since X and $\overline{X}$ are $\Pi^1_1(X)$, then $X \leq_m \mathcal{O}^X$ and $\overline{X} \leq_m \mathcal{O}^X$. It follows by transitivity of the many-one reduction that $X \leq_m \mathcal{O}^Y$ and $\overline{X} \leq_m \mathcal{O}^Y$. Since $\mathcal{O}^Y$ is $\Pi^1_1(Y)$, both X and $\overline{X}$ are $\Pi^1_1(Y)$, so X is $\Delta^1_1(Y)$, hence $X \leq_h Y$ by Theorem 11.6.2. ■

37: This is true in general: if X is $\Pi^1_1(Y)$ and Y is $\Delta^1_1(Z)$, then X is $\Pi^1_1(Z)$.

38: The proof that $\mathcal{O}$ is Π^1_1-complete for the many-one reduction relativizes in a strong way: for every set Y and every $\Pi^1_1(Y)$ set X, there is a *computable* function $f : \mathbb{N} \to \mathbb{N}$ such that $X = \{n : f(n) \in \mathcal{O}^Y\}$.

One deduces from the previous theorem that the hyperjump operator is a hyperdegree-theoretic operation. The following theorem states in a relativized form that the notion of computable ordinal is robust, in that any hyperarithmetic ordinal is computable.

Theorem 11.6.6 (Spector [94])
Fix two sets X, Y. If $X \leq_h Y$, then $\omega_1^X \leq \omega_1^Y$.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be the partial Y-computable function witnessing the uniformity of the Σ^1_1 majoration theorem relativized to Y (Theorem 11.6.3), that is, if $A \subseteq \mathcal{O}^Y$ is a $\Sigma^1_1(Y)$ set with $\Sigma^1_1(Y)$-code c, then $f(c) \in \mathcal{O}^Y$ is such that $\sup_{a \in A} |a|_Y \leq |f(c)|_Y$.

We prove, by transfinite induction over the X-constructible codes, the existence of a partial Y-computable function $g : \mathbb{N} \to \mathbb{N}$ such that for every $a \in \mathcal{O}^X$, $g(a) \in \mathcal{O}^Y$ and $|a|_X \leq |g(a)|_Y$. Let $a \in \mathcal{O}^X$.

Suppose first $a = 1$ codes for $\mathbb{0}$. Letting $g(a) = 1$, we have $|a|_X = |g(a)|_Y$.

Suppose now $a = 2^b$ codes for a successor ordinal. Then by induction hypothesis, $g(b) \in \mathcal{O}^Y$ and $|b|_X \leq |g(b)|_Y$. Letting $g(a) = 2^{g(b)}$, we have $|a|_X = |b|_X + \mathbb{1} \leq |g(b)|_Y + \mathbb{1} = |g(a)|_Y$.

Suppose last $a = 3 \cdot 5^e$ codes for a limit ordinal. Then for every n, by induction hypothesis, $g(\Phi_e^X(n)) \in \mathcal{O}^Y$ and $|\Phi_e^X(n)|_X \leq |g(\Phi_e^X(n))|_Y$. Since X is $\Delta^1_1(Y)$, the set $A = \{g(\Phi_e^X(n)) : n \in \mathbb{N}\} \subseteq \mathcal{O}^Y$ is $\Sigma^1_1(Y)$. Furthermore, a $\Sigma^1_1(Y)$-code c of A can be found uniformly in e. Let $g(a) = f(c)$. ■

Last, the following theorem relates the hypercomputation of Kleene's $\mathcal{O}$ to the computation of a non-computable ordinal. It implies in particular that the hyperjump is strictly increasing in the hyperdegrees.

Theorem 11.6.7 (Spector [94])
Let X be a set. Then $X \geq_h \mathcal{O}$ iff $\omega_1^X > \omega_1^{ck}$.[39]

39: This statement relativizes as follows: let X, Y be sets such that $X \geq_h Y$. Then $X \geq_h \mathcal{O}^Y$ iff $\omega_1^X > \omega_1^Y$. In particular, the hypothesis $X \geq_h Y$ is necessary for the equivalence to hold.

11.6.3 Classes of reals

One can define an effective Borel hierarchy for the Cantor space as one did for the discrete topology on $\mathbb{N}$. This yields the notions of Σ^0_α and Π^0_α classes of reals for every $\alpha < \omega_1^{ck}$. The notions of Σ^0_α-code and Π^0_α-code for classes are defined accordingly.

Many previous theorems about the arithmetic hierarchy relativize uniformly in the oracle. They enable to give canonical representations of the effective Borel hierarchy using iterations of the halting set. Recall that every Σ^0_k class of reals is of the form $\{X : n \in X^{(k)}\}$ for some $n \in \mathbb{N}$. The generalization to the transfinite levels yields the following theorem.

Theorem 11.6.8 (Monin and Patey [4])
Fix some $a \in \mathcal{O}$ such that $|a| \geq \omega$. A class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is $\Sigma^0_{|a|}$ iff there is some $n \in \mathbb{N}$ such that $\mathcal{A} = \{X : n \in H^X_{2^a}\}$.[40]

40: Note again the shift in indices between the finite levels and the transfinite levels.

Given a set Y and $\beta < \omega_1^Y$, we let $\mathcal{O}^Y_{<\beta} = \{a \in \mathcal{O} : |a|_Y < \beta\}$. Among the classes of reals, we shall be particularly interested in the following family of classes:

Theorem 11.6.9 (Spector [94])
For every $n \in \mathbb{N}$ and $a \in \mathcal{O}$, the class $\{X : n \in \mathcal{O}^X_{<|a|}\}$ is $\Sigma^0_{|a|+\mathbb{1}}$ uniformly in n and a.

11.7 Transfinite jump control

Transfinite jump control involves different sets of techniques, depending on whether one wants to control a fixed level in the hyperarithmetic hierarchy, or the hyperjump itself. Indeed, α-jump control for a fixed level $\alpha < \omega_1^{ck}$ is achieved by designing a Σ^0_α-preserving forcing question for Σ^0_α-classes, while hyperjump control furthermore requires to consider G-computable ordinals $\alpha < \omega_1^G$, where G is the generic set being built. This section is therefore divided into two parts, each focusing on one problematic.

11.7.1 α-jump control

As usual, we illustrate the technique with the simplest notion of forcing, namely, Cohen forcing, and with α-jump cone avoidance.

Theorem 11.7.1 (Feferman [90])
Fix a non-zero $\alpha < \omega_1^{ck}$ and let C be a non-Δ^0_α set. For every sufficiently Cohen generic filter $\mathcal{F}$, C is not $\Delta^0_\alpha(G_\mathcal{F})$.

Proof. This proof is a generalization of Theorem 11.2.1 to transfinite levels. Contrary to finite levels which can be represented by arithmetic formulas, defining a notion of Σ^0_α-formula for $\alpha \geq \omega$ would require to work with some effective infinitary logic, with effective countable disjunctions and intersections. It is therefore more convenient to define the forcing relation in terms of classes.

Definition 11.7.2. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition, and $\mathcal{B} \subseteq 2^{\mathbb{N}}$ be a Σ^0_α class for $\alpha < \omega_1^{ck}$.[41]

1. For $\alpha = \mathbb{1}$, let $\sigma \mathrel{?\vdash} \mathcal{B}$ hold if there is some $\tau \geq \sigma$ such that $[\tau] \subseteq \mathcal{B}$.
2. For $\alpha > \mathbb{1}$, $\mathcal{B}$ is of the form $\bigcup_n \mathcal{B}_{\beta_n}$ where $\mathcal{B}_{\beta_n}$ is $\Pi^0_{\beta_n}$. Let $\sigma \mathrel{?\vdash} \mathcal{B}$ hold if there is some $\tau \geq \sigma$ and some $n \in \mathbb{N}$ such that $\tau \mathrel{?\vdash} \mathcal{B}_{\beta_n}$.[42] ◇

41: The notation $\sigma \mathrel{?\vdash} \mathcal{B}$ is a shorthand for $\sigma \mathrel{?\vdash} G \in \mathcal{B}$. At finite levels, $\mathcal{B}$ can be written as $\{X \in 2^{\mathbb{N}} : \varphi(X)\}$ for some Σ^0_n-formula φ and $\sigma \mathrel{?\vdash} \mathcal{B}$ iff $\sigma \mathrel{?\vdash} \varphi(G)$.

42: The class $\mathcal{B}_{\beta_n}$ is $\Pi^0_{\beta_n}$, and the forcing question for Π-formulas is induced from the one for Σ-formulas. Thus, $\tau \mathrel{?\vdash} \mathcal{B}_{\beta_n}$ is a shorthand for $\tau \mathrel{?\nvdash} (2^{\mathbb{N}} \setminus \mathcal{B}_{\beta_n})$

We start by proving that the forcing question for Σ^0_α-classes is Σ^0_α-preserving uniformly in its parameters, for $\alpha < \omega_1^{ck}$.

Lemma 11.7.3. For every non-zero $\alpha < \omega_1^{ck}$, every Σ^0_α class $\mathcal{B} \subseteq 2^{\mathbb{N}}$ and every Cohen condition $\sigma \in 2^{<\mathbb{N}}$. The relation $\sigma \mathrel{?\vdash} \mathcal{B}$ is Σ^0_α uniformly in σ and a Σ^0_α-code c of $\mathcal{B}$. ★

Proof. By induction over α. For $\alpha = \mathbb{1}$, $c = \langle 0, e \rangle$ and $\mathcal{B} = \bigcup_{\tau \in W_e} [\tau]$. Thus, $\sigma \mathrel{?\vdash} \mathcal{B}$ iff there is some $\tau \in W_e$ such that $[\sigma] \cap [\tau] \neq \emptyset$, which is a Σ^0_1 relation uniformly in σ and $\langle 0, e \rangle$.

For $\alpha > \mathbb{1}$, $c = \langle 2, e \rangle$ and $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where $\mathcal{B}_n$ is a $\Pi^0_{\beta_n}$ class of $\Pi^0_{\beta_n}$-code $c_n \in W_e$. Then $\sigma \mathrel{?\vdash} \mathcal{B}$ iff there is some $n \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \mathrel{?\nvdash} (2^{\mathbb{N}} \setminus \mathcal{B}_n)$. By induction hypothesis, the relation $\tau \mathrel{?\vdash} (2^{\mathbb{N}} \setminus \mathcal{B}_n)$ is $\Sigma^0_{\beta_n}$ uniformly in a $\Sigma^0_{\beta_n}$-code of $(2^{\mathbb{N}} \setminus \mathcal{B}_n)$, thus $\tau \mathrel{?\vdash} \mathcal{B}_n$ is $\Pi^0_{\beta_n}$ uniformly in a $\Pi^0_{\beta_n}$-code of $\mathcal{B}_n$. Thus, the overall relation is $\Sigma^0_{\sup_n(\beta_n+\mathbb{1})}$, hence is Σ^0_α. ■

The following lemma shows that the definition of the forcing question meets a strong version of its specifications.

Lemma 11.7.4. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition and $\mathcal{B} \subseteq 2^{\mathbb{N}}$ be a Σ^0_α class for $\alpha < \omega_1^{ck}$.

1. If $\sigma \mathrel{?\vdash} \mathcal{B}$, then there is an extension $\tau \geq \sigma$ forcing $G \in \mathcal{B}$.
2. If $\sigma \mathrel{?\nvdash} \mathcal{B}$, then σ forces $G \notin \mathcal{B}$. ★

Proof. We prove simultaneously both items inductively on α.

Base case: $\alpha = \mathbb{1}$. If $\sigma \mathrel{?\vdash} \mathcal{B}$, then, letting $\tau \geq \sigma$ be such that $[\tau] \subseteq \mathcal{B}$, for every filter $\mathcal{F}$ containing τ, $G_{\mathcal{F}} \in \mathcal{B}$. It follows that τ is an extension of σ forcing $G \in \mathcal{B}$. Conversely, if σ does not force $G \notin \mathcal{B}$, then there is a filter $\mathcal{F}$ containing σ such that $G_{\mathcal{F}} \in \mathcal{B}$. Then, since $\mathcal{B}$ is open in Cantor space, there is a finite $\tau \prec G_{\mathcal{F}}$ such that $[\tau] \subseteq \mathcal{B}$. Since $\sigma \prec G_{\mathcal{F}}$, by taking τ long enough, one has $\sigma \prec \tau$, thus $\sigma \mathrel{?\vdash} \mathcal{B}$.

Inductive case: $\alpha > \mathbb{1}$. Say $\mathcal{B} = \bigcup_n \mathcal{B}_n$, where $\mathcal{B}_n$ is $\Pi^0_{\beta_n}$. If $\sigma \mathrel{?\vdash} \mathcal{B}$, then there is some $n \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \mathrel{?\vdash} \mathcal{B}_n$. By induction hypothesis, there is some $\rho \geq \tau$ forcing $G \in \mathcal{B}_n$. In particular, ρ is an extension of σ forcing $G \in \mathcal{B}$. If $\sigma \mathrel{?\nvdash} \mathcal{B}$, then for every $n \in \mathbb{N}$ and every $\tau \geq \sigma$, $\tau \mathrel{?\nvdash} \mathcal{B}_n$. By induction hypothesis, for every $n \in \mathbb{N}$ and every $\tau \geq \sigma$, there is some $\rho \geq \tau$ forcing $G \notin \mathcal{B}_n$. In other words, for every $n \in \mathbb{N}$, the set of all ρ forcing $G \notin \mathcal{B}_n$ is dense below σ. Thus, for every sufficiently generic filter $\mathcal{F}$ containing σ and for every $n \in \mathbb{N}$, there is some $\rho \in \mathcal{F}$ forcing $G \notin \mathcal{B}_n$, hence $G \notin \bigcup_n \mathcal{B}_n$. In other words, σ forces $G \notin \mathcal{B}$. ■

The following diagonalization lemma is a straightforward generalization of Lemma 3.2.2. Fix some $a \in \mathcal{O}$ such that $|a| = \alpha$. Recall that a set is H_a^Y-computable iff $\alpha < \omega$ and it is $\Delta^0_{\alpha+\mathbb{1}}(Y)$, or $\alpha \geq \omega$ and it is $\Delta^0_\alpha(Y)$. For simplicity, we shall handle only the case $\alpha \geq \omega$, since the finite case is Lemma 11.2.4.

Lemma 11.7.5. For every Cohen condition $\sigma \in 2^{<\mathbb{N}}$ and every Turing index e, there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{H_a^G} \neq C$. ⋆

Proof. Consider the following set[43]

$$U = \{(x, v) \in \mathbb{N} \times 2 : p \mathrel{?\vdash} \{X : \Phi_e^{H_a^X}(x){\downarrow} = v\}\}$$

43: By Corollary 11.5.8, for $\alpha \geq \omega$, the following class is Σ^0_α uniformly in x and v:

$$\mathcal{B}_{x,v} = \{X : \Phi_e^{H_a^X}(x){\downarrow} = v\}$$

Since the forcing question is Σ^0_α-preserving, the set U is Σ^0_α. There are three cases:

- Case 1: $(x, 1 - C(x)) \in U$ for some $x \in \mathbb{N}$. By Lemma 11.7.4(1), there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{H_a^G}(x){\downarrow} = 1 - C(x)$.
- Case 2: $(x, C(x)) \notin U$ for some $x \in \mathbb{N}$. By Lemma 11.7.4(2), there is an extension $\tau \geq \sigma$ forcing $\Phi_e^{H_a^G}(x){\uparrow}$ or $\Phi_e^{H_a^G}(x){\downarrow} \neq C(x)$.
- Case 3: None of Case 1 and Case 2 holds. Then U is a Σ^0_α graph of the characteristic function of C, hence C is Δ^0_α. This contradicts our hypothesis. ■

We are now ready to prove Theorem 11.7.1. Let $\mathcal{F}$ be a sufficiently generic filter for Cohen forcing, and let $G_\mathcal{F} = \bigcup \mathcal{F}$. By genericity of $\mathcal{F}$, $G_\mathcal{F}$ is an infinite binary sequence. If $\alpha < \omega$, by Lemma 11.2.4 $C \not\leq G_\mathcal{F}^{(\alpha-1)}$. If $\alpha \geq \omega$, by Lemma 11.7.5, $C \not\leq_T H_a^{G_\mathcal{F}}$. In both cases, C is not $\Delta^0_\alpha(G_\mathcal{F})$. This completes the proof of Theorem 11.7.1. ■

Exercise 11.7.6. Let $(\mathbb{P}, \leq)$ be the primitive recursive Jockusch-Soare forcing, that is, $\mathbb{P}$ is the set of all infinite primitive recursive binary trees $T \subseteq 2^{<\mathbb{N}}$, partially ordered by inclusion. Fix a non-zero $\alpha < \omega_1^{ck}$.

1. Adapt the proof of Theorem 9.4.1 to design a Σ^0_α-preserving forcing question for Σ^0_α-formulas.
2. Deduce that for every non-Δ^0_α set C and every sufficiently generic $\mathbb{P}$-filter $\mathcal{F}$, C is not $\Delta^0_\alpha(G_\mathcal{F})$. ⋆

11.7.2 Hyperjump control

Hyperjump control can be seen as the higher counterpart of first-jump control. Recall that the hyperjump of a set X is the set $\mathcal{O}^X$, that is, Kleene's O relative to X. The goal of this section is to develop a set of tools to prove that, given a sufficiently generic filter $\mathcal{F}$, $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. From this, it follows that the levels of the relativized hyperarithmetic hierarchy are left unchanged, reducing hyperjump control to α-jump control for every $\alpha < \omega_1^{ck}$.

For this, we first need to define sets and classes slightly more complex than the hyperarithmetic hierarchy, but still in the Borel realm. Recall that, although the notion of Σ^0_α-code can be defined for every ordinal α, by the Σ^1_1 majoration theorem, the corresponding hierarchy collapses at the level of ω_1^{ck}, that is, every Σ^0_α set is Σ^0_β for some $\beta < \omega_1^{ck}$. One can however extend the family of sets and classes by considering effective unions along Π^1_1 sets of ordinals. A *hyperarithmetic code* is a Σ^0_α-code for some $\alpha < \omega_1^{ck}$, and a Π^1_1-*code* of a set $A \subseteq \mathbb{N}$ is a code of a Π^1_1-formula defining A.

Definition 11.7.7.

1. A $\Sigma^0_{\omega_1^{ck}}$-*code* of a class $\mathcal{B} \subseteq 2^{\mathbb{N}}$ is a pair $\langle 3, e\rangle$, where e is Π^1_1-code of set $A \subseteq \mathbb{N}$ such that $\mathcal{B} = \bigcup_{e \in A} \mathcal{B}_e$, where $\mathcal{B}_e$ is the class of hyperarithmetic code e.[44]
2. A $\Pi^0_{\omega_1^{ck}}$-*code* of a class $\mathcal{B} \subseteq 2^{\mathbb{N}}$ is a pair $\langle 1, e\rangle$, where e is a $\Sigma^0_{\omega_1^{ck}}$-code of the class $2^{\mathbb{N}} \setminus \mathcal{B}$.
3. A $\Sigma^0_{\omega_1^{ck}+1}$-*code* of a class $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is a pair $\langle 2, e\rangle$ where W_e is non-empty and enumerates $\Pi^0_{\omega_1^{ck}}$-codes of the classes $\mathcal{B}_n$. ◇

44: As explained, this notion does not coincide with the naive definition of $\Sigma^0_{\omega_1^{ck}}$ in terms of effective countable union of hyperarithmetic sets. The set of hyperarithmetic codes of the union must be non-Σ^1_1 in order to properly extend the hyperarithmetic hierarchy.

A class $\mathcal{B} \subseteq 2^{\mathbb{N}}$ is $\Sigma^0_{\omega_1^{ck}}$ ($\Pi^0_{\omega_1^{ck}}$, $\Sigma^0_{\omega_1^{ck}+1}$) if it admits a corresponding code. One can define the notions of $\Sigma^0_{\omega_1^{ck}}$, $\Pi^0_{\omega_1^{ck}}$ and $\Sigma^0_{\omega_1^{ck}+1}$ for sets accordingly. In the case of sets, Π^1_1 and $\Sigma^0_{\omega_1^{ck}}$ sets coincide. For classes on the other hand, every $\Sigma^0_{\omega_1^{ck}}$ class is Π^1_1, but the converse is not true.[45]

45: From a topological viewpoint, every $\Sigma^0_{\omega_1^{ck}+1}$ class is Borel. The Borel hierarchy does not collapse on the Cantor space, and there exists effectively co-analytic (Π^1_1) classes which are not Borel. On the other hand, as mentioned before, every set of integers is open in the discrete topology on $\mathbb{N}$, so there is no contradiction to the equivalence between Π^1_1 and $\Sigma^0_{\omega_1^{ck}}$ sets.

It will be sometimes more convenient to represent a $\Sigma^0_{\omega_1^{ck}}$ class as a countable union along $\mathcal{O}$. The following lemma shows that the two definitions are equivalent.

Lemma 11.7.8. A class $\mathcal{B} \subseteq 2^{\mathbb{N}}$ is $\Sigma^0_{\omega_1^{ck}}$ iff $\mathcal{B} = \bigcup_{a \in \mathcal{O}} \mathcal{D}_a$, where $\mathcal{D}_a$ is hyperarithmetic uniformly in a.[46] ★

46: Note that one can computably switch from one representation to the other.

Proof. Suppose first $\mathcal{B} = \bigcup_{e \in A} \mathcal{B}_e$, where A is Π^1_1 and $\mathcal{B}_e$ is the class of hyperarithmetic code e. Since $\mathcal{O}$ is Π^1_1-complete for the many-one reduction, there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $e \in A$ iff $f(e) \in \mathcal{O}$. One can furthermore suppose that f is injective and increasing, since given a code $a \in \mathcal{O}$ and $n \in \mathbb{N}$, $2^a_n \in \mathcal{O}$ iff $a \in \mathcal{O}$.[47] In particular, the range of f is computable. For every $a \in \mathcal{O}$, $\mathcal{D}_a = \mathcal{B}_{f^{-1}(a)}$ if a is in the range of f, and $\mathcal{D}_a = \emptyset$ otherwise. Note that $\mathcal{D}_a$ is Σ^0_β for some $\beta < \omega_1^{ck}$, and a Σ^0_β-code of $\mathcal{D}_a$ can be found uniformly in a. By construction, $\mathcal{B} = \bigcup_{a \in \mathcal{O}} \mathcal{D}_a$.

47: The function $(a, n) \mapsto 2^a_n$ is defined inductively by $2^a_0 = a$ and $2^a_{n+1} = 2^{2^a_n}$.

Suppose now $\mathcal{B} = \bigcup_{a \in \mathcal{O}} \mathcal{D}_a$, where $\mathcal{D}_a$ is hyperarithmetic uniformly in a. Let $f : \mathbb{N} \to \mathbb{N}$ be a partial computable function such that $f(a)$ is a hyperarithmetic code of $\mathcal{D}_a$ for every $a \in \mathcal{O}$. Here again, one can suppose that f is injective and increasing, since one can computably transform a hyperarithmetic code into a larger hyperarithmetic code of the same class. Let $A = \{f(a) : a \in \mathcal{O}\}$. The set A is Π^1_1 as it is the image of a Π^1_1 set by a computable injective function. Thus $\mathcal{B} = \bigcup_{e \in A} \mathcal{B}_e$, where $\mathcal{B}_e$ is the class of hyperarithmetic code e. ■

As usual, Cohen forcing provides a simple example to illustrate the use of the forcing question. We therefore prove that Cohen genericity preserves ω_1^{ck}.

Theorem 11.7.9 (Feferman [90])
For every sufficiently Cohen generic filter $\mathcal{F}$, $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$.

Proof. Suppose $\omega_1^G > \omega_1^{ck}$, then there is an element $a \in \mathcal{O}^G$ which codes for ω_1^{ck}. Since ω_1^{ck} is a limit ordinal, $a = 3 \cdot 5^e$, where $\forall n \Phi^G_e(n)\downarrow \in \mathcal{O}^G_{<\omega_1^{ck}}$ and

48: The set $\mathcal{O}^G_{<\alpha}$ is the set of all codes $a \in \mathcal{O}^G$ such that $|a|_G < \alpha$. Note that $\mathcal{O}^G_{<\omega_1^{ck}} \neq \mathcal{O}$ in general. We can however assume for convenience that $\mathcal{O} \subseteq \mathcal{O}^G_{<\omega_1^{ck}}$.

with $\sup_n |\Phi_e^G(n)|_G = \omega_1^{ck}$.[48] We shall therefore naturally work with $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ classes. We first extend the forcing question to $\Sigma^0_{\omega_1^{ck}}$ and $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ classes, assuming the existence of a Σ^0_α-preserving forcing question for Σ^0_α-formulas (see the proof of Theorem 11.7.1).

Definition 11.7.10. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition, and $\mathcal{B} = \bigcup_{a \in \mathcal{O}} \mathcal{B}_a$ be a $\Sigma^0_{\omega_1^{ck}}$ class.[49] Let $\sigma \mathrel{?\vdash} \mathcal{B}$ hold if there is some $a \in \mathcal{O}$ and some $\tau \geq \sigma$ such that $\tau \mathrel{?\vdash} \mathcal{B}_a$. ◊

49: By Lemma 11.7.8, $\mathcal{B}$ can be written of this form.

The forcing question for a $\Sigma^0_{\omega_1^{ck}}$-class $\mathcal{B}$ is $\Sigma^0_{\omega_1^{ck}}$ uniformly in a $\Sigma^0_{\omega_1^{ck}}$-code of $\mathcal{B}$. One easily proves that the forcing question meets its specifications. The proof is left as an exercise.

Exercise 11.7.11. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition, and $\mathcal{B} = \bigcup_{a \in \mathcal{O}} \mathcal{B}_a$ be a $\Sigma^0_{\omega_1^{ck}}$ class. Prove that

1. if $\sigma \mathrel{?\vdash} \mathcal{B}$, then there is an extension of σ forcing $G \in \mathcal{B}$;
2. if $\sigma \mathrel{?\nvdash} \mathcal{B}$, then there is an extension of σ forcing $G \notin \mathcal{B}$. ★

We now extend the forcing question to $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ classes.

Definition 11.7.12. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition, and $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ class. Let $\sigma \mathrel{?\vdash} \mathcal{B}$ hold if there is some $n \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \mathrel{?\vdash} \mathcal{B}_n$.[50] ◊

50: The class $\mathcal{B}_n$ is $\Pi^0_{\omega_1^{ck}}$, so $\tau \mathrel{?\vdash} \mathcal{B}_n$ is a shorthand for $\tau \mathrel{?\nvdash} (2^{\mathbb{N}} \setminus \mathcal{B}_n)$. The forcing question for $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$-classes is $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$-preserving, but we are not going to use this fact in the proof.

The forcing question for $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ classes meets its specification, but one can actually prove a stronger version of it, in the negative case. Recall that, given a set Y and $\beta < \omega_1^Y$, we let $\mathcal{O}^Y_{<\beta} = \{a \in \mathcal{O} : |a|_Y < \beta\}$.

Lemma 11.7.13. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition, and $\mathcal{B} = \bigcup_n \bigcap_{a \in \mathcal{O}} \mathcal{B}_{n,a}$ be a $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ class, where $\mathcal{B}_{n,a}$ is hyperarithmetic uniformly in n and a.[51]

51: Every $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$ class can be written of this form thanks to Lemma 11.7.8.

1. If $\sigma \mathrel{?\vdash} \mathcal{B}$, then there is an extension of σ forcing $G \in \mathcal{B}$;
2. If $\sigma \mathrel{?\nvdash} \mathcal{B}$, then there is some $\beta < \omega_1^{ck}$ and an extension of σ forcing $G \notin \bigcup_n \bigcap_{a \in \mathcal{O}_{<\beta}} \mathcal{B}_{n,a}$.[52] ★

52: Note that $\mathcal{B} \subseteq \bigcup_n \bigcap_{a \in \mathcal{O}_{<\beta}} \mathcal{B}_{n,a}$.

Proof. Suppose $\sigma \mathrel{?\vdash} \mathcal{B}$. Then there is some $n \in \mathbb{N}$ and some $\tau \geq \sigma$ such that $\tau \mathrel{?\vdash} \bigcap_{a \in \mathcal{O}} \mathcal{B}_{n,a}$. By Exercise 11.7.11, there is an extension $\rho \geq \tau$ forcing $G \in \bigcap_{a \in \mathcal{O}} \mathcal{B}_{n,a}$, hence forcing $G \in \mathcal{B}$.

Suppose $\sigma \mathrel{?\nvdash} \mathcal{B}$. For every n and every $\tau \geq \sigma$, $\tau \mathrel{?\nvdash} \bigcap_{a \in \mathcal{O}} \mathcal{B}_{n,a}$, in other words, $\tau \mathrel{?\vdash} \bigcup_{a \in \mathcal{O}} (2^{\mathbb{N}} \setminus \mathcal{B}_{n,a})$. Unfolding the definition, for every n, and every $\tau \geq \sigma$, there is some $\rho \geq \tau$ and some $a \in \mathcal{O}$ such that $\rho \mathrel{?\vdash} (2^{\mathbb{N}} \setminus \mathcal{B}_{n,a})$. Given $n \in \mathbb{N}$ and $\tau \geq \sigma$, let $f(n, \tau) = a$ for some $a \in \mathcal{O}$ such that there some $\rho \geq \tau$ for which $\rho \mathrel{?\vdash} (2^{\mathbb{N}} \setminus \mathcal{B}_{n,a})$. The function f is Π^1_1 and total, so by Corollary 11.6.4, there is some $\beta < \omega_1^{ck}$ such that $\sup_{n, \tau \geq \sigma} |f(n, \tau)| < \beta$. Thus, for every $n \in \mathbb{N}$ and every $\tau \geq \sigma$, there is some $\rho \geq \tau$ and some $a \in \mathcal{O}_{<\beta}$ such that $\rho \mathrel{?\vdash} (2^{\mathbb{N}} \setminus \mathcal{B}_{n,a})$, and by definition of the forcing question, there is some $\mu \geq \rho$ forcing $G \notin \mathcal{B}_{n,a}$. For every n, let D_n be the set of μ such that for some $a \in \mathcal{O}_{<\beta}$, μ forces $G \notin \mathcal{B}_{n,a}$. The set D_n is dense below σ for every $n \in \mathbb{N}$, so for every sufficiently generic filter $\mathcal{F}$ containing σ, $\mathcal{F} \cap D_n \neq \emptyset$, and thus $G_{\mathcal{F}} \notin \bigcup_n \bigcap_{a \in \mathcal{O}_{<\beta}} \mathcal{B}_{n,a}$. ■

The following lemma is an immediate application of Lemma 11.7.13. The core argument actually lies in Lemma 11.7.13 rather than Lemma 11.7.14.

Lemma 11.7.14. Let $\sigma \in 2^{<\mathbb{N}}$ be a Cohen condition and Φ_e be a Turing functional. There is an extension $\tau \geq \sigma$ forcing one of the following:

1. $\exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e^G(n) \notin \mathcal{O}^G_{<\alpha}$;
2. $\exists \beta < \omega_1^{ck}\ \forall n\ \Phi_e^G(n) \in \mathcal{O}^G_{<\beta}$. ★

Proof. By Spector [94], the class $\mathcal{B}_{n,a} = \{X : \Phi_e^X(n) \notin \mathcal{O}^X_{<|a|}\}$ is hyperarithmetic uniformly in $n \in \mathbb{N}$ and $a \in \mathcal{O}$. It follows that the class $\mathcal{B} = \bigcup_n \bigcap_{a \in \mathcal{O}} \mathcal{B}_{n,a}$ is $\Sigma^0_{\omega_1^{ck}+1}$. If $\sigma\ ?\vdash \mathcal{B}$, then by Lemma 11.7.13(1), there is an extension forcing $G \in \mathcal{B}$, in other words forcing $\exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e^G(n) \notin \mathcal{O}^G_{<\alpha}$. If $\sigma\ ?\nvdash \mathcal{B}$, then by Lemma 11.7.13(2), there is some $\beta < \omega_1^{ck}$ and an extension of σ forcing $G \notin \bigcup_n \bigcap_{a \in \mathcal{O}_{<\beta}} \mathcal{B}_{n,a}$, in other words forcing $\forall n \Phi_e^G(n) \in \mathcal{O}^G_{<\beta}$. ■

We are now ready to prove Theorem 11.7.9. Let $\mathcal{F}$ be a sufficiently generic filter for Cohen forcing. Suppose for the contradiction that $\omega_1^{G_\mathcal{F}} > \omega_1^{ck}$. Then there is some $a \in \mathcal{O}^{G_\mathcal{F}}$ which codes for ω_1^{ck}. Since ω_1^{ck} is a limit ordinal, $a = 3 \cdot 5^e$, where $\forall n \Phi_e^{G_\mathcal{F}}(n) \downarrow \in \mathcal{O}^{G_\mathcal{F}}_{<\omega_1^{ck}}$ and with $\sup_n |\Phi_e^{G_\mathcal{F}}(n)|_G = \omega_1^{ck}$. By Lemma 11.7.14, either $\exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e^{G_\mathcal{F}}(n) \notin \mathcal{O}^{G_\mathcal{F}}_{<\alpha}$, or $\exists \beta < \omega_1^{ck}\ \forall n\ \Phi_e^{G_\mathcal{F}}(n) \in \mathcal{O}^{G_\mathcal{F}}_{<\beta}$, in which case $\sup_n |\Phi_e^G(n)|_G \leq \beta < \omega_1^{ck}$. In both cases, this yields a contradiction, so $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. This completes the proof of Theorem 11.7.9. ■

Combining Theorem 11.7.9 and Theorem 11.7.1, we obtain cone avoidance for the hyperarithmetic reduction.

Corollary 11.7.15 (Feferman [90])
Let C be a non-hyperarithmetic set. For every sufficiently generic Cohen filter $\mathcal{F}$, $C \not\leq_h G_\mathcal{F}$.

Proof. Let $\mathcal{F}$ be a sufficiently generic Cohen filter. By Theorem 11.7.1, C is not $\Delta^0_\alpha(G_\mathcal{F})$ for any $\alpha < \omega_1^{ck}$, and by Theorem 11.7.9, $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. It follows that C is not $\Delta^0_\alpha(G_\mathcal{F})$ for any $\alpha < \omega_1^{G_\mathcal{F}}$, thus $C \not\leq_h G_\mathcal{F}$. ■

The following contains the core property to prove that every sufficiently generic filter preserves ω_1^{ck}.

Definition 11.7.16. Given a notion of forcing $(\mathbb{P}, \leq)$, a forcing question is $\Sigma^0_{\omega_1^{ck}+1}$*-majoring* if for every $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_n \bigcap_{a \in \mathcal{O}} \mathcal{B}_{n,a}$ where $\mathcal{B}_{n,a}$ is hyperarithmetic uniformly in n and a, for every condition $p \in \mathbb{P}$ such that $p\ ?\nvdash \mathcal{B}$, there is some $\beta < \omega_1^{ck}$ and an extension $q \leq p$ forcing $G \notin \bigcup_n \bigcap_{a \in \mathcal{O}_{<\beta}} \mathcal{B}_{n,a}$. ◊

We leave the abstract theorem as an exercise.

Exercise 11.7.17. Let $(\mathbb{P}, \leq)$ be a notion of forcing, with a $\Sigma^0_{\omega_1^{ck}+1}$-majoring forcing question. Prove that for every sufficiently generic filter $\mathcal{F}$, $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. ★

Exercise 11.7.18. Let $(\mathbb{P}, \leq)$ be the primitive recursive Jockusch-Soare forcing, that is, $\mathbb{P}$ is the set of all infinite primitive recursive binary trees $T \subseteq 2^{<\mathbb{N}}$, partially ordered by inclusion.

1. Show the existence of a $\Sigma^0_{\omega_1^{ck}+\mathbb{1}}$-majoring forcing question.
2. Deduce that for every sufficiently generic filter $\mathcal{F}$, $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. ⋆

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