Ramsey-like theorems for the Schreier barrier

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Abstract

The family of finite subsets s of the natural numbers such that $|s| = 1 + \min s$ is known as the Schreier barrier in combinatorics and Banach Space theory, and as the family of exactly ω-large sets in Logic. We formulate and prove the generalizations of Friedman's Free Set and Thin Set theorems and of Rainbow Ramsey's theorem to colorings of the Schreier barrier. We analyze the strength of these theorems from the point of view of Computability Theory and Reverse Mathematics. Surprisingly, the exactly ω-large counterparts of the Thin Set and Free Set theorems can code $\emptyset^{(\omega)}$, while the exactly ω-large Rainbow Ramsey theorem does not code the halting set.

1 Introduction and motivation

Ramsey's theorems have been widely investigated from the point of view of Computability Theory, Proof Theory and Reverse Mathematics (see [16] for details and references). In his seminal paper, Jockusch [17] gave a deep analysis of Ramsey's theorem using tools from Computability Theory, which established this theorem as an important bridge between Combinatorics and Computability. The effective and logical strength of many consequences and variants of Ramsey's theorem have since been investigated. Among those, the Free Set, Thin Set and Rainbow Ramsey Theorems have attracted significant interest in recent decades (see, e.g., [4, 34, 27, 3, 21]), due to the peculiar behavior of these theorems when compared to Ramsey's theorem. The Free Set Theorem (denoted FS^n), introduced in the context of Reverse Mathematics by Harvey Friedman [12], states that for every coloring of the n-subsets of the natural numbers in unboundedly many colors, there exists an infinite set H of natural numbers such that for all n-subsets s of H, the color of s is either not in H or else is in s itself. Such a set is called free for the coloring. The Thin Set Theorem (denoted TS^n_ω or TS^n) is a weak variant of the Free Set Theorem, asserting that for any coloring of the n-subsets of the natural numbers there is an infinite set H of natural numbers such that the n-subsets of H avoid at least one color. The Rainbow Ramsey Theorem (denoted RRT^n_k) asserts that for every coloring of the n-subsets of the natural numbers in which each color is used at most

k times, there is an infinite set H of natural numbers such that the coloring assigns different colors to different n-subsets of H. While Ramsey's theorem for colorings of 3-subsets already codes the halting set, none of these principles does the same for any dimension $n \geq 3$. This surprising result is due to Wang [34].

In the present paper we consider generalizations of the Free Set, Thin Set and Rainbow Ramsey Theorems to colorings of objects of unbounded dimension. More precisely, we focus on the natural extensions of these principles to colorings of so-called exactly ω-large sets, i.e., finite sets s of natural numbers such that $|s| = 1 + \min s$.[1]

The concept of ω-large (or relatively large) finite subset of the natural numbers is well-known in the proof theory of Arithmetic, as it is the basic ingredient for the celebrated Paris-Harrington independence result for Peano Arithmetic [25]. In this context, a finite set s of natural numbers is called relatively large (or relatively ω-large) if $|s| \geq \min s$.

Relatively large sets also naturally arise in Ramsey Theory for purely combinatorial reasons. It is well-known, and easy to prove, that the natural generalization of Ramsey's theorem to finite colorings of *all finite sets* is a false principle. This is easily witnessed by coloring according to the parity of the size of the set. The following weakening is also false: for every finite coloring f of the finite subsets of the natural numbers there exists an infinite H of natural numbers such that for infinitely many n, f is constant on the n-subsets of H. Interestingly, a counterexample is given by the coloring that assigns one color to all relatively large sets and the opposite color to all other sets.

While relatively ω-large sets provide a counterexample to the natural extension of Ramsey's theorem to colorings of all finite sets, exactly ω-large sets also provide a way to obtain true versions of Ramsey's theorem for colorings of families of finite sets containing elements of unbounded size. Weakening the requirement of homogeneity from all finite sets to all exactly ω-large sets results in a true principle, sometimes called the Large Ramsey Theorem, which we denote by $\mathsf{RT}^{!\omega}$ following [2]. This principle is arguably the simplest example of a true version of Ramsey's theorem for colorings of objects of *unbounded dimensions*, whereas, as noted above, Ramsey's theorem fails for all finite subsets of natural numbers. $\mathsf{RT}^{!\omega}_2$ is also the base case of a far-reaching generalization of Ramsey's theorem due to Nash-Williams, which ensures monochromatic sets for every coloring of families of finite subsets of the natural numbers satisfying some properties and called *barriers* [33]. In this context the family of exactly ω-large sets is known as the Schreier barrier [33]. The Large Ramsey Theorem has been studied from the perspective of Computability Theory and Reverse Mathematics by Carlucci and Zdanowski [2], and its generalization to barriers in Computability Theory by Clote [5]. They proved that it is computationally and proof-theoretically stronger than the usual Ramsey's theorem for each fixed finite dimension (RT^n_k) and even stronger than Ramsey's theorem for all finite dimensions ($\forall n \mathsf{RT}^n_k$). From the point of view of Computability, the theorem

[1] The inessential variant with $|s| = \min s$ is also common in the literature.

corresponds to $\emptyset^{(\omega)}$; in Reverse Mathematics terms, it is equivalent to ACA_0^+ over RCA_0 (where ACA_0^+ extends RCA_0 by the axiom of closure under the ω-th Turing jump).

In the present paper, we formulate and prove the natural generalization of the Free Set, Thin Set and Rainbow Ramsey Theorems to colorings of exactly ω-large sets and we investigate their effective and logical strength.

1.1 Framework

We shall study our statements using two frameworks: Reverse Mathematics and Weihrauch analysis.

Reverse Mathematics is a foundational program whose goal is to find optimal axioms to prove ordinary theorems. It uses the framework of subsystems of second-order arithmetic, with a base theory, RCA_0 (Recursive Comprehension Axiom), capturing *computable mathematics*. More precisely, RCA_0 consists of the axioms of Robinson arithmetic, together with the Σ^0_1-induction scheme and the Δ^0_1-comprehension scheme. The Σ^0_1-*induction scheme* states, for every Σ^0_1-formula $\varphi(x)$,

$$[\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1))] \to \forall y \varphi(y)$$

The Δ^0_1-*comprehension scheme* states, for every Σ^0_1-formula $\varphi(x)$ and every Π^0_1-formula $\psi(x)$,

$$[\forall x(\varphi(x) \leftrightarrow \psi(x))] \to \exists Y \forall x(x \in Y \leftrightarrow \varphi(x))$$

There exist four other subsystems which, together with RCA_0, calibrate the strength of most theorems. These theorems are known as the "Big Five" (see Montálban [23]). Among these, ACA_0 (Arithmetic Comprehension Axiom) extends the axioms of RCA_0 with the comprehension scheme for all arithmetic formulas. Based on the correspondence between computability and definability, it is equivalent to stating the existence of the Turing jump of any set ($\forall X \exists Y (Y = X')$).

We shall also consider two lesser-known stronger variants of ACA_0, namely, ACA_0' and ACA_0^+. The system ACA_0' extends RCA_0 with the axiom stating closure under all finite jumps; the system ACA_0^+ extends RCA_0 with the axiom stating the existence of the ω-jump of any set. The ω-jump of a set X is the set $X^{(\omega)} = \bigoplus_n X^{(n)}$. The system ACA_0^+ is famous for being the best known upper bound to Hindman's theorem.

Some of our results are expressed in terms of reductions. All principles studied in this paper are of the following logical form: $\forall X(I(X) \to \exists Y S(X,Y))$, where $I(X)$ and $S(X,Y)$ are arithmetical formulas and X and Y are set variables. We refer to such theorems as $\forall\exists$-principles. For principles P of this form we call any X that satisfies I an *instance* of P and any Y that satisfies $S(X,Y)$ a *solution to* P *for* X. We will use the following notion of computable reducibility whose variants became of central interest in Computability Theory and Reverse Mathematics in recent years (see [10] for background and motivation).

Definition 1.1. Q *is* strongly Weihrauch reducible *to* P *(denoted* $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$*) if there exist Turing functionals* Φ *and* Ψ *such that for every instance* X *of* Q *we have that* $\Phi(X)$ *is an instance of* P*, and if* Y *is a solution to* P *for* $\Phi(X)$ *then* $\Psi(Y)$ *is a solution to* Q *for* X*.*

1.2 Organization of the paper

In Section 2 we recall the usual, finite-dimensional versions of the Free Set, Thin Set and Rainbow Ramsey theorems and we observe that these principles fail when generalized to all finite sets. For this, we use the notion of exactly ω-large set. Then, in Section 3, we recall known facts about the extension of Ramsey's theorem to colorings of exactly ω-large sets and show that the generalized Free Set, Thin Set and Rainbow Ramsey Theorem are consequences of the corresponding generalization of Ramsey's theorem. This is analogous to the finite-dimensional case, but some of the proofs require non-trivial adaptations. In Section 4 we show that, contrary to their finite-dimensional counterparts which do not code any non-computable set, the Free Set and Thin Set theorems for exactly ω-large sets code $\emptyset^{(\omega)}$ and imply ACA_0^+. In Section 5, we generalize the statements to a more robust version in terms of barriers, in order to prove upper bounds on a larger class of instances. In Section 6 we prove a cone avoidance result for the Rainbow Ramsey Theorem for exactly ω-large sets and, more generally, for colorings of a larger class of barriers of order type ω^ω. This result entails that none of these principles code the halting set or imply ACA_0. This difference of behavior at the exactly ω-large level is surprising, given the equivalence between the statements $\bigcup_{n\in\omega} \mathsf{RRT}^n$ and $\bigcup_{n\in\omega} \mathsf{FS}^n$ in Reverse Mathematics (see [26, 34]). Last, in Section 7, we conclude and open the discussion to future research directions.

2 Free sets, thin sets and rainbows

The main goal of this section is to recall the definitions of the usual finite-dimensional Free Set, Thin Set and Rainbow Ramsey Theorem and motivate their generalizations to colorings of exactly ω-large sets in analogy to the case of Ramsey's theorem.

Let us first fix some notation and recall the definition of these principles for colorings of finite subsets of fixed size. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}^+$ the set of positive natural numbers. For $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we denote by $[X]^n$ the set of all n-subsets of X. We denote by $[X]^{<\omega}$ the set of all finite subsets of X. We always assume that sets are presented in increasing order and make no distinction between a set and the sequence of its elements in increasing order. We identify a natural number with the set of its predecessors, so that, if $k \geq 1$ we can write $f : X \to k$ to declare a function from X to $\{0, 1, \dots, k-1\}$. A function of this type is often called a coloring of X in k colors.

We start by recalling the classical Ramsey's theorem for finite colorings of the n-subsets of a countable set.

Definition 2.1 (Ramsey Theorem). *Let $n, k \in \mathbb{N}^+$. For every coloring $f : [\mathbb{N}]^n \to k$ there exists an infinite set $H \subseteq \mathbb{N}$ such that $|f([H]^k)| = 1$. The set H is called homogeneous (or monochromatic) for f. We abbreviate this statement by RT^n_k.*

Ramsey's theorem was first studied in Computability Theory by Jockusch [17], who proved that every computable instance admits an arithmetic solution, and constructed a computable instance of RT^3_2 such that every solution computes the halting set. The formalization of Jockusch's proofs in Reverse Mathematics by Simpson [32] yields that RT^1_k is provable over RCA_0 for any standard $k \in \mathbb{N}$, and RT^n_k is equivalent to ACA_0 over RCA_0 for $n \geq 3$ and any $k \geq 2$. The case $n = 2$ was a long-standing open question, until Seetapun [31] proved that no computable instance of RT^2_k codes the halting set, hence that RT^2_k is strictly weaker than ACA_0 over RCA_0.

The following Free Set Theorem has been introduced by Harvey Friedman [12] and first studied in [4].

Definition 2.2 (Free Set Theorem). *Let $n \in \mathbb{N}^+$. For every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that for every set $s \in [H]^n$, if $f(s) \in H$ then $f(s) \in s$. The set H is called* free *for f. We abbreviate this statement by FS^n.*

The notion of free set in Friedman's Free Set Theorem is the same as the one used in a combinatorial characterization theorem for the $\aleph_n$ cardinals by Kuratowski [18]. Here, the following property of an $n+1$-subset U of a set X is with respect to a function $f : [X]^n \to [X]^{<\omega}$ is considered: for all $x \in U$ we have $x \notin f(U \setminus \{x\})$. For the particular case of a function mapping in (singletons from) X this means that for all $x \in U$ we have $f(U \setminus \{x\}) \neq x$. This is the same as asking that for all n-subset s of U, if $f(s) \in U$ then $f(s) \in s$, which means that U is free for f. Interestingly, in Section 6 we are lead to consider extensions of the Free Set Theorem with colors in $[\mathbb{N}]^{<\omega}$.

The next theorem is a weakening of the Free Set Theorem, introduced in [12] and first studied in [4].

Definition 2.3 (Thin Set Theorem). *For $n \geq 1$. For every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $f([H]^n) \neq \mathbb{N}$. The set H is called* thin *for f. We abbreviate this statement by TS^n.*

The notion of free set might seem ad-hoc at first sight, but can be better understood in the light of the Thin Set Theorem. Indeed, an infinite set $H \subseteq \mathbb{N}$ is f-free if and only if, for every $x \in H$, the set $H \setminus \{x\}$ is f-thin with witness color x. Thus, the Free Set Theorem is a natural generalization of the Thin Set Theorem. The next principle is sometimes called an anti-Ramsey theorem or the Rainbow Ramsey Theorem. We first need the following definition.

Definition 2.4 (k-bounded function)**.** *Let $k \in \mathbb{N}^+$ and X be a set. A function $f : X \to \mathbb{N}$ is k-bounded if for all $i \in \mathbb{N}$, $|f^{-1}(i)| \leq k$.*

Definition 2.5 (Rainbow Ramsey Theorem)**.** *Let $n, k \in \mathbb{N}^+$. For all k-bounded colorings $f : [\mathbb{N}]^n \to \mathbb{N}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that f is injective on $[H]^n$. The set H is called a* rainbow *for f. We abbreviate this statement by RRT^n_k.*

The principles $\mathsf{TS}^n, \mathsf{FS}^n$ and RRT^n_k have been thoroughly investigated from the point of view of Computability Theory and Reverse Mathematics by a number of authors ([4, 27, 3, 21]). The general picture that emerged is that these principles are computationally and combinatorially very weak consequences of Ramsey's theorem ($\mathsf{RT}^n \to \mathsf{FS}^n \to \mathsf{TS}^n \wedge \mathsf{RRT}^n$): while RT^3 codes the jump and implies ACA_0, Wang [34] showed that the Free Set, Thin Set and Rainbow Ramsey Theorem do not code the halting set and satisfy the so-called strong cone avoidance property, which entails that they do not imply ACA_0.

In the remainder of this section we observe that, similarly to the case of Ramsey's theorem, the natural generalizations of the Free Set, Thin Set and Rainbow Ramsey Theorems to colorings of all finite subsets of $\mathbb{N}$ *fail*.

Let $\mathsf{FS}^{<\omega}$ be the following principle: For all $f : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ there exists an infinite free set, where we denote by $[\mathbb{N}]^{<\omega}$ the set of all finite subsets of $\mathbb{N}$ which we identify with finite increasing sequences. Analogously, let $\mathsf{TS}^{<\omega}$ be the following principle: for all $f : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ there exists an infinite thin set. Finally, let $\mathsf{RRT}^{<\omega}_k$ be the following statement: For all k-bounded coloring $f : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ there exists an infinite set $H \subseteq \mathbb{N}$ such that f is injective on $[H]^{<\omega}$ (i.e., H is a rainbow for f).

Proposition 2.6. *There exists a coloring of the finite subsets of the natural numbers that admits no infinite thin set (i.e., $\mathsf{TS}^{<\omega}$ is false).*

Proof. Let $f : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ be defined by setting $f(s) = |s|$. Let X be an infinite subset of $\mathbb{N}$. Then $f([X]^{<\omega}) = \mathbb{N}$. $\square$

Similarly to the finite dimensions case [4], the Free Set Theorem for all finite sets implies the Thin Set Theorem for all finite sets. We formulate this fact in terms of reductions since the exact same argument applies to other principles of interest in this paper. The observation that the proof of the next proposition (see proof Theorem 3.2 in [4]) applies to barriers was one of the starting points of the present work.

Proposition 2.7. $\mathsf{TS}^{<\omega} \leq_{\mathrm{sW}} \mathsf{FS}^{<\omega}$.

Proof. Let $f : [\mathbb{N}]^{<\omega} \to \mathbb{N}$. Let A be an infinite free set for f. Let B be a non-empty subset of A such that $A \setminus B$ is infinite. We claim that $A \setminus B$ is thin for f. Assume, by way of contradiction, that for all $n \in \mathbb{N}$ there exists an $s_n \in [A \setminus B]^{<\omega}$ such that $f(s_n) = n$. Take $n \in B$. Thus, $n \in A$. Since A is free for f, it must be the case that $n \in s_n$, contradicting the fact that s_n was chosen in $A \setminus B$. $\square$

As a corollary we obtain the following proposition.

Proposition 2.8. *There exists a coloring of the finite subsets of the natural numbers that admits no infinite free set (i.e.,* $\mathsf{FS}^{<\omega}$ *is false).*

We next show that the natural generalization of the Rainbow Ramsey Theorem to colorings of all finite sequences fails. The proof features the notion of exactly ω-large set, which is central for the present paper.

Definition 2.9 ((Exactly) ω-large sets)**.** *A finite* $s \subseteq \mathbb{N}$ *is* ω*-large (a.k.a. relatively large) if* $|s| \geq 1 + \min s$ *and is exactly* ω*-large if* $|s| = 1 + \min s$*.*

For an infinite $X \subseteq [\mathbb{N}]$ we denote by $[X]^{!\omega}$ the family of all exactly ω-large subsets of X. The family $[\mathbb{N}]^{!\omega}$ coincides with the famous Schreier barrier used in better quasi ordering theory and Banach Space Theory [33].

Proposition 2.10. *There exists a coloring of the finite subsets of the natural numbers that admits no infinite rainbow (i.e.,* $\mathsf{RRT}^{<\omega}_2$ *is false).*

Proof. A finite set $s \subseteq \mathbb{N}$ is called *quasi-exactly* ω*-large* if there exists an exactly ω-large set t such that $s = t \setminus \{\min t\}$. Obviously if t is exactly ω-large then $t \setminus \{\min t\}$ is quasi exactly ω-large. Also it is easy to see that if s is quasi exactly ω-large then there exists a unique exactly ω-large t such that $s = t \setminus \{\min t\}$. Let $b : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ be a bijection. We define a 2-bounded coloring $f : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ as follows. If s is neither exactly ω-large nor quasi exactly ω-large then $f(s) = b(s)$. If s is exactly ω-large then $f(s) = f(s \setminus \{\min s\}) = b(s)$. Let $H \subseteq \mathbb{N}$ be infinite. Then for any $s \in [H]^{!\omega}$ we have $s \setminus \{\min s\} \in [H]^{<\omega}$. But for any such s we have $f(s) = f(s \setminus \{\min s\})$. Thus H is not a rainbow for f.

□

3 Colorings of exactly ω-large sets

In this section, we introduce and prove the generalizations of the Free Set, Thin Set and Rainbow Ramsey theorems to colorings of exactly ω-large sets. We also establish some basic implications and relations among those principles.

As mentioned, Ramsey's theorem fails when generalized to the family of all finite sets. On the other hand, there exists a natural generalization of Ramsey's theorem to some families of finite sets of *unbounded* size that is central for our investigation. The following is the generalization of Ramsey's theorem to colorings of exactly ω-large sets, as a particular case of more general theorems by Pudlák and Rödl [30] and by Farmaki and Negrepontis [11]. Besides, it is a particular case of Nash-Williams' generalization of Ramsey's theorem to barriers, which in turn is a consequence of the Clopen Ramsey Theorem (see [32]).

Definition 3.1 (Large Ramsey Theorem)**.** *Let* $k \geq 1$ *be an integer. For every coloring* $f : [\mathbb{N}]^{!\omega} \to k$*, there exists an infinite set* $H \subseteq \mathbb{N}$ *such that* $|f([H]^{!\omega})| = 1$*. We abbreviate this statement by* $\mathsf{RT}^{!\omega}_k$*.*

The classical Ramsey Theorem is a statement about cardinality, and RT^n_k proves over RCA_0 the following stronger statement "For every infinite set $X \subseteq \mathbb{N}$ and every coloring $f : [X]^n \to k$, there is an infinite f-homogeneous set $H \subseteq X$." The situation is more complex in the case of Large Ramsey Theorem, as there is no clear bijection between $[X]^{!\omega}$ and $[\mathbb{N}]^{!\omega}$. On the other hand, the classical and computability-theoretic proofs of Large Ramsey Theorem hold for the strongest version of the statement. Because of this, Large Ramsey Theorem can arguably be considered as a non-robust statement. A first solution, adopted by Carlucci and Zdanowski [2], consisted in directly studying the stronger formulation. However, making the domain part of the instance raises some issues when considering strong cone avoidance as we do in Section 6, since the domain can be chosen to be sparse enough to compute any hyperarithmetic set. We shall therefore adopt another approach, and prove our lower bounds in terms of the weaker version, while proving the upper bounds on a generalized version formulated in terms of barriers, that will be presented in Section 6. All these versions are equivalent over RCA_0 to ACA_0^+, so either formulation can be chosen.

Carlucci and Zdanowski [2] established the following bounds on the effective content and logical strength of the Large Ramsey Theorem.

Theorem 3.2 (Carlucci-Zdanowski [2])**.**

1. *All computable finite colorings of $[\mathbb{N}]^{!\omega}$ admit an infinite monochromatic set computable in $\emptyset^{(\omega)}$.*
2. *There exists a computable coloring of $[\mathbb{N}]^{!\omega}$ in 2 colors such that all infinite monochromatic sets compute $\emptyset^{(\omega)}$.*
3. *$\mathsf{RT}^{!\omega}_2$ is equivalent to ACA_0^+ over RCA_0.*

Clote [6] proved point 1. for a larger family of colorings and point 2. for colorings of a family closely related to $[\mathbb{N}]^{!\omega}$ (see Theorem 5.5 below).

It is quite natural to ask if the natural generalizations of the Free Set, Thin Set and Rainbow Ramsey Theorems to colorings of exactly ω-large sets hold. The following is the natural generalization of the Free Set Theorem to colorings of exactly ω-large sets.

Definition 3.3 (Large Free Set Theorem)**.** *For every coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ there exists an infinite set $H \subseteq \mathbb{N}$ such that for every set $s \in [H]^{!\omega}$, $f(s) \notin (H \setminus s)$. The set H is called* free *for f. We abbreviate this statement by $\mathsf{FS}^{!\omega}$.*

There exists a direct combinatorial proof of $\mathsf{FS}^{!\omega}$, as for the proof of $\mathsf{RT}^{!\omega}_k$, involving countable applications of RT^n_k and FS^n for $n \in \mathbb{N}^+$ and a final application of FS^1. A computability-theoretic analysis of this proof yields a solution computable in the ω-jump of the instance. We rather establish $\mathsf{FS}^{!\omega}$ by reduction to $\mathsf{RT}^{!\omega}$. The proof combines ideas from the proof of FS^n from RT^n_{2n+2} (Theorem 5.2 and Corollary 5.3 in [4]) and from the proof of Theorem 4.1 in [2].

Theorem 3.4. $\mathsf{FS}^{!\omega} \leq_{\mathrm{sW}} \mathsf{RT}^{!\omega}_2$.

Proof. For $s = \{s_0, \dots, s_{s_0}\}$ an exactly ω-large set with $s_0 > 0$, and $s_0 < \dots < s_{s_0}$ define $s \ominus 1$ to be the following exactly ω-large set: $\{s_0 - 1, s_1 - 1, \dots, s_{s_0-1} - 1\}$. The idea is to keep one degree of freedom: s_{s_0}, as in the proof of Carlucci and Zdanowski [2, Proposition 4.1].

Let $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be an instance of $\mathsf{FS}^{!\omega}$. Consider the following function $g : [\mathbb{N}]^{!\omega} \to 2$ defined by induction. Note that the function g is called recursively on lexicographically smaller parameters, so the induction is well-defined since the lexicographic order is a well-order. For $s = \{s_0, \dots, s_{s_0}\}$ an exactly ω-large set:

$$g(s) = \begin{cases} 0 & \text{if } f(s \ominus 1) = s_i - 1 \textit{ for some } i < s_0 \\ 1 - g(f(s \ominus 1) + 1, s_1, \dots, s_{f(s\ominus 1)+1}) & \\ & \text{if } f(s \ominus 1) < s_0 - 1 \\ 1 - g(s_0, \dots, s_i, f(s \ominus 1) + 1, s_{i+2}, \dots, s_{s_0}) & \\ & \text{if } f(s \ominus 1) \in (s_i - 1, s_{i+1} - 1) \textit{ for some } i < s_0 - 1 \\ 0 & \text{if } f(s \ominus 1) \in (s_{s_0-1} - 1, s_{s_0} - 1) \\ 1 & \text{otherwise } (\textit{if } f(s \ominus 1) \geq s_{s_0}) \end{cases}$$

Let $H = \{x_0, x_1, \dots\} \subseteq \mathbb{N}^+$ be an infinite g-homogeneous set. We claim that $H' = \{x_0 - 1, x_1 - 1, \dots\}$ is f-free. Consider some exactly ω-large set $\{s_0 - 1, \dots, s_{s_0-1} - 1\} \subseteq H'$. Then $\{s_0, \dots, s_{s_0-1}\} \subseteq H$. There are two cases:

Case 1: H is homogeneous for the color 0. Then, take s_{s_0} to be the next element of H after s_{s_0-1} and write $s = \{s_0, \dots, s_{s_0}\}$, then $g(s) = 0$. There are four subcases:

Subcase 1.1: $f(s \ominus 1) = s_i - 1$ for some $i < s_0$. In that case we are done.

Subcase 1.2: $f(s \ominus 1) \in (s_{s_0-1} - 1, s_{s_0} - 1)$. In that case, by definition of s_{s_0}, $f(s \ominus 1)$ is not in H'.

Subcase 1.3: $f(s \ominus 1) < s_0 - 1$ and $g(f(s \ominus 1) + 1, s_1, \dots, s_{f(s\ominus 1)+1}) = 1$. If $f(s \ominus 1) \in H'$ this contradicts the fact that H is g-homogeneous for the color 0.

Subcase 1.4: $f(s \ominus 1) \in (s_i - 1, s_{i+1} - 1)$ for some $i < s_0 - 1$ and $g(s_0, \dots, s_i, f(s \ominus 1) + 1, s_{i+2}, \dots, s_{s_0}) = 1$. If $f(s \ominus 1) \in H'$ this contradicts the fact that H is g-homogeneous for the color 0.

Case 2: H is homogeneous for the color 1. Take $s_{s_0} \in H$ bigger than s_{s_0-1} and write $s = \{s_0, \dots, s_{s_0}\}$, then $g(s) = 1$. There are three subcases:

Subcase 2.1: $f(s \ominus 1) \geq s_{s_0}$. This case is impossible, indeed, as H is infinite, there exists some element $x \in H$ such that $x > f(s \ominus 1) + 1$ and therefore $f(s \ominus 1) \in (s_{s_0} - 1, x - 1)$, which leads to $g(s_0, \dots, s_{s_0-1}, x) = 0$ contradicting the fact that H is g-homogeneous for the color 1.

Subcase 2.2: $f(s \ominus 1) < s_0 - 1$ and $g(f(s \ominus 1) + 1, s_1, \dots, s_{f(s\ominus 1)+1}) = 0$. If $f(s \ominus 1) \in H'$ this contradicts the fact that H is g-homogeneous for the color 1.

Subcase 2.3: $f(s \ominus 1) \in (s_i - 1, s_{i+1} - 1)$ for some $i < s_0 - 1$ and $g(s_0, \dots, s_i, f(s \ominus 1) + 1, s_{i+2}, \dots, s_{s_0}) = 0$. If $f(s \ominus 1) \in H'$ this contradicts the fact that H is g-homogeneous for the color 1.

Notice that g is uniformly computable in f and that H' is uniformly computable in H. Therefore, $\mathsf{FS}^{!\omega} \leq_{\mathrm{sW}} \mathsf{RT}^{!\omega}_2$.

□

The above proof uses the fact that the exactly ω-large sets are well-ordered under lexicographic ordering. Since the order type of this ordering is ω^ω and the statement "ω^ω is well-ordered" implies the consistency of RCA_0 (see [14, 15]), the above proof is not formalizable in RCA_0. Yet, since $\mathsf{RT}^{!\omega}_2$ implies ACA_0 and the latter proves that ω^ω is well-ordered we obtain the following corollary.

Corollary 3.5. $\mathsf{RCA}_0 \vdash \mathsf{RT}^{!\omega}_2 \to \mathsf{FS}^{!\omega}$.

We next introduce the generalization of the Thin Set Theorem to colorings of exactly ω-large sets. For the fixed-dimension case, Cholak et al. [4] have proved in RCA_0 that for all $k \geq 2$, FS^k implies TS^k (see Theorem 3.2 in [4]; the proof yields a strong Weihrauch reduction). A completely analogous argument establishes that the Thin Set Theorem follows from (and is reducible to) the Free Set Theorem for colorings of exactly ω-large sets.

Definition 3.6 (Large Thin Set Theorem)**.** *For every coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $f([H]^{!\omega}) \neq \mathbb{N}$. The set H is called* thin *for f. We abbreviate this statement by $\mathsf{TS}^{!\omega}$.*

Theorem 3.7. $\mathsf{TS}^{!\omega} \leq_{\mathrm{sW}} \mathsf{FS}^{!\omega}$ *and* $\mathsf{RCA}_0 \vdash \mathsf{FS}^{!\omega} \to \mathsf{TS}^{!\omega}$.

Proof. Completely analogous to the proof of Proposition 2.7. □

The following proposition states that requiring that one color is omitted is equivalent to requiring that infinitely many colors are omitted. The result is the analogue of Theorem 3.5 in [4] and can be proved by exactly the same proof.

Proposition 3.8. *For every coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ there exists an infinite set $X \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus f([X]^{!\omega})$ is infinite. Moreover, the just stated principle is strongly Weihrauch-equivalent to $\mathsf{TS}^{!\omega}$ and provably equivalent to the latter over RCA_0.*

Proof. See proof of Theorem 3.5 in [4]. □

We now turn to the generalization of the Rainbow Ramsey Theorem.

Definition 3.9 (Large Rainbow Ramsey Theorem)**.** *Let $k \in \mathbb{N}^+$. For every k-bounded coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ there exists an infinite set $H \subseteq \mathbb{N}$ such that f is injective on $[H]^{!\omega}$. The set H is called a* rainbow *for f. We abbreviate this statement by $\mathsf{RRT}^{!\omega}_k$.*

For the fixed dimension case, Galvin gave a reduction to Ramsey's theorem. His argument is formalizable in RCA_0 and yields that for each $n, k \in \mathbb{N}$, $\mathsf{RRT}^n_k \leq_{\mathrm{sW}} \mathsf{RT}^n_k$ (see [7], proof of Theorem 5.2). Csima and Mileti also showed that for each $n, k \in \mathbb{N}^+$, RRT^n_k follows from RRT^{n+1}_k over RCA_0 (the proof of Theorem 5.3 in [7] yields a strong Weihrauch reduction).

We first observe that Galvin's argument adapts to the case of colorings of exactly ω-large sets and establishes a strong Weihrauch reduction.

Theorem 3.10. *For all $k \geq 1$, $\mathsf{RRT}^{!\omega}_k \leq_{\mathrm{sW}} \mathsf{RT}^{!\omega}_k$. Moreover $\mathsf{RCA}_0 \vdash \forall k(\mathsf{RT}^{!\omega}_k \to \mathsf{RRT}^{\omega}_k)$.*

Proof. Let $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be k-bounded and fix a computable bijection $b(\cdot) : [\mathbb{N}]^{!\omega} \to \mathbb{N}$. Define $g : [\mathbb{N}]^{!\omega} \to k$ as follows:

$$g(s) = |\{t \in [\mathbb{N}]^{!\omega} : b(t) < b(s) \text{ and } f(s) = f(t)\}|.$$

The fact that g is a k-coloring depends on the hypothesis that f is k-bounded. Let H be an infinite set such that g is constant on $[H]^{!\omega}$, as given by $\mathsf{RT}^{!\omega}_k$. Let $s, t \in [H]^{!\omega}$. Since $g(s) = g(t)$ and either $b(s) < b(t)$ or $b(t) < b(s)$ we have that $f(s) \neq f(t)$. Thus H is rainbow for f. □

The following is an adaptation of a result by Wang [34], who showed that for every n, $\mathsf{RRT}^n_2 \leq_{\mathrm{sW}} \mathsf{FS}^n$.

Proposition 3.11. $\mathsf{RRT}^{!\omega}_2 \leq_{\mathrm{sW}} \mathsf{FS}^{!\omega}$.

Proof. Fix a computable bijection $b(\cdot) : [\mathbb{N}]^{!\omega} \to \mathbb{N}$. Let $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be 2-bounded. Define $g : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ as follows:

$$g(s) = \begin{cases} \min(t \setminus s) & \text{if there is a } t \in [\mathbb{N}]^{!\omega} \text{ such that } b(t) < b(s) \text{ and } f(s) = f(t), \\ 0 & \text{otherwise.} \end{cases}$$

Since f is 2-bounded, if t exists in the definition of g then it is unique. If t and s are distinct exactly ω-large sets then $(t \setminus s) \neq \emptyset$, since $t \subseteq s$ is impossible. Let A be an infinite g-free set. We claim that A is a rainbow for f. Suppose otherwise, by way of contradiction, as witnessed by $s, t \in [A]^{!\omega}$ such that $f(s) = f(t)$. Without loss of generality we can assume $b(t) < b(s)$. Then $g(s) = \min(t \setminus s) \in A \setminus s$, contradicting that A is g-free. □

Wang [34] proved that the Free Set, Thin Set and Rainbow Ramsey Theorems for fixed-sized sets satisfy cone-avoidance. This entails that none of these principles codes the halting set or implies ACA_0. A natural question is whether the same is true of their versions for exactly ω-large sets. We will show that $\mathsf{FS}^{!\omega}$ and $\mathsf{TS}^{!\omega}$ code $\emptyset^{(\omega)}$ and imply the much stronger system ACA_0^+, while $\mathsf{RRT}^{!\omega}$ admits cone avoidance.

Some first lower bounds on our principles can be obtained by adapting results from the finite-dimensional case. For example, the Rainbow Ramsey Theorem with internal quantification over all fixed finite dimensions $\forall n \mathsf{RRT}^n_2$ follows from the Rainbow Ramsey Theorem for exactly ω-large sets.

Proposition 3.12. *For each $k, n \in \mathbb{N}^+$, $\mathsf{RRT}^{!\omega}_k \geq_{\mathrm{sW}} \mathsf{RRT}^n_k$. Moreover, $\mathsf{RCA}_0 \vdash \forall k(\mathsf{RRT}^{!\omega}_k \to \forall n \mathsf{RRT}^n_k)$.*

Proof. Let $n \in \mathbb{N}^+$ and $f : [\mathbb{N}]^n \to \mathbb{N}$ be k-bounded. Define $g : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ as follows: $g(s) = \langle 0, f(s_0, s_1, \ldots, s_{n-1})\rangle$ if $s_0 \geq n$, and $\langle 1, s\rangle$ otherwise (where $\langle\cdot\rangle : \mathbb{N}^{<\omega} \to \mathbb{N}$ is a fixed computable bijection). Let A be an infinite rainbow for g. Then $A \cap [n, \infty)$ is an infinite rainbow for f. □

Analogous results can be obtained for the Large Free Set and the Large Thin Set Theorem, so that $\mathsf{P}^{!\omega}$ implies $\forall n \mathsf{P}^n$ over RCA_0 for $\mathsf{P} \in \{\mathsf{FS}, \mathsf{TS}\}$. Both implications are also witnessed by strong Weihrauch reductions. We omit the proofs since these results are superseded by the results of the next section where we prove that $\mathsf{FS}^{!\omega}$ and $\mathsf{TS}^{!\omega}$ imply ACA_0^+.

4 Large Thin and Free Set Theorems code $\emptyset^{(\omega)}$

In this section we establish strong lower bounds on $\mathsf{FS}^{!\omega}$ and $\mathsf{TS}^{!\omega}$ showing that both these principles code $\emptyset^{(\omega)}$ and imply ACA_0^+. This should be contrasted with the fact that neither $\forall n \mathsf{FS}^n$ nor $\forall n \mathsf{TS}^n$ imply ACA_0.

The first goal is to prove the existence of a computable instance of $\mathsf{TS}^{!\omega}$ such that every solution uniformly computes $\emptyset^{(\omega)}$. In particular, $\mathsf{TS}^{!\omega}$ admits the same lower bound as $\mathsf{RT}^{!\omega}_2$. Since $\mathsf{TS}^{!\omega}$ is strongly Weihrauch reducible to $\mathsf{FS}^{!\omega}$ by Theorem 3.7, it follows that there is a computable instance of $\mathsf{FS}^{!\omega}$ such that every solution computes $\emptyset^{(\omega)}$. By results in the previous section, this bound is optimal, since every computable instance of $\mathsf{FS}^{!\omega}$ admits a solution computable in $\emptyset^{(\omega)}$ by Theorem 3.4 and Theorem 3.2. The following definition of thinness is technically convenient.

Definition 4.1. *Let C be any non-empty set. Given a coloring $f : [\mathbb{N}]^n \to C$, a set $H \subseteq \mathbb{N}$ is* f-thin for color $c \in C$ *if $c \notin f([H]^n)$. A set $H \subseteq \mathbb{N}$ is f-thin (or thin for f) if H is f-thin for some color $c \in C$.*

The definition of f-thin set depends on the choice of codomain of the function. It will always be clear from the context.

The following version of the Thin Set Theorem for finite colorings was introduced in [8] and is useful for our purposes.

Definition 4.2 (Thin Set Theorem for finite colorings)**.** *Let $n \geq 1$ and $k \geq 2$. For every coloring $f : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is thin for f. We abbreviate this statement by TS^n_k.*

Dorais et al. [8, Proposition 5.5] proved the existence, for every $n \in \mathbb{N}^+$, of a computable coloring $f : [\mathbb{N}]^{n+2} \to 2^n$ such that every infinite f-thin set computes $\emptyset'$. However, their proof is not uniform, which is a required feature for our construction to code $\emptyset^{(\omega)}$. We prove the existence, for every $n \geq 2$, of a computable coloring $f : [\mathbb{N}]^{n+1} \to n$ such that every infinite f-thin set uniformly computes $\emptyset'$.

Lemma 4.3. *There exists two computable arrays $(e_{n,k})_{n,k\in\mathbb{N}}$ and $(d_{n,k})_{n,k\in\mathbb{N}}$ of Turing indexes such that for every $n \geq 2$ and $k \geq 1$, $\Phi_{e_{n,k}}^{\emptyset^{(k)}}$ is a coloring $f_n^k : [\mathbb{N}]^n \to n$ such that for every infinite f_n^k-thin set H, $\Phi_{d_{n,k}}^H = \emptyset^{(k)}$.*

Proof. Let $k \in \mathbb{N}^+$. Let g_k be a uniform modulus of the set $\emptyset^{(k)}$. Note that Δ^0_{k+1}-indexes for each function g_k can be found computably, uniformly in k.

For $n \geq 2$ let $f_n^k(x_0, \dots, x_{n-1}) = n-1$ if $g_k(x_0) \leq x_1$ and if $g_k(x_0) > x_1$, let $f_n^k(x_0, \dots, x_{n-1}) = i$ for $i < n-1$ the largest value such that $g_k(x_0) > x_{i+1}$.

Let H be an infinite f_n^k-thin set for some color c. The color c cannot be $n-1$ as for a given $x_0 \in H$ there exists $x_1 < \dots < x_{n-1} \in H \setminus \{0, \dots, g_k(x_0)\}$, hence $f_n^k(x_0, \dots, x_{n-1}) = n-1$.

We claim H is thin for color $n-2$. Indeed, assume by contradiction that there exists some tuple $x_0 < \dots < x_{n-1} \in H$ such that $g_k(x_0) > x_{n-1}$ so that f_n^k takes color $n-2$ on $\{x_0, \dots, x_{n-1}\}$. Since $c < n-2$ and H is infinite, there exists some $y_{c+2} < \dots < y_{n-1} \in H \setminus \{0, \dots, g_k(x_0)\}$, therefore $f_n^k(x_0, \dots, x_{c+1}, y_{c+2}, \dots, y_{n-1}) = c$, contradicting our assumption that H is f_n^k-thin for c.

Write $H = \{x_0 < x_1 < \dots\}$. For every $i \in \mathbb{N}$, $g_k(x_i) \leq x_{i+n-1}$, hence H computes a function dominating g and therefore computes $\emptyset^{(k)}$. This computation can be done uniformly in the set H, k and n. □

Lemma 4.4. *There exists a functional Γ such that for every set X and every index e, if $\Phi_e^{X'}$ is a coloring $f : [\mathbb{N}]^n \to \ell$, then $\Gamma^X(e)$ is a coloring $g : [\mathbb{N}]^{n+1} \to \ell$ such that if an infinite set H is g-thin for a color c, then H is f-thin for the same color c.*

Proof. Consider a set X and an index e such that $\Phi_e^{X'}$ is a coloring $f : [\mathbb{N}]^n \to \ell$ for some $n, \ell \in \mathbb{N}^+$. Let $(f_s)_{s\in\mathbb{N}}$ be a $\Delta^0_2(X)$-approximation of f (indexes for such an approximation, where the f_s are seen as X-computable functions, can computably be found uniformly in X and e). Finally, consider the following X-computable coloring g defined by $g(x_0, \dots, x_n) = f_{x_n}(x_0, \dots, x_{n-1})$ (again, the construction is uniform).

Let $H \subseteq \mathbb{N}$ be an infinite set. If for some $x_0 < \dots < x_{n-1} \in H$ we have $f(x_0, \dots, x_{n-1}) = c$ for some color c, then $f_s(x_0, \dots, x_{n-1}) = c$ for every s bigger than a certain threshold. Thus, as H is infinite, there exists some $x_n \in H$ such that $g(x_0, \dots, x_n) = c$. By contrapositive, if H is g-thin for c, then it is also f-thin for c. □

Combining these two lemmas, we get the following, where the array $(d_{n,k})_{n,k\in\mathbb{N}}$ is as in Lemma 4.3.

Lemma 4.5. *There exists a uniformly computable sequence of colorings $f_{n,k} : [\mathbb{N}]^{n+k} \to n$, for $n, k \geq 1$, such that for every infinite $f_{n,k}$-thin set H, $\Phi_{d_{n,k}}^H = \emptyset^{(k)}$.*

The following theorem states that $\mathsf{TS}^{!\omega}$ does not admit cone avoidance in a strong sense: there exists a single computable instance of $\mathsf{TS}^{!\omega}$ that computes $\emptyset^{(\omega)}$.

Theorem 4.6. *There exists a computable coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ such that every infinite f-thin set H computes $\emptyset^{(\omega)}$. Moreover, this computation is uniform in H and an avoided color c.*

Proof. Let $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be defined as follows (where the functions $f_{n,k}$ are from Lemma 4.5): for $x_0 < \cdots < x_{x_0}$ in $\mathbb{N}^+$, let $f(x_0, \ldots, x_{x_0}) = f_{x_0-k,k}(x_1, \ldots, x_{x_0})$ for $k = \lceil \frac{x_0}{2} \rceil$. By Lemma 4.5, f is indeed computable.

Let H be an infinite f-thin set for some color c and let $n \in \mathbb{N}$. Since H is infinite, there exists some $x_0 \in H$ such that, by letting $k = \lceil \frac{x_0}{2} \rceil$, we have $x_0 - k > c$ and $k \geq n$. The set $G = H \setminus \{0, \ldots, x_0\}$ is infinite and f-thin for c. By definition of f, G is $f_{x_0-k,k}$-thin for the color c. Note that c is part of the range of $f_{x_0-k,k}$ as $x_0 - k > c$. By Lemma 4.5, since $k \geq n$, $G \geq_T \emptyset^{(n)}$. Since G is obviously H-computable, we have $H \geq_T \emptyset^{(n)}$. This computation can be done uniformly in H, c and n, thus $H \geq_T \emptyset^{(\omega)}$ uniformly in H and c. □

Corollary 4.7. *There exists a computable coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ such that every infinite f-free set computes $\emptyset^{(\omega)}$.*

Proof. Immediate by Theorem 4.6 and the fact that $\mathsf{TS}^{!\omega} \leq_{\mathrm{sW}} \mathsf{FS}^{!\omega}$ by Theorem 3.7. □

Since the proof of Theorem 4.6 relativizes and is formalizable in RCA_0 we obtain the following Reverse Mathematics corollary.

Corollary 4.8. *Each of $\mathsf{TS}^{!\omega}$ and $\mathsf{FS}^{!\omega}$ implies ACA_0^+ over RCA_0.*

The above corollary coupled with Theorem 3.2 and Corollary 3.5 implies the equivalence over RCA_0 of $\mathsf{RT}^{!\omega}_2$, $\mathsf{TS}^{!\omega}$ and $\mathsf{FS}^{!\omega}$. We do not know whether $\mathsf{RT}^{!\omega}_2$ is reducible to $\mathsf{TS}^{!\omega}$ or $\mathsf{FS}^{!\omega}$.

The remaining question is whether $\mathsf{RRT}^{!\omega}$ codes $\emptyset^{(\omega)}$. In the next section, we shall answer this question negatively in a strong sense: $\mathsf{RRT}^{!\omega}$ does not code any non-computable set. Some computability-theoretic weak anti-basis results for $\mathsf{RRT}^{!\omega}$ can be obtained by streamlining results from the finite dimensional case. For instance, Csima and Mileti [7] proved the following theorem:

Theorem 4.9 (Csima-Mileti [7])**.** *For all $n, k \geq 2$ there exists a computable k-bounded $f : [\mathbb{N}]^n \to \mathbb{N}$ with no infinite Σ^0_n rainbow.*

The proof of Theorem 4.9 is uniform in n. Using this uniformity we obtain the following.

Theorem 4.10. *There is a computable instance of $\mathsf{RRT}^{!\omega}_2$ with no arithmetical solution.*

Proof. For each $n \geq 2$ let $f_n : [\mathbb{N}]^n \to \mathbb{N}$ be the 2-bounded instance of RRT^n_2 with no infinite Σ^0_n rainbow given by Theorem 4.9. Let $g : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be defined as follows:

$$g(n, x_0, \ldots, x_{n-1}) = \langle n, f_n(x_0, \ldots, x_{n-1}) \rangle$$

The function g is clearly 2-bounded since each f_n is 2-bounded. Then for every g-rainbow H and every n in H, the set $H \setminus [0, n]$ is an f_n-rainbow, so H is not Σ^0_n by Theorem 4.9. □

Patey [26] proved that $\mathsf{RCA}_0 \vdash (\forall n)[\mathsf{RRT}^{n+1}_2 \to \mathsf{TS}^n]$ and for every $n \in \mathbb{N}^+$, $\mathsf{RCA}_0 \vdash \mathsf{RRT}^{2n+1}_2 \to \mathsf{FS}^n$. The argument almost translates in the exactly ω-largeness setting as follows. A set $s \subseteq \mathbb{N}$ is *exactly* $(\omega+1)$*-large* if $s \setminus \{\min s\}$ is exactly ω-large. Let $\mathsf{RRT}^{!(\omega+1)}_2$ be the Rainbow Ramsey Theorem for 2-bounded colorings of the exactly $(\omega+1)$-large sets.

Proposition 4.11. $\mathsf{TS}^{!\omega} \leq_{\mathrm{sW}} \mathsf{RRT}^{!(\omega+1)}_2$.

Proof. Fix an instance $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ of $\mathsf{TS}^{!\omega}$. Then, consider the following f-computable instance g of $\mathsf{RRT}^{!(\omega+1)}_2$: for every $s \in [\mathbb{N}]^{!\omega}$ and every $x \in \mathbb{N}$, if $f(s) = \langle x, y\rangle$ with $x < y < \min s$, let $g(x, s) = g(y, s)$ and otherwise assign to $g(x, s)$ a fresh color. The construction of g is uniform in f.

Let H be an infinite rainbow for g and let $x, y \in H$ with $x < y$. The set $H_1 = H \setminus [0, y]$ is f-thin for the color $\langle x, y\rangle$. Indeed, for every $s \in [H_1]^{!\omega}$, by definition of H_1, $x < y < \min s$, thus, if $f(s) = \langle x, y\rangle$, then $g(x, s)$ would be equal to $g(y, s)$, contradicting the fact that H is a rainbow for g. The definition of H_1 is uniform in H, thus $\mathsf{TS}^{!\omega} \leq_{\mathrm{sW}} \mathsf{RRT}^{!(\omega+1)}_2$. ☐

Corollary 4.12. *There exists a computable instance of* $\mathsf{RRT}^{!(\omega+1)}_2$ *such that every solution computes* $\emptyset^{(\omega)}$.

Proof. Immediate by Theorem 4.6 and Proposition 4.11. ☐

5 Coloring barriers of order type ω^ω

In the previous sections we established lower bounds showing that the Free Set Theorem and the Thin Set Theorem for the Schreier barrier code $\emptyset^{(\omega)}$. In this section we develop a more robust generalization of the principles of interest, based on the notion of barrier. The main motivation is to prove upper bounds on a generalization of the Large Rainbow Ramsey theorem.

Ramsey's theorem for exactly ω-large sets ($\mathsf{RT}^{!\omega}_k$) is arguably the simplest generalization of Ramsey's theorem to collections of finite sets of arbitrary size, which is combinatorially true. However, the restriction $|s| = 1 + \min s$ is somewhat arbitrary, and could be replaced by any restriction of the form $|s| = h(\min s)$ for a computable function $h : \mathbb{N} \to \mathbb{N}$, while leaving the computational lower bounds and upper bounds of the statement unchanged. In this section, we therefore generalize the previous statements about exactly ω-large sets to a family of statements satisfying better closure properties.

There exist two possible approaches to relate $\mathsf{RT}^{!\omega}_k$ to existing theorems. The bottom-up approach, already explained, consists in considering $\mathsf{RT}^{!\omega}_k$ as a generalization of Ramsey's theorem to larger families of finite sets. The top-down approach, which we explore now, consists in seeing $\mathsf{RT}^{!\omega}_k$ as a particular case of the clopen Ramsey theorem by Galvin and Prikry [13]. In what follows, we write $[X]^\omega$ for the class of all infinite subsets of X. This notation should not be confused with the set $[X]^{!\omega}$ of all exactly ω-large subsets of X.

Theorem 5.1 (Clopen Ramsey Theorem)**.** *Fix $k \in \mathbb{N}^+$. For every continuous coloring $f : [\mathbb{N}]^\omega \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ which is f-*homogeneous*, that is, such that $|f([H]^\omega)| = 1$.*

Every coloring $f : [\mathbb{N}]^{!\omega} \to k$ can be considered as a continuous coloring $g : [\mathbb{N}]^\omega \to k$ defined by $g(X) = f(X \upharpoonright_{1+\min X})$, and every g-homogeneous set is f-homogeneous. On the other hand, given a continuous coloring $g : [\mathbb{N}]^\omega \to k$, there exists a prefix-free set $B \subseteq [\mathbb{N}]^{<\omega}$ and a coloring $f : B \to k$ such that

(1) for every infinite set $X \in [\mathbb{N}]^\omega$, there exists a (unique) $s_X \in B$ such that $s_X \prec X$ (where $\prec$ denotes proper initial segment);

(2) for every $X \in [\mathbb{N}]^\omega$, $g(X) = f(s_X)$.

Any set B satisfying the above properties is called a *block*. Marcone [22] proved over RCA_0 that such a set B can always be assumed to satisfy slightly stronger structural properties, called *barrier*. Given a set $B \subseteq [\mathbb{N}]^{<\omega}$, we write base$(B)$ for the set $\{n \in \mathbb{N} : (\exists s \in B)(n \in \operatorname{ran}(s))\}$. In the following definition $\preceq$ denotes the initial segment relation and $\subset$ denotes the proper subset relation.

Definition 5.2. *A set $B \subseteq [\mathbb{N}]^{<\omega}$ is a* barrier *if*

(1) base(B) *is infinite;*

(2) for every $X \in [\text{base}(B)]^\omega$, there is some $s \in B$ such that $s \preceq X$;

(3) for every $s, t \in B$, $s \not\subset t$.

Note that the last item is stronger than asking for B to be prefix-free. The simplest notions of barriers are the families $[\mathbb{N}]^n$ for $n \in \mathbb{N}$. Barriers were introduced by Nash-Williams [24] in order to study *better quasi-orders* (bqo), a strengthening of well-quasi-orders (wqo) with better closure properties. Barriers were studied by Marcone [22] in the context of Reverse Mathematics.

Theorem 5.3 (Barrier Ramsey Theorem)**.** *Fix a barrier $B \subseteq [\mathbb{N}]^{<\omega}$ and some $k \in \mathbb{N}^+$. For every coloring $f : B \to k$, there exists an infinite set $H \subseteq$ base(B) such that $|f([H]^{<\omega} \cap B)| = 1$.*

Marcone [22] proved over RCA_0 that the Barrier Ramsey Theorem is equivalent to ATR_0. Thus, the Barrier Ramsey Theorem is much stronger than Ramsey's theorem for exactly ω-large sets, which stands at the level of ACA_0^+.

Barriers can be classified based on the order type of their lexicographic order. Given a barrier $B \subseteq [\mathbb{N}]^{<\omega}$ and $s, t \in B$, let $s <_{\text{lex}} t$ if $s(x) < t(x)$ for the least x such that $s(x) \neq t(x)$, if it exists. Here, we identify s and t with finite increasing sequences over $\mathbb{N}$. Note that since B is prefix-free, $<_{\text{lex}}$ is total on B. The lexicographic orders are not in general well-orders, but Pouzet [29] proved that they are on barriers. Assous [1] characterized the order types of lexicographic orders on barriers, and proved that they are either of the form ω^n for some $n \in \mathbb{N}^+$, or $\omega^\alpha \cdot k$ for some $\alpha \geq \omega$ and $k \in \mathbb{N}^+$.

Definition 5.4. *The order type of a barrier is the order type of its lexicographic order.*

Clote [5] proved that the order type of barriers is relevant to the computability-theoretic analysis of the Barrier Ramsey Theorem, by conducting a level-wise analysis of its solutions in the hyperarithmetic hierarchy based on the order type of the barrier. He proved in particular the following theorem:

Theorem 5.5 (Clote [5]). *Fix $k \in \mathbb{N}^+$.*

- *For every computable barrier $B \subseteq [\mathbb{N}]^{<\omega}$ of order type at most ω^ω and every computable coloring $f : B \to k$, there is an infinite f-homogeneous set computable in $\emptyset^{(\omega)}$.*
- *There exists a computable barrier $B \subseteq [\mathbb{N}]^{<\omega}$ of order type ω^ω and a computable coloring $f : B \to 2$ such that every infinite f-homogeneous set computes $\emptyset^{(\omega)}$.*

The Schreier barrier is a simple example of barrier of order type ω^ω. Carlucci and Zdanowski [2] showed that the lower bound of Clote is witnessed by the Schreier barrier. Based on Clote's analysis, it is natural to conjecture that the Free set, Thin set and Rainbow Ramsey theorems for barriers of order type ω^ω are the robust counterpart of their versions for exactly ω-large sets. Actually, we shall see that, arguably, the right notion is the restriction of the statements to a sub-class of barriers of order type ω^ω.

Given a set X and $n \in \mathbb{N}$, we write $[X]^{\leq n}$ for the set of all subsets $s \subseteq X$ such that $|s| \leq n$, and $[X]^{\leq !\omega}$ for the set of all subsets $s \subseteq X$ such that $|s| \leq 1 + \min s$. By convention, for $n = 0$, $[X]^{\leq n}$ is the singleton $\{\emptyset\}$. Given a function $h : \mathbb{N} \to \mathbb{N}$, we write $[X]^{\leq h(\cdot)}$ for the set of all finite $s \subseteq X$ such that $|s| \leq h(\min s)$.

Definition 5.6. *A set $B \subseteq [\mathbb{N}]^{<\omega}$ is* ω-bounded *if and only if $B \subseteq [\mathbb{N}]^{\leq h(\cdot)}$ for some function $h : \mathbb{N} \to \mathbb{N}$. It is* computably ω-bounded *if furthermore h is computable.*

Lemma 5.7. *A barrier B has order type at most ω^ω if and only if B is ω-bounded.*

Proof. Suppose first $B \subseteq [\mathbb{N}]^{\leq h(\cdot)}$ for some function $h : \mathbb{N} \to \mathbb{N}$. For every $s \in B$, let $\alpha_s = \sum_{i<|s|} \omega^{h(\min s)-i} s(i)$. Note that $s <_{\text{lex}} t$ if and only if $\alpha_s < \alpha_t$, so B has order type at most ω^ω.

Suppose now B has order type at most ω^ω. Let $x \in \mathbb{N}$ and $B_x = \{s : x \cdot s \in B\}$. Then B_x is a barrier of order type at most ω^{n_x} for some $n_x \in \mathbb{N}^+$. By Assous [1, Proposition II.1], a barrier B has order type at most ω^n if and only if $B \subseteq [\mathbb{N}]^{\leq n}$, so $B_x \subseteq [\mathbb{N}]^{\leq n_x}$. Let $h(x) = n_x$. Then $B \subseteq [\mathbb{N}]^{\leq h(\cdot)}$. □

A function $h : \mathbb{N} \to \mathbb{N}$ is *left-c.e.* if there is a uniformly computable sequence of functions $h_0, h_1, \dots$ such that for every $x, i \in \mathbb{N}$, $h_i(x) \leq h_{i+1}(x)$ and $\lim_i h_i(x) = h(x)$. The sequence $(h_i)_{i\in\mathbb{N}}$ is then called a *left-c.e. approximation* of h. If B is a computable barrier of order type at most ω^ω, then it is ω-bounded by a left-c.e. function. This bound is tight, as there exist computable barriers of order type ω^ω which are not computably ω-bounded.

Let us first define a generalized version of Rainbow Ramsey Theorem for subsets of $[\mathbb{N}]^{<\omega}$, and show that its restriction to computable barriers of order type ω^ω codes the jump.

Definition 5.8. *Let $B \subseteq [\mathbb{N}]^{<\omega}$. A coloring $f : B \to \mathbb{N}$ is* k-bounded *if for every $c \in \mathbb{N}$, $|f^{-1}(c)| \leq k$. A set $H \subseteq \text{base}(B)$ is an* f-rainbow *if for every $s, t \in B \cap [H]^{<\omega}$ such that $s \neq t$, $f(s) \neq f(t)$.*

In this paper, we shall consider only sets B such that $\text{base}(B) = \mathbb{N}$.

Definition 5.9 (Generalized Rainbow Ramsey Theorem)**.** *Given a set $B \subseteq [\mathbb{N}]^{<\omega}$ and $k \in \mathbb{N}$, let RRT^B_k be the statement "For every k-bounded coloring $f : B \to \mathbb{N}$, there exists an infinite f-rainbow".*

As seen in Proposition 2.10, the statement RRT^B_k is not mathematically true for $B = [\mathbb{N}]^{<\omega}$. However, its restriction to barriers follows from Ramsey's theorem for barriers (the proof is completely analogous to the proof of Theorem 3.10 above). Moreover, we shall see in Section 6 that RRT^B_k holds for every ω-bounded set $B \subseteq [\mathbb{N}]^{<\omega}$. The following proposition shows that RRT^B_2 restricted to computable barriers of order type ω^ω codes the halting set.

Proposition 5.10. *There exists a computable barrier B of order type ω^ω and a computable 2-bounded function $f : B \to \mathbb{N}$ such that every f-rainbow computes $\emptyset'$.*

Proof. Let $g : \mathbb{N} \to \mathbb{N}$ be the modulus of $\emptyset'$ and let $(g_n)_{n\in\mathbb{N}}$ be a left-c.e. approximation of g. It can be assumed that g_n is non-decreasing for each $n \in \mathbb{N}$.

Let B be defined as follows: for $x, y \in \mathbb{N}$ with $x < y$ and $s \subseteq \mathbb{N}$ with $y < \min s$, let $x \cdot y \cdot s \in B$ if and only if $|s| = (g_{\min s}(x))^2$.

B is a barrier with base $\mathbb{N}$, indeed, for every infinite set $X = \{x_0, x_1, \dots\}$, $(x_0, x_1, \dots, x_{g_{x_2}(x_0)+1}) \in B$ and if $x \cdot y \cdot s, x' \cdot y' \cdot s' \in B$ satisfies $x \cdot y \cdot s \subseteq x' \cdot y' \cdot s'$ then $x \geq x'$ and $\min s \geq \min s'$, hence $|s| = (g_{\min s}(x))^2 \geq (g_{\min s'}(x'))^2 = |s'|$, but also $|s| \leq |s'|$, so $|s| = |s'|$ and therefore $x \cdot y \cdot s = x' \cdot y' \cdot s'$. The order type of B is ω^ω by Lemma 5.7 as B is ω-bounded by $g^2 + 1$.

Let $h_n : [\mathbb{N}]^{n+1} \to n$ be the computable instance of TS^{n+1}_n obtained in Lemma 4.5 such that every infinite h_n-thin set computes $\emptyset'$. Consider also the computable bijection $k : \mathbb{N} \to \{(y, x) \in \mathbb{N}^2 : y > x\}$ that list all such pairs lexicographically (the order type of that set is ω).

Let $f : B \to \mathbb{N}$ be defined as follows: for every $x \cdot y \cdot s \in B$, if $k(h_{|s|-1}(s)) = (z - x, y - x)$ for some $y < z < \min s$, then let $f(x, y, s) = f(x, z, s)$ and otherwise give a fresh new color for $f(x, y, s)$.

Let H be an infinite rainbow for f. For every $x \in H$, there exists some bound $b_x > g(x)$ such that $g_{b_x}(x) = g(x)$. For every $s \subseteq H \setminus [0, b_x]$ of cardinality $g(x)^2$, we have $h_{g(x)^2-1}(s) \in [0, g(x)^2 - 2)$. There are two cases:

Case 1: $H\setminus[0,b_x]$ is $h_{g(x)^2-1}$-thin for some $x \in H$, in that case, by definition of $h_{g(x)^2-1}$, $H \geq_T \emptyset'$.

Case 2: $H \setminus [0,b_x]$ is not $h_{g(x)^2-1}$-thin for every $x \in H$, in that case, for every $x \in H$ and every pair $y < z \in (x, x+g(x))$, there exists some $s \subseteq H\setminus[0,b_x]$ such that $k(h_{g(x)^2-1}(s)) = (z-x, y-x)$. Indeed, $h_{g(x)^2-1}(s)$ takes every value in $[0, g(x)^2-2)$, so every such couple $(z-x, y-x)$ is reached and therefore $f(x,y,s) = f(x,z,s)$. Since H is an f-rainbow, y and z cannot be both in H. So $H \geq \emptyset'$ as for every $x < y < z \in H$, z is bigger than $g(x)$, so H computes a function dominating g.

□

We shall however see that for every computable, computably ω-bounded barrier $B \subseteq [\mathbb{N}]^{<\omega}$ and every $k \in \mathbb{N}^+$, RRT^B_k admits strong cone avoidance (see below). Note that the Schreier barrier is an example of a computable, computably ω-bounded barrier.

Because of this, Proposition 5.10 cannot be improved to code more than $\emptyset'$. Indeed, every computable barrier of order type ω^ω is $\emptyset'$-computably ω-bounded, so for any non-$\emptyset'$-computable set D, every computable barrier of order type ω^ω, and every computable k-bounded function $f : B \to \mathbb{N}$, there exists an infinite f-rainbow which does not compute D.

6 Large Rainbow Ramsey Theorem avoids cones

In this section, we prove that the Rainbow Ramsey theorem for computably ω-bounded barriers admits strong cone avoidance. In particular, this is the case for the Large Rainbow Ramsey theorem since the Schreier barrier is ω-bounded by the computable function $x \mapsto x+1$.

Definition 6.1. *A problem* P *admits* strong cone avoidance *if for every set* Z*, every non-*Z*-computable set* C *and every* P*-instance* X*, there exists a* P*-solution* Y *to* X *such that* $C \not\leq_T Y \oplus Z$*.*

Note that in the previous definition, no computability constraint is given on the P-instance X. Thus, strong cone avoidance reflects the combinatorial weakness of P, in the sense that no matter how complex the instance is, it cannot code in its solutions an infinite binary sequence. We shall use the following two theorems:

Theorem 6.2 (Wang [34])**.** *For every* $n \in \mathbb{N}$*,* FS^n*,* TS^n *and* RRT^n *admit strong cone avoidance.*

Wang [34] introduced and studied the following formal theorem, which is strictly related to the Thin Set theorem.

Definition 6.3 (Achromatic Ramsey Theorem)**.** *Let* $n, k, \ell \in \mathbb{N}^+$*. For all* $f : [\mathbb{N}]^n \to k$*, there exists an infinite* $H \subseteq \mathbb{N}$ *such that* $|f([H]^n)| \leq \ell$*. We denote this statement* $\mathsf{RT}^n_{k,\ell}$*. We write* $\mathsf{RT}^n_{<\infty,\ell}$ *for* $\forall k \mathsf{RT}^n_{k,\ell}$*.*

Note that $\mathsf{RT}^n_{k,k-1}$ is the same as TS^n_k from Definition 4.2.

The following sequence of numbers, known as *Catalan numbers*, is omnipresent in Combinatorics. It is inductively defined as follows:

$$C_0 = 1 \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

This sequence starts with $1, 1, 2, 5, 14, 42, \dots$ (see sequence A000108 in the OEIS). The number C_n admits many characterizations, such as the number of ways of associating n applications of a binary operator. In computability theory, the nth Catalan number C_n surprisingly arose as the exact threshold ℓ at which $\mathsf{RT}^n_{<\infty,\ell}$ admits strong cone avoidance.

Theorem 6.4 (Cholak and Patey [3])**.** *For every $n \in \mathbb{N}^+$, $\mathsf{RT}^n_{<\infty,C_n}$ admits strong cone avoidance.*

In the remaining part of this section we show that the Rainbow Ramsey Theorem for computable, computably ω-bounded barriers admits strong cone avoidance and therefore does not code the jump. To obtain this result, we introduce some variants of the Free Set Theorem for large sets which are of interest in their own right. We first introduce the needed terminology.

Definition 6.5. *Let $B \subseteq [\mathbb{N}]^{<\omega}$ be a set and $f : B \to [\mathbb{N}]^{<\omega}$ be a coloring. A set $H \subseteq \mathrm{base}(B)$ is* f-free *if for every $s \in B \cap [H]^{<\omega}$, $f(s) \cap H \subseteq s$.*

For example, given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, one can let $B = [\mathbb{N}]^n$ and $g : [\mathbb{N}]^n \to [\mathbb{N}]^{<\omega}$ be defined by $g(s) = \{f(s)\}$. Then a set is f-free if and only if it is g-free. Of course, even with $B = [\mathbb{N}]^n$, infinite free sets do not necessarily exist for arbitrary colorings. We need to impose some constraints on the size of the sets in the image of f.

Definition 6.6. *Fix $B \subseteq [\mathbb{N}]^{<\omega}$. A coloring $f : B \to [\mathbb{N}]^{<\omega}$ is* b-constrained *for a bounding function $b : \mathbb{N} \to \mathbb{N}$ if for every $s \in B$, $|f(s)| \leq b(\min s)$. If b is the constant function $x \mapsto k$, then we say that f is* k-constrained.

Definition 6.7 (k-Constrained Free Set theorem)**.** *Fix $B \subseteq [\mathbb{N}]^{<\omega}$ and $k \in \mathbb{N}$ FS^B_k is the statement "For every coloring $f : B \to [\mathbb{N}]^{\leq k}$, there is an infinite f-free set".*

Given $n \in \mathbb{N}$ and a function $h : \mathbb{N} \to \mathbb{N}$, we write $\mathsf{FS}^{\leq n}_k$, $\mathsf{FS}^{\leq !\omega}_k$ and $\mathsf{FS}^{\leq h(\cdot)}_k$ for FS^B_k when B is $[\mathbb{N}]^{\leq n}$, $[\mathbb{N}]^{\leq !\omega}$ and $[\mathbb{N}]^{\leq h(\cdot)}$, respectively. In the extreme case where $B = \{\emptyset\}$, FS^B_k is nothing but the statement "For every finite set $F \in [\mathbb{N}]^{\leq k}$, there is an infinite set $H \subseteq \mathbb{N}$ such that $H \cap F = \emptyset$". We have seen in Corollary 4.7 that there exists a computable instance f of $\mathsf{FS}^{\leq !\omega}_1$ such that every infinite f-free set computes $\emptyset^{(\omega)}$. The case $\alpha < \omega$ is different.

Proposition 6.8. *For every $k, n \in \mathbb{N}$, $\mathsf{FS}^{\leq n}_k$ admits strong cone avoidance.*

Proof. Fix some set Z, some non-Z-computable set C, and some coloring $f : [\mathbb{N}]^{\leq n} \to [\mathbb{N}]^{\leq k}$. For every $m \leq n$ and $j < k$, let $f_{m,j} : [\mathbb{N}]^m \to \mathbb{N}$ be the coloring defined for every $s \in [\mathbb{N}]^m$ by letting $f_{m,j}(s)$ be the jth element of $f(s)$, if it exists, and $f_{m,j}(s) = 0$ otherwise. By finitely many successive applications of strong cone avoidance of FS^m for $m \leq n$ (see Wang [34]), there is an infinite set $H \subseteq \mathbb{N}$ which is simultaneously $f_{m,j}$-free for every $m \leq n$ and $j < k$, and such that $C \not\leq_T H \oplus Z$.

We claim that H is f-free. Suppose for the contradiction that there is some $s \in [H]^{\leq n}$ and some $c \in (f(s) \cap H) \setminus s$. Note that s is necessarily non-empty. Let j be such that c is the jth element of $f(s)$, and let $m = |s|$. Then $f_{m,j}(s) = c$, contradicting $f_{m,j}$-freeness of H. □

Note that the k-constraint cannot be released, even in the case of colorings of singletons, as it would yield a combinatorially false statement:

Proposition 6.9. *There exists a computable function $f : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ such that for every $x \in \mathbb{N}$, $|f(x)| \leq x$, and with no f-free set of size 2.*

Proof. Let $f(x) = [0, x)$. Let $\{x, y\}$ be an f-free set, with $x < y$. Then $x \in f(y) \setminus \{y\}$, contradicting f-freeness of $\{x, y\}$. □

One can however replace the constant constraint by a function when considering a natural sub-class of instances.

Definition 6.10. *Fix $B \subseteq [\mathbb{N}]^{<\omega}$. A function $f : B \to [\mathbb{N}]^{<\omega}$ is* progressive *if for every $s \in B$, either $f(s) = \emptyset$, or $\min f(s) \geq \min s$.*

An easy combinatorial argument shows that the following statement is classically true.

Definition 6.11 (Progressive Free Set theorem)**.** *Fix a set $B \subseteq [\mathbb{N}]^{<\omega}$ and a bounding function $b : \mathbb{N} \to \mathbb{N}$. PFS^B_b is the statement "For every b-constrained progressive coloring $f : B \to [\mathbb{N}]^{<\omega}$, there is an infinite f-free set".*

Here again, given $n \in \mathbb{N}$ and a function $h : \mathbb{N} \to \mathbb{N}$, we write $\mathsf{PFS}^{\leq n}_b$, $\mathsf{PFS}^{\leq !\omega}_b$ and $\mathsf{PFS}^{\leq h(\cdot)}_b$ for FS^B_b when B is $[\mathbb{N}]^{\leq n}$, $[\mathbb{N}]^{\leq !\omega}$ and $[\mathbb{N}]^{\leq h(\cdot)}$, respectively. We now show how the above principle relates to the Rainbow Ramsey Theorem.

Proposition 6.12. *Fix a set $B \subseteq [\mathbb{N}]^{<\omega}$ and some $k \in \mathbb{N}$. For every k-bounded coloring $f : B \to \mathbb{N}$, there is an f'-computable k-constrained progressive coloring $g : B \to [\mathbb{N}]^{<\omega}$ such that every infinite g-free set is an f-rainbow.*

Proof. Let $\leq_{\text{lex}}$ be the lexicographic ordering on B, that is, $s <_{\text{lex}} t$ (seen as finite increasing sequences over $\mathbb{N}^{<\omega}$) if there is some $x < \min(|s|, |t|)$ such that $s(x) < t(x)$, or $s \prec t$. In particular, if $s <_{\text{lex}} t$, then $\min s \leq \min t$, so $\min(t \setminus s) \geq \min s$. Let $g(s) = \{\min(t \setminus s) : f(t) = f(s) \wedge s <_{\text{lex}} t\}$. The coloring g is progressive and k-constrained.

We claim that every infinite g-free set H is an f-rainbow. Suppose for the contradiction that there are some distinct $s, t \in [H]^{<\omega} \cap B$ such that $f(s) = f(t)$. One can suppose without loss of generality that $s <_{\text{lex}} t$. Then $\min(t \setminus s) \in (g(s) \cap H) \setminus s$, contradicting g-freeness of H. □

The following proposition shows the existence of a computable barrier of order type ω^ω for which the Progressive Free Set theorem does not admit cone avoidance. In particular, this barrier is not computably ω-bounded, as we shall prove that the Progressive Free Set theorem for computable barriers which are computably ω-bounded admits strong cone avoidance.

Proposition 6.13. *There exists a computable barrier B of order type ω^ω and a computable progressive coloring $f : B \to \mathbb{N}$ such that every f-free set computes $\emptyset'$.*

Proof. Let $g : \mathbb{N} \to \mathbb{N}$ be the modulus of $\emptyset'$ and let $(g_n)_{n\in\mathbb{N}}$ be a left-c.e. approximation of g. Without loss of generality, it can be assumed that $g_n(x) \geq 1$ for every $x, n \in \mathbb{N}$ and that g_n is non-decreasing for each $n \in \mathbb{N}$.

Let B be defined as follows: for $x \in \mathbb{N}$ and $s \subseteq \mathbb{N}$ with $x < \min s$, let $x \cdot s \in B$ if and only if $|s| = g_{\min s}(x)$.

We claim that B is a barrier with base $\mathbb{N}$: for any infinite subset $X = \{x_0, x_1, \dots\}$ of $\mathbb{N}$, we have $\{x_0, x_1, \dots, x_{g_{x_1}(x_0)}\} \in B$ (since $g_{x_1}(x_0) \geq 1$). Let $x \cdot s, y \cdot t$ be in B and suppose $x \cdot s \subseteq y \cdot t$. Then $x \geq y$ and $\min s \geq \min t$. Moreover $|s| = g_{\min s}(x) \geq g_{\min s}(y) \geq g_{\min t}(y) = |t|$. Thus $s = t$ and $x = y$. By Lemma 5.7, the order type of B is ω^ω.

Let $f : B \to \mathbb{N}$ be defined by $f(x, s) = x + h_{|s|-1}(s) + 1$ where $h_n : [\mathbb{N}]^{n+1} \to n$ is the computable instance of TS^{n+1}_n obtained in Lemma 4.5 such that every infinite h_n-thin set computes $\emptyset'$.

Let H be an infinite f-free set. For every $x \in H$, there exists some bound $b_x > g(x)$ such that $g_{b_x}(x) = g(x)$. For every $s \subseteq H \setminus [0, b_x]$ of cardinality $g(x)$, we have $f(x, s) = x + h_{g(x)-1}(s) + 1$. There are two cases:

Case 1: $H \setminus [0, b_x]$ is $h_{g(x)-1}$-thin for some $x \in H$, in that case, by definition of $h_{g(x)-1}$, $H \geq_T \emptyset'$.

Case 2: $H \setminus [0, b_x]$ is not $h_{g(x)}$-thin for every $x \in H$, in that case, for every $x \in H$ and every $y \in (x, x + g(x))$, there exists some $s \subseteq H \setminus [0, b_x]$ such that $f(x, s) = y$ and therefore, since H is f-free, y is not in H. So $H \geq_T \emptyset'$ as the principal function of H dominates g. □

We now proceed to establish that the $\mathsf{PFS}^{\leq h(\cdot)}_h$ admits strong cone avoidance for every computable function $h : \mathbb{N} \to \mathbb{N}$.

Fix a function $h : \mathbb{N} \to \mathbb{N}$. Let $\mathcal{P}$ be the collection of all progressive colorings of type $[\mathbb{N}]^{\leq h(\cdot)} \to [\mathbb{N}]^{<\omega}$. For a function $b : \mathbb{N} \to \mathbb{N}$ we write $\mathcal{P}_b$ for the class of all b-constrained colorings in $\mathcal{P}$.

Given two colorings $f, g \in \mathcal{P}$, we write $g \leq f$ if for every $s \in [\mathbb{N}]^{\leq h(\cdot)}$, $g(s) \supseteq f(s)$. Note that if H is g-free and $g \leq f$, then H is f-free. Given two colorings $f, g \in \mathcal{P}$, let $f \cup g$ be the coloring defined by $(f \cup g)(s) = f(s) \cup g(s)$. The coloring $f \cup g$ is the greatest lower bound of f and g with respect to $\leq$.

Theorem 6.14. *Fix a set Z, a non-Z-computable set D and a Z-computable function $h : \mathbb{N} \to \mathbb{N}$. For every h-constrained progressive coloring $f : [\mathbb{N}]^{\leq h(\cdot)} \to [\mathbb{N}]^{<\omega}$, there exists an infinite f-free set $G \subseteq \mathbb{N}$ such that $D \not\leq_T G \oplus Z$.*

Proof. For the simplicity, we prove the theorem in a non-relativized form. Relativization is straightforward. Fix a non-computable set D. Consider the following notion of forcing:

Definition 6.15. *A* condition *is a triple* (f, σ, X) *such that* $f \in \mathcal{P}$ *is* b_f*-constrained, for some computable function* $b_f : \mathbb{N} \to \mathbb{N}$*,* $\sigma \in [\mathbb{N}]^{<\omega}$*,* $X \subseteq \mathbb{N}$ *is an infinite set such that* $\max \sigma < \min X$*, and*

(a) for every $s \in [\sigma \cup X]^{\leq h(\cdot)}$ *with* $\min s \in \sigma$*,* $f(s) \cap X \subseteq s$*.*

(b) for every $s \in [\sigma \cup X]^{\leq h(\cdot)}$*,* $f(s) \cap \sigma \subseteq s$*.*

(c) $D \not\leq_T X$*.*

A condition $d = (g, \tau, Y)$ extends $c = (f, \sigma, X)$ *(written* $d \leq c$*) if* $g \leq f$*,* $\tau \succeq \sigma$*,* $Y \subseteq X$ *and* $\tau \setminus \sigma \subseteq X$*.*

The following lemma states that property (a) can be obtained "for free", that is, by restricting the reservoir, and therefore does not impose any constraint on the stem. In what follows, fix a coloring $f \in \mathcal{P}$ which is b_f-constrained, for some computable function $b_f : \mathbb{N} \to \mathbb{N}$.

Lemma 6.16. *For every* $\sigma \in [\mathbb{N}]^{<\omega}$ *and every infinite set* X *such that* $D \not\leq_T X$*, there is an infinite set* $Y \subseteq X$ *such that* $D \not\leq_T Y$ *and for every* $s \in [\sigma \cup Y]^{\leq h(\cdot)}$ *with* $\min s \in \sigma$*,* $f(s) \cap Y \subseteq s$*.*

Proof. For every $t \in [\sigma]^{\leq h(\cdot)}$ with $t \neq \emptyset$, let $f_t : [X]^{\leq h(\min t) - |t|} \to [\mathbb{N}]^{\leq b_f(\min t)}$ be defined for every u by $f_t(u) = f(t, u)$. By finitely many successive applications of Proposition 6.8, there exists an infinite subset $Y \subseteq X$ such that $D \not\leq_T Y$ and such that Y is simultaneously f_t-free for every $t \in [\sigma]^{\leq h(\cdot)}$ with $t \neq \emptyset$.

We claim that Y is our desired set. Indeed, for every $s \in [\sigma \cup Y]^{\leq h(\cdot)}$ with $\min s \in \sigma$, letting $t = s \cap \sigma$ and $u = s \cap Y$, we have $f(s) = f_t(u)$, so by f_t-freeness of Y, $f(s) = f_t(u) \cap Y \subseteq u \subseteq s$. □

In what follows, recall that C_n stands for the nth Catalan number.

Definition 6.17. *A set* X stabilizes σ *if for every* $t \in [\sigma]^{\leq h(\cdot)}$ *with* $t \neq \emptyset$ *and every* $n \leq h(\min t) - |t|$*, there is a set* $I_{t,n} \subseteq [\sigma]^{\leq b_f(\min t) \times C_n}$ *such that for every* $u \in [X]^n$*,* $f(t, u) \cap \sigma \subseteq I_{t,n}$*.*

Lemma 6.18. *For every* $\sigma \in [\mathbb{N}]^{<\omega}$ *and every infinite set* X *such that* $D \not\leq_T X$*, there is an infinite set* $Y \subseteq X$ *stabilizing* σ *and such that* $D \not\leq_T Y$*.*

Proof. For every $t \in [\sigma]^{\leq h(\cdot)}$ with $t \neq \emptyset$ and every $n \leq h(\min t) - |t|$, let $g_{t,n} : [X]^n \to [\sigma]^{\leq b_f(\min t)}$ be defined by $g_{t,n}(u) = f(t, u) \cap \sigma$. By finitely many successive applications of strong cone avoidance of $\mathsf{RT}^n_{<\infty, C_n}$ (see Cholak and Patey [3]), there is an infinite subset $Y \subseteq X$ such that $D \not\leq_T Y$ and for every $t \in [\sigma]^{\leq h(\cdot)}$ with $t \neq \emptyset$ and every $n \leq h(\min t) - |t|$, $|g_{t,n}[Y]^n| \leq C_n$. Note that here, a color is an element of $[\sigma]^{\leq b_f(\min t)}$ instead of a natural number. Let $I_{t,n} = \bigcup g_{t,n}[Y]^n$. Then $|I_{t,n}| \leq b_f(\min t) \times C_n$. By definition, for every $u \in [Y]^n$, $f(t, u) \cap \sigma = g_{t,n}(u) \subseteq I_{t,n}$. □

Given a computable function $b : \mathbb{N} \to \mathbb{N}$, let $b^+ : \mathbb{N} \to \mathbb{N}$ be the computable function defined by $b^+(m) = \sum_{n \leq h(m)} b(m) \times C_n$.

Definition 6.19. *Let X be a reservoir stabilizing $[0, k]$. The* limit coloring *is the function $g_{k,X} : [k]^{\leq h(\cdot)} \to [k]^{<\omega}$ defined by $g_{k,X}(\emptyset) = \emptyset$ and $g_{k,X}(t) = \bigcup_{n \leq h(t)-|t|} I_{t,n}$ otherwise.*

The limit function $g_{k,X}$ is b_f^+-constrained. Note that if $\rho \subseteq [0, k]$ is $g_{k,X}$-free, then it is f-free. Indeed, for every $t \in [0, k]^{\leq h(\cdot)}$, $g_{k,X}(t) = I_{t,0}$ and by definition of stability for $n = 0$, $f(t) \cap [0, k] \subseteq I_{t,0}$.

The following lemma is the core combinatorial lemma which specifies the conditions under which a block of elements ρ can be added to the stem while preserving the property (b).

Lemma 6.20. *Let (f, σ, X) be a condition and $Y \subseteq X$ be an infinite set stabilizing $[0, k]$ for some $k \in \mathbb{N}$ and let $g_{k,Y} : [k]^{\leq h(\cdot)} \to [k]^{<\omega}$ be the limit coloring. Let $\rho \subseteq X \restriction_k$ be a finite $g_{k,Y}$-free set. Then $(f, \sigma \cup \rho, Y)$ satisfies property (b).*

Proof. Fix some $s \in [\sigma \cup \rho \cup Y]^{\leq h(\cdot)}$. We have multiple cases.

- Case 1: $s \cap \sigma \neq \emptyset$. Then by properties (a) and (b) of (f, σ, X), $f(s) \cap (\sigma \cup X) \subseteq s$. Since $(\sigma \cup \rho) \subseteq (\sigma \cup X)$, then $f(s) \cap (\sigma \cup \rho) \subseteq s$.
- Case 2: $s \cap \sigma = \emptyset$ but $s \cap \rho \neq \emptyset$. By property (b) of (f, σ, X), $f(s) \cap \sigma \subseteq s$. Let $t = s \cap \rho$ and $u = s \cap Y$. Since ρ is $g_{k,Y}$-free, $g_{k,Y}(t) \cap \rho \subseteq t$. By definition of $g_{k,Y}(t)$, $I_{t,|u|} \subseteq g_{k,Y}(t)$, so $I_{t,|u|} \cap \rho \subseteq t$. In particular, $f(s) = f(t, u) \in I_{t,|u|}$, so $f(s) \cap \rho \subseteq t \subseteq s$. Thus, $f(s) \cap (\sigma \cup \rho) \subseteq s$.
- Case 3: $s \cap (\sigma \cup \rho) = \emptyset$. Then $s \subseteq Y$. Since $\min Y > \max(\sigma \cup \rho)$, then by progressiveness of f, $f(s) \cap (\sigma \cup \rho) = \emptyset$.

□

One can combine Lemmas 6.16, 6.18 and 6.20 to obtain an extensibility lemma, saying that every sufficiently generic filter induces an infinite set.

Lemma 6.21. *Let (f, σ, X) be a condition. There is an extension $(f, \tau, Y) \leq (f, \sigma, X)$ such that $|\tau| > |\sigma|$.*

Proof. Let $x = \min X$. By Lemma 6.18, there is an infinite subset $Y_0 \subseteq X$ stabilizing $[0, x]$ such that $D \not\leq_T Y_0$. Let g be the limit function. By Lemma 6.16, there is an infinite subset $Y \subseteq Y_0$ such that $(f, \sigma \cup \{x\}, Y)$ satisfies property (a). By Lemma 6.20, $\{x\}$ being vacuously g-free, $(f, \sigma \cup \{x\}, Y)$ satisfies property (b). Thus, $(f, \sigma \cup \{x\}, Y)$ is a valid extension. □

Given a computable function $b : \mathbb{N} \to \mathbb{N}$, the space $\mathcal{P}_b$ is not compact. However, $\mathcal{P}_b$ is in one-to-one correspondence with the effectively compact space $\mathcal{R}_b$ of all relations $R \subseteq [\mathbb{N}]^{\leq h(\cdot)} \times \mathbb{N}$ such that for every $s \in [\mathbb{N}]^{\leq h(\cdot)}$, $|\{y : (\nu, y) \in R\}| \leq b(\min s)$ and for every $(s, y) \in R$, $y \geq \min s$. Indeed, given a function $g \in \mathcal{P}_b$, one can define the relation $R_g \in \mathcal{R}_b$ defined by $R_g = \{(s, y) : y \in g(s)\}$

and given a relation $R \in \mathcal{R}_b$, the function $g_R \in \mathcal{P}_b$ is defined by $g_R(s) = \{y : (s, y) \in R\}$.

Note that the map $g \mapsto R_g$ is computable, but the map $R \mapsto g_R$ is not even continuous. Thankfully, there is a Turing functional which, given R and a finite set ρ, decides whether ρ is g_R-free or not. Indeed, to decide whether ρ is g_R-free, one does not need to know g_R restricted to $[\rho]^{\leq h(\cdot)}$, only to know R restricted to $[\rho]^{\leq h(\cdot)} \times \rho$.

We are now ready to define the forcing question:

Definition 6.22. *Let (f, σ, X) be a condition and $\varphi(G)$ be a Σ^0_1-formula. Let $(f, \sigma, X) ?\!\vdash \varphi(G)$ hold if and only if for every relation $R \in \mathcal{R}_{b_f^+}$, there is a finite g_R-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds.*

The previous formulation of the forcing question is $\Pi^1_1(X)$ as it starts with a second-order universal quantification. However, thanks to the effective compactness of the space $\mathcal{R}_{b_f^+}$, it is equivalent to a $\Sigma^0_1(X)$-formula:

Lemma 6.23. *Let (f, σ, X) be a condition and $\varphi(G)$ be a Σ^0_1-formula. Then $(f, \sigma, X) ?\!\vdash \varphi(G)$ if and only if there is some $k \in \omega$ such that for every b_f^+-constrained progressive function $g : [0, k]^{\leq h(\cdot)} \to [0, k]^{<\omega}$, there is a finite g-free set $\rho \subseteq X \upharpoonright_k$ such that $\varphi(\sigma \cup \rho)$ holds.*

Proof. Suppose first that there is some $k \in \mathbb{N}$ such that for every b_f^+-constrained progressive function $g : [0, k]^{\leq h(\cdot)} \to [0, k]^{<\omega}$, there is a finite g-free set $\rho \subseteq X \upharpoonright_k$ such that $\varphi(\sigma \cup \rho)$ holds. Let $R \in \mathcal{R}_{b_f^+}$ be a relation, and let $g_R \in \mathcal{P}_{b_f^+}$ be the corresponding function. Then, letting $g : [0, k]^{\leq h(\cdot)} \to [0, k]^{<\omega}$ be defined by $g(s) = g_R(s) \cap [0, k]$, the function g is b_f^+-constrained and progressive, so there is some finite g-free set $\rho \subseteq X \upharpoonright_k$ such that $\varphi(\sigma \cup \rho)$ holds. In particular, ρ is g_R-free. Since there is such a ρ for every $R \in \mathcal{R}_{b_f^+}$, then $(f, \sigma, X) ?\!\vdash \varphi(G)$ holds.

Suppose now that for every $k \in \mathbb{N}$, there is a b_f^+-constrained progressive function $g_k : [0, k]^{\leq h(\cdot)} \to [0, k]^{<\omega}$ such that for every g_k-free set $\rho \subseteq X \upharpoonright_k$, $\varphi(\sigma \cup \rho)$ does not hold. Let $\mathcal{T}$ be the tree which, at level k, contains all such functions g_k, and which is ordered by the function extension relation. The tree $\mathcal{T}$ is finitely branching, so by König's lemma, there is an infinite path $g \in \mathcal{T}$. This path is a function $g \in \mathcal{P}_{b_f^+}$ such that for every finite g-free set $\rho \subseteq X$, $\varphi(\sigma \cup \rho)$ does not hold. Then the relation R_g witnesses that $(f, \sigma, X) ?\!\vdash \varphi(G)$ does not hold. $\square$

The following lemma states that the forcing question meets its specification.

Lemma 6.24. *Let $p = (f, \sigma, X)$ be a condition and $\varphi(G)$ be a Σ^0_1-formula.*

1. *If $p ?\!\vdash \varphi(G)$, then there is an extension of p forcing $\varphi(G)$.*

2. *If $p ?\!\nvdash \varphi(G)$, then there is an extension of p forcing $\neg\varphi(G)$.*

Proof. Suppose first $p \mathrel{?}\!\vdash \varphi(G)$ holds. By Lemma 6.23, there is some $k \in \mathbb{N}$ such that for every b_f^+-constrained progressive function $g : [0,k]^{\leq h(\cdot)} \to [0,k]^{<\omega}$, there is a finite g-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. By Lemma 6.18, there is an infinite subset $Y_0 \subseteq X$ stabilizing $[0,k]$ and such that $D \not\leq_T Y_0$. Let $g_{k,Y_0} : [0,k]^{\leq h(\cdot)} \to [0,k]^{<\omega}$ be the limit function. Note that g_{k,Y_0} is b_f^+-constrained and progressive, so there is a finite g_{k,Y_0}-free set $\rho \subseteq X$ such that $\varphi(\sigma \cup \rho)$ holds. By Lemma 6.20, $(f, \sigma \cup \rho, Y_0)$ satisfies (b). By Lemma 6.16, there is an infinite subset $Y \subseteq Y_0$ such that $(f, \sigma \cup \rho, Y)$ satisfies (a) and $D \not\leq_T Y$. Thus $(f, \sigma \cup \rho, Y)$ is a valid extension of p. By choice of ρ, it forces $\varphi(G)$.

Suppose $p \mathrel{?}\!\nvdash \varphi(G)$ holds. Then there is a relation $R \in \mathcal{R}_{b_f^+}$ such that for every finite g_R-free set $\rho \subseteq X$, $\varphi(\sigma \cup \rho)$ does not hold. Let $g_R \in \mathcal{P}_{b_f^+}$ be the corresponding function. Let $\hat{f} : [\mathbb{N}]^{\leq h(\cdot)} \to [\mathbb{N}]^{<\omega}$ be defined by $\hat{f}(s) = f(s) \cup (g_R(s) \setminus \sigma)$. Note that $\hat{f} \leq f$, and every $\hat{f}$-free subset $\rho \subseteq X$ is g_R-free. Moreover, $\hat{f}$ is $(b_f + b_f^+)$-constrained and progressive. Since (f, σ, X) satisfies (b) and $\hat{f}$ does not contain any element of σ, then $(\hat{f}, \sigma, X)$ satisfies (b). By Lemma 6.16, there is an infinite subset $Y \subseteq X$ such that $D \not\leq_T Y$ and $(\hat{f}, \sigma, Y)$ satisfies (a). Thus $q = (\hat{f}, \sigma, Y)$ is a valid extension of p.

We claim that q forces $\neg\varphi(G)$. Indeed, suppose there is an extension $(\tilde{f}, \tau, Z) \leq q$ such that $\varphi(\tau)$ holds. Then by property (b) of $(\tilde{f}, \tau, Z)$, τ is $\tilde{f}$-free, and since $\tilde{f} \leq \hat{f}$, τ is $\hat{f}$-free. Let $\rho = \tau \setminus \sigma$. By definition of $\hat{f}$, ρ is a g_R-free subset of X, contradicting our choice of g_R. ☐

We can now prove our diagonalization lemma.

Lemma 6.25. *Let $p = (f, \sigma, X)$ be a condition and Φ_e be a Turing functional. There is an extension of p forcing $\Phi_e^G \neq D$.*

Proof. Let $U = \{(x, v) \in \mathbb{N} \times 2 : p \mathrel{?}\!\vdash \Phi_e^G(x) \downarrow = v\}$. We have three cases:

- Case 1: $(x, 1 - D(x)) \in U$ for some $x \in \mathbb{N}$. By Lemma 6.24, there is an extension of p forcing $\Phi_e^G(x) \downarrow = 1 - D(x)$, hence forcing $\Phi_e^G \neq D$.
- Case 2: $(x, D(x)) \notin U$ for some $x \in \mathbb{N}$. By Lemma 6.24, there is an extension of p forcing $\neg(\Phi_e^G(x) \downarrow = D(x))$, hence forcing $\Phi_e^G \neq D$.
- Case 3: U is the graph of the characteristic function of D. By Lemma 6.23, the set U is $\Sigma_1^0(X)$, so $D \leq_T X$, contradiction.

☐

We are now ready to prove Theorem 6.14. Let $f : [\mathbb{N}]^{\leq h(\cdot)} \to [\mathbb{N}]^{<\omega}$ be an h-constrained, progressive coloring, for a computable function $h : \mathbb{N} \to \mathbb{N}$. Let $\mathcal{F}$ be a sufficiently generic filter containing $(f, \emptyset, \mathbb{N})$, and let $G_{\mathcal{F}} = \bigcup\{\sigma : (g, \sigma, X) \in \mathcal{F}\}$. By definition of a forcing condition, $G_{\mathcal{F}}$ is f-free. By Lemma 6.21, $G_{\mathcal{F}}$ is infinite, and by Lemma 6.25, $D \not\leq_T G_{\mathcal{F}}$. This completes the proof of Theorem 6.14. ☐

Note that the previous theorem is tight in many senses. First, if h is non-computable, then by Proposition 6.13, $\mathsf{PFS}_1^{\leq h(\cdot)}$ does not admit cone avoidance. The following proposition shows that if b is allowed to be non-computable, then $\mathsf{PFS}_b^{\leq 1}$ does not admit strong cone avoidance in general.

Proposition 6.26. *There exists a $\emptyset'$-computable function $b : \mathbb{N} \to \mathbb{N}$ and a $\emptyset'$-computable b-constrained progressive function $f : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ such that every infinite f-free set computes $\emptyset'$.*

Proof. Let $b : \mathbb{N} \to \mathbb{N}$ be the modulus of $\emptyset'$, and let $f(x) = [x+1, \dots, b(x)]$. Let H be an infinite f-free set. Then given $x < y \in H$, $y > b(x)$, so one can H-compute a function dominating b, hence H-compute $\emptyset'$. □

Remark 6.27. *The proof of Theorem 6.14 can be adapted to prove many other notions of avoidance or preservation. For instance, one can prove that for every computable function $h : \mathbb{N} \to \mathbb{N}$, $\mathsf{PFS}_h^{\leq h(\cdot)}$ admits strong PA avoidance (see Liu [19]), strong constant-bound enumeration avoidance (see Liu [20]) or strong preservation for k hyperimmunities for every $k \in \mathbb{N}$ (see Patey [28]). In particular, for every computable function $h : \mathbb{N} \to \mathbb{N}$ and $k \in \mathbb{N}$, $\mathsf{PFS}_h^{\leq h(\cdot)}$ does not imply any of WKL_0, WWKL_0, RT^2_2, $\mathsf{RT}^2_{<\infty,k}$ over ω-models.*

Corollary 6.28. *For every $k \geq 1$, $\mathsf{RRT}_k^{\leq !\omega}$ admits strong cone avoidance.*

Proof. Immediate by Proposition 6.12 and Theorem 6.14. □

7 Conclusions and perspectives

The analysis of the exactly ω-large counterparts to the Ramsey, Free Set and Thin Set theorems from a computable perspective, gave the exact same tight bound, namely, $\emptyset^{(\omega)}$, translating in reverse mathematical terms by an equivalence with ACA_0^+. This equivalence is to be put in contrast with the finite-dimensional cases, where RT^n_2 coincides with ACA_0 for $n \geq 3$, while FS^n and TS^n both admit strong cone avoidance. On the other hand, the Rainbow Ramsey theorem for exactly ω-large sets ($\mathsf{RRT}_k^{!\omega}$) still has no coding power, and admits strong cone avoidance. Figure 1 and Figure 2 summarize the relationship between the studied statements, in Reverse Mathematics and over strong Weihrauch reducibility, respectively.

Many questions remain open around the generalization of combinatorial statements to exactly ω-large sets and barriers.

By Theorem 3.10, $\mathsf{RRT}_k^{!\omega} \leq_{\mathrm{sW}} \mathsf{RT}_k^{!\omega}$, so every computable instance of $\mathsf{RRT}_k^{!\omega}$ admits a $\emptyset^{(\omega)}$-computable solution. By Corollary 4.7, here exists a computable instance of $\mathsf{FS}^{!\omega}$ such that every solution computes $\emptyset^{(\omega)}$. It follows that $\mathsf{RRT}_k^{!\omega}$ is computably reducible to $\mathsf{FS}^{!\omega}$ in the sense of Dzhafarov [9]. We gave a direct combinatorial reduction in Proposition 3.11 in the case $k = 2$, and leave the general case open.

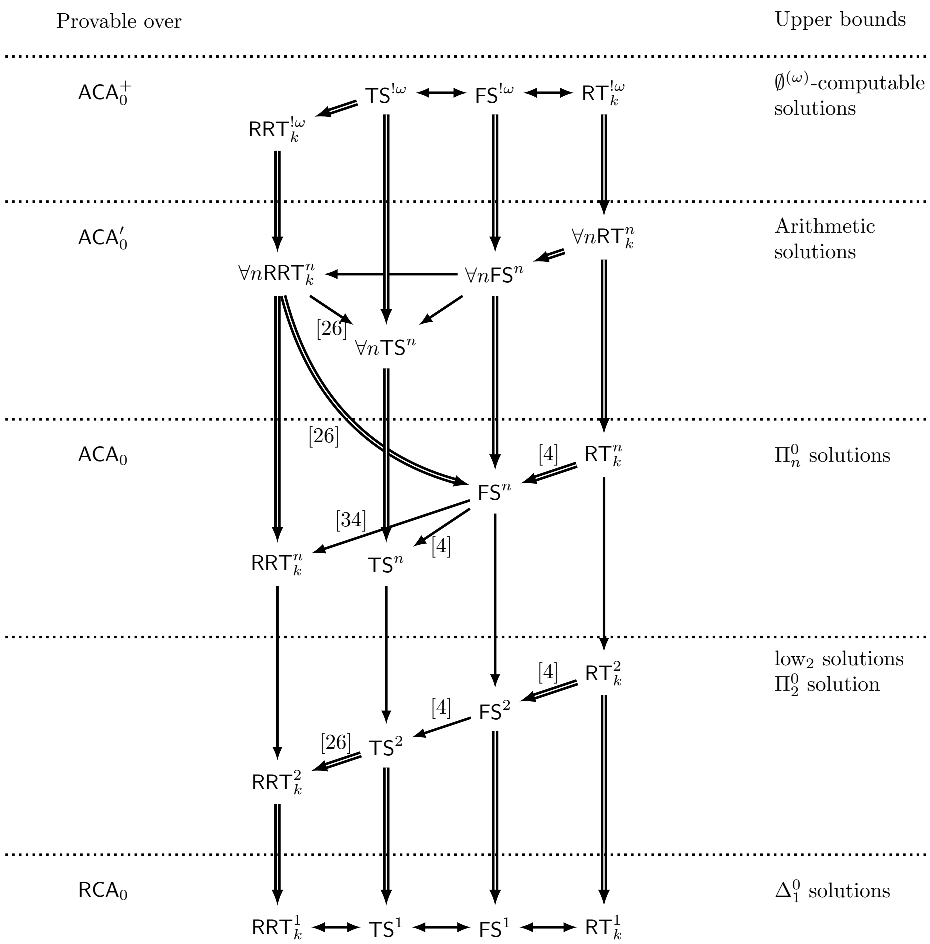

Figure 1: Studied statements from the perspectives of Computability Theory and Reverse Mathematics. A simple (double) arrow represents a (strict) implication over RCA_0.

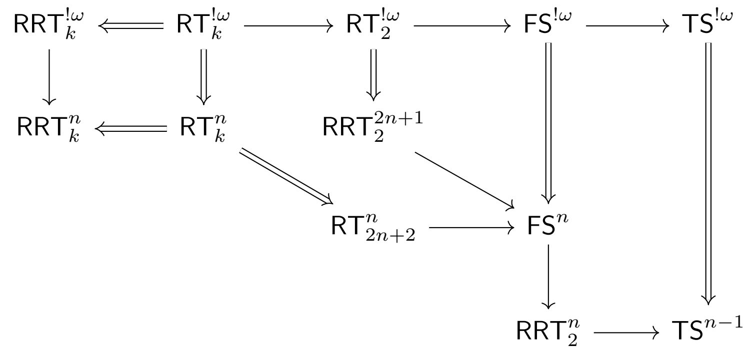

Figure 2: Studied statements from the perspective of Weihrauch analysis. A simple (double) arrow represents a (strict) strong Weihrauch reduction.

Question 7.1. *Does* $\mathsf{RRT}^{!\omega}_k \leq_{\mathrm{sW}} \mathsf{FS}^{!\omega}$ *for every* $k \in \mathbb{N}^+$*?*

While $\mathsf{RT}^{!\omega}, \mathsf{FS}^{!\omega}$ and $\mathsf{TS}^{!\omega}$ code $\emptyset^{(\omega)}$ and are all equivalent to ACA_0^+, we don't know of a direct proof of $\mathsf{RT}^{!\omega}_2$ from either $\mathsf{TS}^{!\omega}$ or $\mathsf{FS}^{!\omega}$. More precisely, we don't know if $\mathsf{RT}^{!\omega}_2$ is reducible to either $\mathsf{TS}^{!\omega}$ or $\mathsf{FS}^{!\omega}$.

Question 7.2. *Does* $\mathsf{RT}^{!\omega}_2 \leq_{\mathrm{sW}} \mathsf{TS}^{!\omega}$*? Does* $\mathsf{RT}^{!\omega}_2 \leq_{\mathrm{sW}} \mathsf{FS}^{!\omega}$*?*

A natural continuation of the line of research of the present paper is to consider the Free Set, Thin Set and Rainbow Ramsey theorems for arbitrary barriers and to inquire into their effective and logical strength. We plan to give in future work a complete layered analysis of the strength of these principles based on the complexity of the barrier. While weak anti-basis results relative to the hyperarithmetical hierarchy can be obtained along the lines of Clote's [5] results for Ramsey's Theorem, several questions remain to be answered to get a full picture.

In particular, the analysis of the Rainbow Ramsey theorem for barriers revealed a subtlety in the correspondence between the order type of a barrier and the computability-theoretic analysis of the corresponding theorem. Indeed, RRT^B_k admits strong cone avoidance when B is a computably ω-bounded, computable barrier, while it does not in general when B is a computable barrier of order type ω^ω (or equivalently an ω-bounded barrier). This subtle distinction does not arise in the analysis of Ramsey's theorem for barriers.

Barriers or order type ω^ω admit a simple combinatorial characterization as the ω-bounded barriers. Then, computable ω-bounded barriers can be considered as barriers of *effective order type* ω^ω. Is there an appropriate counterpart to the notion of "effective order type" for larger ordinals?

Clote [5] proved lower bounds on the Barrier Ramsey Theorem by defining his own notion of *canonical barrier* for every order type. This notion was also used in [6] to prove a generalization of Shoenfield's limit lemma to the hyperarithmetic hierarchy. On the other hand, Carlucci and Zdanowski [2] showed that in the case of barriers of order type ω^ω, the lower bounds could be witnessed by the Schreier barrier, that is, the barrier of exactly ω-large sets. The notion of ω-large set admits a natural generalization to any computable ordinal (see Hájek and Pudlák [15]). It is thus natural to ask whether exact α-largeness can be used instead of Clote's canonical barriers to witness his lower bounds for the Barrier Ramsey Theorem and to give a layered analysis of the Free Set, Thin Set and Rainbow Ramsey Theorem for barriers. We plan to address these questions in future work.

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