# The reverse mathematics of the Ordered Variable Word theorem

Lu Liu Ludovic Patey

July 2, 2026

**Abstract**

In this article, we study the reverse mathematics of variable word theorems used in the proof of the Dual Ramsey theorem. We prove that the Ordered Variable Word theorem does not imply Ramsey's theorem for pairs and that every computable instance admits a solution of $\mathrm{low}_2$ degree. This is used to prove the Carlson-Simpson Lemma and the Open Dual Ramsey theorem over $\mathsf{ACA}_0$, thereby answering some 40-years old open questions. These results have consequences in the reverse mathematics of structural Ramsey theory.

## 1 Introduction

Celebrated Ramsey's theorem [43] ($\mathsf{RT}^n$) states the existence, for every $k$-coloring of $[\mathbb{N}]^n$, of an infinite set $H \subseteq \mathbb{N}$ on which $[H]^n$ is monochromatic, for $k, n \geq 1$. Here, we write $[X]^n$ for the collection of all $n$-subsets of $X$. In 1984, Carlson and Simpson [6] proved a dual version of Ramsey's theorem, in which $k$-colorings of $\mathbb{N}$ are colored instead of $[\mathbb{N}]^n$. Given $\alpha \leq \omega$, let $(\mathbb{N})^\alpha$ denotes the class of partitions of $\mathbb{N}$ into exactly $\alpha$ non-empty parts, and given $p \in (\mathbb{N})^\omega$, let $(p)^\alpha$ denotes the class of coarsenings of $p$ into exactly $\alpha$ many parts, that is, given $q \in (p)^\alpha$ and $x, y \in \mathbb{N}$, if $p(x) = p(y)$, then $q(x) = q(y)$.

**Theorem 1.1** (Dual Ramsey Theorem [6])**.** *For all finite $k, \ell \geq 1$, for every Borel $\ell$-coloring of $(\mathbb{N})^k$, there exists some $p \in (\mathbb{N})^\omega$ such that $(p)^k$ is monochromatic.*

We write $\mathsf{DRT}^k$ and $\mathsf{ODRT}^k$ for the Dual Ramsey theorem restricted to Borel and open colorings of $(\mathbb{N})^k$, respectively. The core combinatorial content of the Open Dual Ramsey theorem is known as the *Carlson-Simpson Lemma* and can be formulated in terms of variable word theorems.

A finite or infinite *word* over a finite alphabet $A$ is a map $w : \alpha \to A$ for some $\alpha \leq \omega$. We write $|w|$ for its *length* $\alpha$. Given $\beta \leq \omega$, a $\beta$-variable word over $A$ is a word $w$ over the alphabet $A \sqcup \{x_i : i < \beta\}$ where each $x_i$ appears at least once, and the first occurrence of $x_i$ appears before the first occurrence of $x_{i+1}$. We write $A^{\alpha,\beta}$ and $A^{<\alpha,\beta}$ for the sets of all $\beta$-variable word of length $\alpha$,

and smaller than $\alpha$, respectively. Given $\gamma \leq \beta \leq \omega$, a $\beta$-variable word $w$ over $A$ and a $\gamma$-variable word $u$ of length $|u| \leq \beta$, we write $w[u]$ for the $\gamma$-variable word over $A$ where each occurrence of $x_j$ is replaced with $u(j)$, and cut before the first occurrence of $x_{|u|}$. If $|u| = \beta$, then $|w[u]| = |w|$.

**Lemma 1.2** (Carlson-Simpson Lemma [6])**.** *Fix a finite alphabet $A$ and some $n, \ell \in \mathbb{N}$. For every $\ell$-coloring of $A^{<\omega,n}$, there exists an $\omega$-variable word $W$ over $A$ such that $\{W[u] : u \in A^{<\omega,n}\}$ is monochromatic.*

We write $\mathsf{CSL}^n$ for the Carlson-Simpson Lemma restricted to $\ell$-colorings of $A^{<\omega,n}$ for any finite alphabet $A$ and any $\ell \geq 1$. Since then, the Carlson-Simpson Lemma found other applications in proving the existence of finite big Ramsey degree in structural Ramsey theory [29], such as reproving a result by Dobrinen [11] for the triangle-free Henson graph. The Carlson-Simpson Lemma admits an inductive proof along $n$ which is similar to the proof of Ramsey's theorem. For the base case, Carlson and Simpson [6, Theorem 6.3] proved a stronger statement in terms of *ordered* variable words. Given $\beta \leq \omega$, an ordered $\beta$-variable word over $A$ is a word $w$ over the alphabet $A \sqcup \{x_i : i < \beta\}$ where each $x_i$ appears at least once, and the *last* occurrence of $x_i$ appears before the first occurrence of $x_{i+1}$. Note that contrary to the unordered version, each variable kind appears only finitely often in an ordered $\omega$-variable word.

**Theorem 1.3** (Ordered Variable Word theorem [6])**.** *Fix a finite alphabet $A$ and some $\ell \in \mathbb{N}$. For every $\ell$-coloring of $A^{<\omega,0}$, there exists an ordered $\omega$-variable word $W$ over $A$ such that $\{W[u] : u \in A^{<\omega,0}\}$ is monochromatic.*

We write $\mathsf{OVW}^0$ for the Ordered Variable Word theorem restricted to $\ell$-colorings of $A^{<\omega,0}$ for any finite alphabet $A$ and any $\ell \geq 1$. There exists actually an ordered counterpart of the Carlson-Simpson Lemma proven by Carlson [5] but for which no elementary proof is known.

We are interested in the meta-mathematics of the Dual Ramsey theorem and its core lemmas from the viewpoint of computability theory and reverse mathematics.

## 1.1 Reverse mathematics

Reverse mathematics is a foundational program started by Harvey Friedman, whose goal is to find optimal axioms to prove ordinary theorems. It uses the framework of subsystems of second-order arithmetic, with a base theory, $\mathsf{RCA}_0$, capturing *computable mathematics*. More precisely, $\mathsf{RCA}_0$ consists of the axioms of a discrete ordered commutative semiring, together with the $\Sigma^0_1$-induction scheme and the $\Delta^0_1$-comprehension scheme. See Simpson [50] or Dzhafarov and Mummert [18] for a reference in reverse mathematics. Overall this article, we shall be interested in the following subsystems, which are listed in increasing axiomatic strength:

- $\mathsf{WKL}_0$, standing for *Weak König's lemma*, is $\mathsf{RCA}_0$ augmented with the statement "Every infinite binary tree admits an infinite path". It can be thought of as capturing compactness arguments.

- $\mathsf{ACA}_0$, standing for *Arithmetic Comprehension Axiom*, is $\mathsf{RCA}_0$ together with the comprehension scheme for arithmetic formulas (where set parameters are allowed). It is a $\Pi^1_1$-conservative extension of Peano arithmetic. From a computability-theoretic viewpoint, it can be equivalently defined as $\mathsf{RCA}_0$ augmented with the statement "$\forall X \exists Y (Y = X')$", where $X' = \{e : \Phi_e^X(e)\downarrow\}$ is the Turing jump of $X$.
- $\mathsf{ACA}'_0$ is $\mathsf{RCA}_0$ augmented with the statement $\forall n \forall X \exists Y (Y = X^{(n)})$, where $X^{(n)}$ is the $n$th iterate of the Turing jump, defined by $X^{(0)} = X$ and $X^{(n+1)} = (X^{(n)})'$. By Yokoyama [54] (see also [39]), $\mathsf{ACA}'_0$ is equivalent to the $\Pi^1_2$-reflection principle for $\mathsf{ACA}_0$, which is the statement "every $\Pi^1_2$-formula provable in $\mathsf{ACA}_0$ is true".
- $\mathsf{ACA}^+_0$ is $\mathsf{RCA}_0$ augmented with the statement $\forall X \exists Y (Y = X^{(\omega)})$, where $X^{(\omega)} = \bigoplus_n X^{(n)}$ is the $\omega$-jump of $X$. Given the Cantor pairing function $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$, the operator $\bigoplus_n X_n = \{\langle a, b \rangle : a \in X_n\rangle\}$ is the effective union of the sequence $(X_n)_{n \in \mathbb{N}}$. The system $\mathsf{ACA}^+_0$ was popularized as being the weakest system known to prove Hindman's theorem.
- $\Pi^1_1\text{–}\mathsf{CA}_0$ is $\mathsf{RCA}_0$ augmented with the comprehension scheme for $\Pi^1_1$-formulas. As such, it is an impredicative system of extremely high axiomatic strength compared to the systems above.

We shall mostly study $\Pi^1_2$-*problems*, which are $\Pi^1_2$-sentences of the form

$$\mathsf{P} = \forall X [\Phi(X) \to \exists Y \Psi(X, Y)]$$

where $\Phi$ and $\Psi$ are arithmetic formulas. A $\mathsf{P}$-*instance* is any set $X$ such that $\Phi(X)$ holds, and a $\mathsf{P}$-*solution to* $X$ is a set $Y$ such that $\Psi(X, Y)$ holds. For example, an $\mathsf{OVW}^0$-*instance* is a coloring $f : A^{<\omega,0} \to \ell$ for a finite alphabet $A$ and some $\ell \geq 1$. An $\mathsf{OVW}^0$-*solution* to $f$ is an ordered $\omega$-variable word $W$ such that $\{W[u] : u \in A^{<\omega,0}\}$ is monochromatic.

In their article, Carlson and Simpson [6] asked about the proof-theoretic and computability-theoretic strength of the Dual Ramsey theorem and its combinatorial lemmas. In an unpublished paper, Slaman [51] noticed that $\mathsf{DRT}^k_\ell$ holds over $\Pi^1_1\text{–}\mathsf{CA}_0$. Miller and Solomon [37] proved that $\mathsf{ODRT}^{k+1}$ implies $\mathsf{RT}^k$ over $\mathsf{RCA}_0$, hence that $\mathsf{ODRT}^4$ implies $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$. They also proved that $\mathsf{OVW}^0$ is not provable over $\mathsf{WKL}_0$, even for binary alphabets and 2-colorings. Erhard [20] proved that $\mathsf{OVW}^0$ is not provable by Ramsey's theorem for pairs restricted to stable colorings over $\mathsf{RCA}_0$. Dzhafarov, Flood, Solomon and Westrick [16] studied extensively the Dual Ramsey theorem and its variable word lemmas. They showed in particular that the full statement of the Dual Ramsey theorem is sensitive to representation, in terms of Borel or Baire codes. Liu, Monin and Patey [35] proved $\mathsf{OVW}^0$ over $\mathsf{ACA}_0$ for binary alphabets and constructed a computable 2-coloring of $\{0,1\}^{<\omega,0}$ with no $\Delta^0_2$ $\mathsf{OVW}^0$-solution. More recently, Anglès d'Auriac, Mignoty, Liu and Patey [1] proved two versions of $\mathsf{OVW}^0$: a piecewise syndetic and a density version, the latter being based on Dodos, Kanellopoulos and Tyros [14]. They deduced that the Carlson-Simpson Lemma holds over $\mathsf{ACA}^+_0$.

## 1.2 Main contributions

In this article, we give a new proof of $\mathsf{OVW}^0$ in the style of Towsner [53], and derive from its combinatorics a new parametric notion of effective forcing for building solutions to $\mathsf{OVW}^0$ with a good first-jump control. Thanks to this notion of forcing, we prove the following theorem:

**Main Theorem 1.4.** $\mathsf{OVW}^0$ *implies neither* $\mathsf{RT}^2_2$ *nor* $\mathsf{ACA}_0$ *over* $\mathsf{RCA}_0$.

Moreover, such a separation is witnessed by an $\omega$-*model*, that is, a structure $\mathcal{M} = (M, S, 0, 1, +, \cdots, <)$ in which the first-order part consists of the standard integers $\omega$, equipped with the natural operations and order. This answers partially [38, Question 11] and [37, Problem 2] and answers negatively [37, Problem 1] and [35, Question 4.9].

Second, we prove that every computable instance of $\mathsf{OVW}^0$ admits a solution of $\mathrm{low}_2$ degree by constructing a $\emptyset'$-computable 1-generic filter for this notion of forcing. Formalizing the construction over $\mathsf{ACA}_0$, we use this as the base case of an inductive proof $\mathsf{CSL}^n$ over $\mathsf{ACA}_0$. We therefore obtain an exact characterization of the strength of the Carlson-Simpson Lemma and the Open Dual Ramsey Theorem.

**Main Theorem 1.5.** *For every* $n \in \omega$, $\mathsf{ACA}_0 \vdash \mathsf{CSL}^n$ *and* $\mathsf{ACA}_0 \vdash \mathsf{ODRT}^{n+2}$. *Moreover, a reversal holds for every* $n \geq 2$.

This therefore gives a partial answer to the original question of Carlson and Simpson [6] and a complete answer to [37, Problem 3]. Based on the work of Hubička [29], formalized in reverse mathematics by Anglès d'Auriac et al. [1], Main Theorem 1.5 shows over $\mathsf{ACA}_0$ that the triangle-free Henson graph has finite big Ramsey degree. Note that this fact was recently proven over $\mathsf{ACA}_0$ by Cholak, Dobrinen, and Towsner [8] for every Henson graph, using a Milliken-style proof for coding trees, yielding exact bounds for finite big Ramsey degree contrary to the approach of Hubička [29].

Last, we complete the picture by showing that $\mathsf{CSL}^1$ and $\mathsf{ODRT}^3$ are logically weak, with the following theorem:

**Main Theorem 1.6.** *Neither* $\mathsf{CSL}^1$, *nor* $\mathsf{ODRT}^3$ *implies* $\mathsf{ACA}_0$ *over* $\mathsf{RCA}_0$.

Main Theorem 1.6 is by far the most complicated part of the article, involving three non-trivial steps: we first prove that an infinite variant of the Hales-Jewett theorem admits so-called strong preservation of hyperimmunity, then use it to prove that the Ordered Variable Word theorem admits the same property, and last, we prove that a cohesive version of $\mathsf{CSL}^1$ admits preservation of hyperimmunity.

Main Theorem 1.6 shows that the statement of the indivisibility of the triangle-free Henson graph does not imply $\mathsf{ACA}_0$, answering a question of Gill [22] and Cholak, Dobrinen and McCoy [7, Question 48].

### 1.3 Organization of the paper

We start with Section 2, which provides a simple proof of $\mathsf{OVW}^0$ over $\mathsf{ACA}_0$ in the style of Towsner [53]. This settles the combinatorial objects which are then used in Section 3 to design a parametric notion of effective forcing with good first-jump control. In particular, we prove a $\text{low}_2$ basis theorem for $\mathsf{OVW}^0$. Then, in Section 4, we prove that $\mathsf{OVW}^0$ admits so-called preservation of $\omega$ hyperimmunities, and deduce Main Theorem 1.4. In Section 5, we prove that an infinite version of the hales-jewett theorem admits so-called strong preservation of $\omega$ hyperimmunities. We use this result in Section 6 to prove that $\mathsf{OVW}^0$ admits strong preservation of 1 hyperimmunity. In Section 7, a formalization of the $\text{low}_2$ basis theorem for $\mathsf{OVW}^0$ from Section 3 is used to prove Main Theorem 1.5. In Section 8, we prove that a cohesive version of $\mathsf{CSL}^1$ admits preservation of $\omega$ hyperimmunities and, combined with Section 6, we deduce Main Theorem 1.6. Last, in Section 9, we derive consequences of Main Theorem 1.5 and Main Theorem 1.6 to structural Ramsey theory.

### 1.4 Notation

*Computability theory.* We shall use the standard notations of computability theory. See Soare [52] for a reference. In particular, we let $\Phi_0, \Phi_1, \dots$ be an enumeration of all Turing functionals, and write $\Phi_e^A(x) \downarrow = y$ if the $e$th functional with oracle $A$ halts on input $x$ and outputs $y$. Otherwise, we say it diverges on input $x$, and write $\Phi_e^A(x) \uparrow$.

*Words.* Given two (finite or infinite) words $w_0, w_1$, we write $w_0 \preceq w_1$ if $w_0$ is a prefix of $w_1$. Given a finite or infinite word $W$ and some length $t \leq |W|$, we write $W \upharpoonright_t$ for the initial segment of $W$ of length $t$. More generally, given $t_0 \leq t_1 \leq |W|$, we write $W \upharpoonright_{t_0}^{t_1}$ for the sequence of length $t_1 - t_0$ such that $W \upharpoonright_{t_0} \cdot W \upharpoonright_{t_0}^{t_1} = W \upharpoonright_{t_1}$. We write $\epsilon$ for the empty word. In particular, given a variable word $w$, $w[\epsilon]$ is the initial segment of $w$ cut before the first occurrence of $x_0$.

*Combinatorial spaces.* Throughout this article, we shall work with two different combinatorial spaces: ordered variable words (in Sections 2 to 6) for $\mathsf{OVW}^0$, and unordered variable words (in Sections 7 and 8) for $\mathsf{CSL}$. To avoid introducing too many notations at once, we shall introduce the specific definitions for the corresponding spaces in Section 2.1 and Section 7.1, respectively, when they are first used.

## 2 A simple proof of $\mathsf{OVW}^0$

Anglès d'Auriac et al. [1] gave proofs of the piecewise syndetic and the density versions of the zero-dimensional Ordered Variable Word theorem, respectively, yielding two proofs of $\mathsf{OVW}^0$ over $\mathsf{ACA}_0$. Both proofs were based on finitary versions of the respective statements, one of which was proven by Dodos et al [14]. In this section, we give an arguably more natural proof of $\mathsf{OVW}^0$ over $\mathsf{ACA}_0$,

using the Hales-Jewett theorem below as a bootstrap. The combinatorics will be then reused in later sections to show that $\mathsf{OVW}^0$ admits a weakly low basis (Section 3) and preservation of $\omega$ hyperimmunities (Section 4).

Towsner [53] gave a simple combinatorial proof of the Finite Union Theorem over $\mathsf{ACA}_0^+$ by introducing the concepts of half-match and full-match. His technique was then used to give a combinatorial proof Carlson's theorem for words [4] over $\mathsf{ACA}_0^+$. We will give a Towsner-style proof of $\mathsf{OVW}^0$. Before this, we introduce some notation.

## 2.1 Notation

Given $\beta \leq \alpha \leq \omega$, we write $A_<^{\alpha,\beta}$ and $A_<^{\leq\alpha,\beta}$ for the sets of all ordered $\beta$-variable word of length $\alpha$, or at most $\alpha$, respectively. When $\beta = 0$, we simply write $A^\alpha$ and $A^{\leq\alpha}$. Given ordered variable words $u$ and $v$, when doing the concatenation $u \cdot v$, we will always assume that the variable kinds of $u$ and $v$ are from disjoint spaces, so that there is no variable kind clash. In particular, if $u$ is an ordered $n$-variable word and $v$ is a ordered $\beta$-variable word, then $u \cdot v$ is an ordered $\beta + n$-variable word. Given $\gamma \leq \beta \leq \omega$ and an ordered $\beta$-variable word $w$, we let $\mathsf{OSub}_A^\gamma(w)$ be the set of all ordered $\gamma$-variable words generated by $w$, that is,

$$\mathsf{OSub}_A^\gamma(w) = \{w[u] : u \in A_<^{\leq\beta,\gamma}\}$$

Note that if $\beta = \omega$ and $\gamma < \omega$, then $\mathsf{OSub}_A^\gamma(w)$ contains ordered $\gamma$-variable words of finite and of infinite length. On the other hand, $\mathsf{OSub}_A^\omega(w)$ contains only ordered $\omega$-variable words, hence of infinite length. Most of the time, the alphabet $A$ will be clear from context, and we shall simply write $\mathsf{OSub}^\gamma(w)$ for $\mathsf{OSub}_A^\gamma(w)$. Let

$$\mathsf{OSub}(w) := \bigcup_{0\leq\gamma\leq\beta} \mathsf{OSub}^\gamma(w). \tag{1}$$

Note that $\mathsf{OSub}(w)$ includes in particular $\mathsf{OSub}^0(w)$, that is, the set of words generated by $w$. Since most theorems we shall consider are colorings of words, we will write $\mathsf{WSub}(w)$ for $\mathsf{OSub}^0(w)$. Last, we define

$$\mathsf{OSub}^=(w) := \big\{v \in \mathsf{OSub}(w) : |v| = |w|\big\}$$
$$\mathsf{OSub}^*(w) := \big\{v \in \mathsf{OSub}(w) : |v| < |w|\big\}.$$

The sets $\mathsf{OSub}^{\gamma,=}(w)$, $\mathsf{OSub}^{\gamma,\star}(w)$, $\mathsf{WSub}^=(w)$ and $\mathsf{WSub}^*(w)$ are defined accordingly.

## 2.2 Hales-Jewett theorem

The following theorem is known as the Hales-Jewett theorem [24]. The original proof used double induction and yielded a bound $HJ$ growing as fast as the Ackermann function. Shelah [48] gave a new proof yielding primitive recursive bounds:

**Theorem 2.1** (Hales-Jewett, $\mathsf{RCA}_0$)**.** *There exists a primitive recursive function $HJ(k,\ell)$ such that for every $n \geq HJ(k,\ell)$, every alphabet $A$ of size $k$ and every coloring $f : A^n \to \ell$, there exists some 1-variable word $v$ over $A$ of length $n$ such that $\mathsf{WSub}^{=}(v) = \{v[a] : a \in A\}$ is $f$-homogeneous.*

For simplicity, given a finite alphabet $A$ of size $k$ and some $\ell \geq 1$, we might simply write $HJ(A,\ell)$ for $HJ(\operatorname{card} A, \ell)$.

## 2.3 Proof of $\mathsf{OVW}^0$

We now turn to the proof of $\mathsf{OVW}^0$ over $\mathsf{ACA}_0$. In what follows, fix an finite alphabet $A$ of size $k \geq 1$ and a number of colors $\ell \geq 1$. Unless mentioned, the ordered variable words will be over $A$.

**Definition 2.2.** Let $w$ be an ordered $n$-variable word, $W$ be an ordered $\omega$-variable word, and $f : \mathsf{WSub}^\star(w \cdot W) \to \ell$ be a coloring.

1. *$w$ half-matches $W$* if for every $u \in \mathsf{WSub}^\star(W)$, there is some $v \in \mathsf{OSub}^{1,=}(w)$ such that $\mathsf{WSub}^{=}(v \cdot u)$ is $f$-monochromatic (in other words, there is some color $c < \ell$ such that for every $a \in A$, $f(v[a] \cdot u) = c$).

2. *$w$ full-matches $W$* if for every $u \in \mathsf{WSub}^\star(W)$, there is some $v \in \mathsf{OSub}^{1,=}(w)$ such that $\mathsf{WSub}(v \cdot u)$ is $f$-monochromatic (in other words, for every $a \in A$, $f(v[a] \cdot u) = f(v[\epsilon])$, where $v[\epsilon]$ is the initial segment of $v$ up to the first occurrence of the variable).

The idea behind full-matches is to be able, given an $\mathsf{OVW}^0$-solution within $W$, to merge it with an element of $\mathsf{OSub}^{1,=}(w)$ to obtain a larger $\mathsf{OVW}^0$-solution. It follows that if $w$ full-matches $W$, then at least one element of $\mathsf{OSub}^{1,=}(w)$ can be extended into an infinite $\mathsf{OVW}^0$-solution. The following lemma will not be used in our proof of $\mathsf{OVW}^0$ as it already assumes that $\mathsf{OVW}^0$ holds, but is useful to give an intuition:

**Lemma 2.3** ($\mathsf{RCA}_0 + \mathsf{OVW}^0$)**.** *Let $w$ be an ordered $n$-variable word, $W$ be an ordered $\omega$-variable word, and $f : \mathsf{WSub}^\star(w \cdot W) \to \ell$ be a coloring. Suppose that $w$ full-matches $W$, then there is some $u \in \mathsf{OSub}^{1,=}(w)$ extensible into an infinite $\mathsf{OVW}^0$-solution to $f$.*

*Proof.* Let $g : A^{<\omega} \to \ell \times \operatorname{card} \mathsf{OSub}^{1,=}(w)$ be defined by $g(v) = \langle c, u\rangle$ where $u \in \mathsf{OSub}^{1,=}(w)$ is such that $\mathsf{WSub}(u \cdot W[v])$ is $f$-monochromatic for color $c$. By $\mathsf{OVW}^0$, there is some ordered $\omega$-variable word $U$ such that $\mathsf{WSub}^\star(U)$ is $g$-monochromatic for some color $\langle c, u\rangle$. By definition of $g$, for every $v \in A^{<\omega}$ and every $a \in A$,

$$f(u[\epsilon]) = f(u[a] \cdot W[U[v]]) = c \tag{2}$$

We claim that $u \cdot W[U]$ is an $\mathsf{OVW}^0$-solution to $f$. Indeed, given $v \in \mathsf{WSub}^\star(u \cdot W[U])$, we have two cases. Case 1: $|v| = 0$. Then, $f((u \cdot W[U])[v]) = f(u[\epsilon]) = c$ by Equation (2). Case 2: $|v| > 1$. Say $v = a \cdot v'$ with $a \in A$. Then $f((u \cdot W[U])[v]) = f(u[a] \cdot W[U[v']]) = c$ still by Equation (2). □

Once proven the existence, for every coloring, of a full-match, then one can define an $\omega$-sequence of nested full-matches to produce a tree containing at least one path forming an infinite solution. Towsner proved the existence of full-matches over $\mathsf{ACA}_0$ by defining an $\omega$-sequence of nested half-matches. The existence of half-matches itself was proven over $\mathsf{RCA}$, an extension of $\mathsf{RCA}_0$ will the full induction scheme. However, the proof was non-uniform. Because of this, the $\omega$-sequence of nested half-matches was only shown to exist over $\mathsf{ACA}_0$, and the $\omega$-sequence of nested full-matches needed $\mathsf{ACA}_0^+$. In our case, the existence of half-matches is an immediate consequence of the Hales-Jewett theorem, and yields a uniform bound. Indeed, Shelah [48] proved that the function $HJ$ is primitive recursive, and therefore provably total over $\mathsf{RCA}_0$.

**Lemma 2.4** ($\mathsf{RCA}_0$)**.** *Fix $A$ and $\ell \geq 1$. Let $w$ be an ordered $HJ(A,\ell)$-variable word, $W$ be an ordered $\omega$-variable word, and $f : \mathsf{WSub}^\star(w \cdot W) \to \ell$ be a coloring for some $\ell \geq 1$. Then $w$ half-matches $W$.*

*Proof.* Fix some $u \in \mathsf{WSub}^\star(W)$. Let $g : A^{HJ(A,\ell),0} \to \ell$ be defined by $g(v) = f(w[v] \cdot u)$. By Theorem 2.1, there is some 1-variable word $v \in A_<^{HJ(A,\ell),1}$ such that $\mathsf{WSub}^=(v)$ is $g$-homogeneous for some color $c$. In particular, $w[v] \in \mathsf{OSub}^{1,=}(w)$ is such that $\mathsf{WSub}^=(w[v] \cdot u)$ is $f$-homogeneous for color $c$. □

Before proving the existence of a full-match over $\mathsf{WKL}$, we define the notion of *Graham-Rothschild tree* which corresponds to an $\omega$-sequence of nested half-matches, and will be of fundamental importance over the article. Let $(\ell_n)_{n\in\mathbb{N}}$ be inductively defined by

$$\ell_0 = \ell \text{ and } \ell_{n+1} = \ell_n \times \operatorname{card} A_<^{HJ(A,\ell_n),1}$$

Here, $\operatorname{card} A_<^{HJ(A,\ell_n),1}$ is the number of ordered 1-variable words of length $HJ(A,\ell_n)$ over $A$. Then, let $(t_n)_{n\in\mathbb{N}}$ be inductively defined by

$$t_0 = 0 \text{ and } t_{n+1} = t_n + HJ(A,\ell_n)$$

The *Graham-Rothschild tree for $\ell$-colorings over alphabet $A$* is the tree $\mathcal{T}_{A,\ell}$ whose level $n$ consists of all ordered $n$-variable words $w$ of length $t_n$, such that for every $s < n$, the variable $x_s$ appears only within the positions $[t_n, t_{n+1})$. This tree is partially ordered by the prefix relation. Note that this tree is computably bounded, and every infinite path is an ordered $\omega$-variable word over $A$. The fundamental property of this Graham-Rothschild tree is the following:

**Lemma 2.5** ($\mathsf{RCA}_0$)**.** *Fix $A$ and $\ell \geq 1$. Let $f : A^{<\omega} \to \ell$ be a coloring. For every $u \in A^{<\omega}$ and $n \in \mathbb{N}$, there is some ordered $n$-variable word $w$ at level $n$ in $\mathcal{T}_{A,\ell}$ such that $\mathsf{WSub}^=(w \cdot u)$ is $f$-monochromatic.*

*Proof.* For every $n \in \mathbb{N}$, let $f_n : A^{<\omega} \to \ell_n$ be defined inductively as follows: First, $f_0 : A^{<\omega} \to \ell_0$ is simply $f$. Having defined $f_n$, let $f_{n+1} : A^{<\omega} \to \ell_{n+1}$ be defined by $f_{n+1}(u) = \langle f_n(w[a] \cdot u), w\rangle$ for the least (in any fixed order)

$w \in A_<^{HJ(A,\ell_n),1}$ such that $f_n$ is monochromatic on $\{w[a] \cdot u : a \in A\}$. Note that such a $w$ exists by definition of $HJ(A, \ell_n)$.

By $\Pi^0_1$-induction, for any $n$ and $u \in A^{<\omega}$, $f_n(u)$ can be written of the form $\langle c, w_0, \dots, w_{n-1}\rangle$, where $c < \ell$, $w_s \in A_<^{HJ(A,\ell_s),1}$, and

$$S^u_n = \{w_0[a_0] \cdots w_{n-1}[a_{n-1}] \cdot u : a_0, \dots, a_{n-1} \in A\}$$

is $f$-monochromatic for color $c$. Let $v_n$ be the ordered $n$-variable word

$$w_0[x_0] \cdots w_{n-1}[x_{n-1}]$$

Note that $S^u_n = \mathsf{WSub}^=(v_n \cdot u)$ and $v_n$ is at level $n$ in $\mathcal{T}_{A,\ell}$. □

Note that by the uniformity of the notion of half-match for $\mathsf{OVW}^0$, the Graham-Rothschild tree does not depend on the coloring $f : A^{<\omega} \to \ell$. This has to be put in contrast with Towsner's proof of Hindman's theorem, where the nested sequence of half-matches depends on the coloring. However, we shall not use this uniformity in this article, except for the computability of the Graham-Rothschild tree, while its Hindman counterpart is computable in $\emptyset''$.

**Remark 2.6.** The Graham-Rothschild tree $\mathcal{T}_{A,\ell}$ actually satisfies the following stronger universality property, provable using only $\Delta^0_0$-induction on $n$:

> For every $n \in \mathbb{N}$ and every coloring $g : A^{t_n} \to \ell$, there is an ordered $n$-variable word $w$ at level $n$ in $\mathcal{T}_{A,\ell}$ such that $\mathsf{WSub}^=(w)$ is $g$-monochromatic.

The statement of Lemma 2.5 is the restriction of this universality property to the sub-class $(f^n_u)_{n\in\mathbb{N},u\in A^{<\omega}}$ of *shifts* of $f$, defined for each $n \in \mathbb{N}$ and $u \in A^{<\omega}$ by

$$f^n_u : A^{t_n} \ni v \mapsto f(v \cdot u).$$

However, to emphasize the general translation from Towsner-style proofs to notions of forcing with good first-jump control, we shall only use the statement of Lemma 2.5.

We now turn to the proof of the existence of full-matches over $\mathsf{WKL}$, an extension of $\mathsf{WKL}_0$ with the full induction scheme. Given a function $h : \mathbb{N} \to \mathbb{N}$, we write $h^{<\omega}$ for the set of finite words $w$ such that for every $i < |w|$, $w(i) < h(i)$. A tree is *bounded* if it is a sub-tree of $h^{<\omega}$ for some function $h : \mathbb{N} \to \mathbb{N}$. $\mathsf{WKL}_0$ proves König"s lemma for bounded infinite trees (see Simpson [50, Lemma IV.1.4]). As in Towsner's proof, we need the following intermediary step:

**Lemma 2.7** ($\mathsf{WKL}_0$)**.** *Fix $A$ and $\ell \geq 1$. Let $f : A^{<\omega} \to \ell$ be a coloring. One of the following holds:*

*(1) There is some $n \in \mathbb{N}$ such that $x_0 \cdots x_n$ full-matches $x_{n+1}x_{n+2} \cdots$;*

*(2) There is some ordered $\omega$-variable word $W$ such that $\mathsf{WSub}^\star(W)$ $f$-avoids a color.*

*Proof.* Suppose that (1) does not hold. Let $\mathcal{T}_{A,\ell}$ be the Graham-Rothschild tree for $\ell$-colorings and alphabet $A$.

**Claim 1**: For every $n \in \mathbb{N}$, there is some $w$ at level $n$ in $\mathcal{T}_{A,\ell}$ such that $\mathsf{WSub}^\star(w)$ $f$-avoids some color. Because (1) does not hold, there is some $u_0 \in A^{<\omega}$ such that for every $v \in A^{t_n,1}_<$, $\mathsf{WSub}(v \cdot u_0)$ is not $f$-monochromatic. By Lemma 2.5 applied to $f$ and $u_0$, there is some $w$ at level $n$ in $\mathcal{T}_{A,\ell}$ such that $\mathsf{WSub}^=(w \cdot u_0)$ is $f$-monochromatic for some color $c$. Given $u \in \mathsf{WSub}^\star(w)$, there is some $v \in \mathsf{OSub}^{1,=}(w)$ such that $v[\epsilon] = u$. In particular, $v \in A^{t_n,1}_<$, so $\mathsf{WSub}(v \cdot u_0)$ is not $f$-monochromatic. So $c \neq f(v[\epsilon]) = f(u)$. So $\mathsf{WSub}^\star(w)$ $f$-avoids $c$. This proves our Claim 1.

Let $\mathcal{S}$ be the sub-tree of $\mathcal{T}_{A,\ell}$ of all $w$ witnessing Claim 1. By $\mathsf{WKL}_0$, there is an infinite path $U$ through $\mathcal{S}$. In particular, $U$ is an ordered $\omega$-variable word such that $\mathsf{WSub}^\star(U)$ $f$-avoids some color $c$. □

**Lemma 2.8** ($\mathsf{WKL}$)**.** *Fix $A$ and $\ell \geq 1$. Let $W$ be an ordered $\omega$-variable word and $f : \mathsf{WSub}^\star(W) \to \ell$ be a coloring. There is some ordered $n$-variable word $w_1$ for some $n \geq 1$, and some ordered $\omega$-variable word $W_1$ such that $w_1 \cdot W_1 \in \mathsf{OSub}^\omega(W)$ and $w_1$ full-matches $W_1$.*

*Proof.* By $\Pi^1_2$-induction over $\ell$. Let $g : A^{<\omega} \to \ell$ be defined by $g(u) = f(W[u])$. By Lemma 2.7, there are two cases: Case 1: there is some $n \in \mathbb{N}$ such that $x_0 \cdots x_n$ full-matches $x_{n+1}x_{n+2}\cdots$ for $g$. Let $w_1 = W[x_0 \dots x_n]$ and $W_1$ be such that $W = w_1 \cdot W_1$. Then $w_1$ full-matches $W_1$ for $f$. Case 2: there is some ordered $\omega$-variable word $U$ such that $\mathsf{WSub}^\star(U)$ $g$-avoids a color. Unfolding the definition, $\mathsf{WSub}^\star(W[U])$ $f$-avoids a color. By induction hypothesis, there is some ordered $n$-variable word $w_1$ for some $n \geq 1$ and some ordered $\omega$-variable word $W_1$ such that $w_1 \cdot W_1 \in \mathsf{OSub}^\omega(W[U])$ and $w_1$ full-matches $W_1$. □

Note that Lemma 2.8 is proven over $\mathsf{WKL}_0$ augmented with the full induction scheme. The proof shows that actually, $\Pi^1_2$-induction suffices. The following conservation theorems states that not only $\mathsf{WKL}$ is $\Pi^1_2$-conservative over $\mathsf{ACA}_0$, but one can even bound the computational complexity of the existential witness of the statement. It appears to be part of the folklore.

**Proposition 2.9.** *Let $\Phi(X, Y)$ be an arithmetic statement. Suppose that $\mathsf{WKL} \vdash \forall X \exists Y \Phi(X, Y)$, then $\mathsf{ACA}_0 \vdash \forall X \exists Y$ low over $X$ such that $\Phi(X, Y)$.*

*Proof.* Fix $\Phi(X, Y)$. By the compactness theorem, since $\mathsf{WKL} \vdash \forall X \exists Y \Phi(X, Y)$, then there is a finite set of axioms $T$ of $\mathsf{WKL}$ such that $T \vdash \forall X \exists Y \Phi(X, Y)$. Let $\mathcal{M} = (M, S)$ be a model of $\mathsf{ACA}_0$, and $X \in S$. By Simpson [50, Chapter VIII] and the formalized low basis theorem, there exists a set $Z = \bigoplus_n Z_n$ with $Z' \leq_T X'$, such that $\mathcal{N} = (M, \{Z_n : n \in M\}) \models \mathsf{WKL}_0$ and $X = Z_0$. Note that any second-order formula over $\mathcal{N}$ can be transformed into a first-order formula over $\mathcal{M}$, so since $\mathsf{ACA}_0$ contains arithmetic induction, letting $\Gamma$ be the finite fragment of formulas such that $T \subseteq \mathsf{WKL}_0 + \mathsf{I}\Gamma$, we have $\mathcal{N} \models \mathsf{WKL}_0 + \mathsf{I}\Gamma$. It follows that $\mathcal{N} \models \exists Y \Phi(X, Y)$, so there exists some $n \in Z_n$ such that $\mathcal{N} \models \Phi(X, Z_n)$. Since $\Phi$ is arithmetic, then $\mathcal{M} \models \Phi(X, Z_n)$. Note that $(X \oplus Z_n)' \leq_T Z' \leq_T X'$. □

Combining Proposition 2.9 with Lemma 2.8, we obtain:

**Lemma 2.10** ($\mathsf{ACA}_0$)**.** *Fix $A$ and $\ell \geq 1$. Let $W$ be an ordered $\omega$-variable word and $f : \mathsf{WSub}^\star(W) \to \ell$ be a coloring. There is some ordered $n$-variable word $w_1$ for some $n \geq 1$, and some ordered $\omega$-variable word $W_1$ such that $w_1 \cdot W_1 \in \mathsf{OSub}^\omega(W)$, $w_1$ full-matches $W_1$ and $(W_1 \oplus W \oplus f)' \leq_T (W \oplus f)'$.*

We are now ready to prove $\mathsf{OVW}^0$ over $\mathsf{ACA}_0$.

**Theorem 2.11** ($\mathsf{ACA}_0$)**.** *Fix $A$ and $\ell \geq 1$. For every coloring $f : A^{<\omega} \to \ell$, there is an ordered $\omega$-variable word $W$ over $A$ such that $\mathsf{WSub}^\star(W)$ is $f$-monochromatic.*

*Proof.* We will define a sequence of tuples $(\ell_n, f_n, w_{n+1}, W_n)_{n\in\omega}$ such that

(1) $f_n : \mathsf{WSub}^\star(W_n) \to \ell_n$;

(2) $w_{n+1}$ full-matches $W_{n+1}$ for $f_n$;

(3) $w_{n+1} \cdot W_{n+1} \in \mathsf{OSub}^\omega(W_n)$;

(4) $(\bigoplus_{s\leq n} f_n \oplus W_n)' \leq_T f'$.

First, $\ell_0 = \ell$, $f_0 = f$ and $W_0 = x_0x_1\cdots$. Assume we have defined $\ell_n, f_n$ and $W_n$ satisfying items (1) and (4). By Lemma 2.10, there is some pair $(w_{n+1}, W_{n+1})$ satisfying items (2-4). Let $\ell_{n+1} = \ell_n \times \operatorname{card} \mathsf{OSub}^{1,=}(w_{n+1})$ and let $f_{n+1} : \mathsf{WSub}^\star(W_{n+1}) \to \ell_{n+1}$ be defined by $f_{n+1}(u) = \langle c, v\rangle$ for some $v \in \mathsf{OSub}^{1,=}(w_{n+1})$ such that $\mathsf{WSub}(v \cdot u)$ is $f_n$-monochromatic for some color $c < \ell_n$.

By $\Pi^0_1$-induction, for any $n \in \mathbb{N}$ and $u \in A^{<\omega}$, $f_n(u)$ can be written in the form $\langle c, v_0, \ldots, v_{n-1}\rangle$, where $c < \ell$, $v_s \in \mathsf{OSub}^{1,=}(w_s)$, and $\mathsf{WSub}(v_0 \cdots v_{s-1} \cdot u)$ is $f$-monochromatic for color $c$.

Let $\mathcal{S}$ be the tree whose level $n$ consists of all ordered $n$-variable words $w \in \mathsf{OSub}^{n,=}(w_1 \cdots w_n)$ such that $\mathsf{WSub}^\star(w)$ is $f$-monochromatic. Note that $\mathcal{S}$ is $f'$-computably bounded. Moreover, $\mathcal{S}$ is infinite. By $\mathsf{WKL}_0$, there is an infinite path $U$ through $\mathcal{S}$. Such a path $U$ is an ordered $\omega$-variable word such that $\mathsf{WSub}^\star(U)$ is $f$-monochromatic. □

# 3 Weakly low basis theorem for OVW

The levels of the arithmetic hierarchy are not closed under relativization, in the sense that for $n \geq 2$, the relation "$X$ is $\Delta^0_n(Y)$" is not transitive. It is therefore useful to work with notions of lowness. A set $X$ is of *low$_n$ degree* over $Y$ if $(X \oplus Y)^{(n)} \leq_T Y^{(n)}$. In particular, the relation "$X$ is of low$_n$ degree over $Y$" is transitive. When $n = 1$, we simply say low for low$_1$.

**Definition 3.1.** A $\Pi^1_2$-problem $\mathsf{P}$ admits a *low$_n$ basis* if for every set $Z$ and every $Z$-computable $\mathsf{P}$-instance $X$, there is a $\mathsf{P}$-solution $Y$ to $X$ such that $(Y \oplus Z)^{(n)} \leq Z^{(n)}$.

For instance, Jockusch and Soare [32] proved that $\mathsf{WKL}$ admits a low basis. Some $\Pi^1_2$-problems are "close" to admit a low basis, in that any PA degree over $\emptyset'$ can decide the jump of a solution. We then say that it admits a *weakly low basis*.

**Definition 3.2.** A $\Pi^1_2$-problem $\mathsf{P}$ admits a *weakly low$_n$ basis* if for every set $Z$, every set $P$ of PA degree over $Z^{(n)}$ and every $Z$-computable $\mathsf{P}$-instance $X$, there is a $\mathsf{P}$-solution $Y$ to $X$ such that $(Y \oplus Z)^{(n)} \leq P$.

For any $Z$, by the low basis theorem [32] relative to $Z^{(n)}$, there is a set $P$ of PA degree over $Z^{(n)}$ such that $P' \leq_T Z^{(n+1)}$. It follows that if $(Y \oplus Z)^{(n)} \leq P$, then $(Y \oplus Z)^{(n+1)} \leq_T P' \leq_T Z^{(n+1)}$, so if $\mathsf{P}$ admits a weakly low$_n$ basis, then it admits a low$_{n+1}$ basis. The most famous theorem admitting a weakly low basis [9], but no low basis [31], is Ramsey's theorem for pairs.

Liu, Monin and Patey [35, Corollary 4.5] constructed a computable instance of $\mathsf{OVW}^0$ with no $\Delta^0_2$ solutions, so *a fortiori*, $\mathsf{OVW}^0$ does not admit a low basis. The goal of this section is to prove the following theorem:

**Theorem 3.3.** $\mathsf{OVW}^0$ *admits a weakly low basis.*

As mentioned, if a $\Pi^1_2$-problem admits a weakly low basis, then its admits a low$_2$ basis.

**Corollary 3.4.** $\mathsf{OVW}^0$ *admits a low$_2$ basis theorem.*

*Proof.* Immediate by Theorem 3.3 and by the low basis theorem [32]. □

Last, note that the construction only uses arithmetical induction, and can be entirely formalized within $\mathsf{ACA}_0$:

**Theorem 3.5** ($\mathsf{ACA}_0$)**.** *Fix a finite alphabet $A$ and $\ell \geq 1$. For every set $Z$ and every $Z$-computable coloring $f : A^{<\omega} \to \ell$, there is an ordered $\omega$-variable word $W$ such that $\mathsf{WSub}^\star(W)$ is $f$-monochromatic, and $(W \oplus Z)'' \leq_T Z''$.*

The rest of this section is dedicated to the proof of Theorem 3.3. For simplicity, instead of constructing an $\mathsf{OVW}^0$-solution, we shall work with a weakening of the Ordered Variable Word theorem where only infinitely levels are required to be monochromatic.

**Definition 3.6.** Let $\exists^\infty$-$\mathsf{OVW}^0$ be the $\Pi^1_2$-problem whose instances are colorings $f : A^{<\omega} \to \ell$ for a finite alphabet $A$ and some number of colors $\ell \geq 1$. A $\exists^\infty$-$\mathsf{OVW}^0$*-solution* to $f$ is an ordered $\omega$-variable word $W$ such that for infinitely many $n \in \omega$, $W[A^n] = \{W[u] : u \in A^n\}$ is monochromatic for $f$.

This weaker variant is actually computably equivalent to $\mathsf{OVW}^0$, and this provably over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$:

**Lemma 3.7** ($\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$)**.** *Let $f : A^{<\omega} \to \ell$ be a coloring. For every $\exists^\infty$-$\mathsf{OVW}^0$-solution $W$ to $f$, $f \oplus W$ computes an $\mathsf{OVW}^0$-solution to $f$.*

*Proof.* By the infinite pigeonhole principle (equivalent to $\mathsf{B}\Sigma^0_2$ over $\mathsf{RCA}_0$), there is a color $c < \ell$ such that the $f \oplus W$-computable set

$$L_c = \{n \in \mathbb{N} : W[A^n] \text{ is } f\text{-monochromatic for color } c\}$$

is infinite. Say $L_c = \{n_0 < n_1 < \dots\}$. Let $\hat{W}$ be obtained from $W$ by merging the variable kinds $\{x_m : n_i \leq m < n_{i+1}\}$ into one variable kind, for each $i \in \mathbb{N}$. Then, $\hat{W} \in \mathsf{OSub}^\omega(W)$, and for every $u \in \mathsf{WSub}^\star(\hat{W})$, $|u| \in L_c$, hence $f(u) = c$. It follows that $\mathsf{WSub}^\star(\hat{W})$ is $f$-monochromatic for color $c$, so $\hat{W}$ is an $\mathsf{OVW}^0$-solution to $f$. □

For notation convenience, we shall prove Theorem 3.3 in an unrelativized form. Throughout this section, fix a computable coloring $f : A^{<\omega} \to \ell$ and a set $P$ of PA degree over $\emptyset'$. We will construct an $\exists^\infty$-$\mathsf{OVW}^0$-solution $G$ such that $G' \leq_T P$ using an effectivization of a notion of forcing.

The idea is to construct an infinite $\emptyset'$-computable sequence of conditions whose upward-closure forms a 1-generic filter. The set of stems of this decreasing sequence will form an infinite $\emptyset'$-computable, $\emptyset'$-computably bounded tree $T$, and any PA degree $P$ over $\emptyset'$ will compute an infinite path $U$. By 1-genericity of the filter, any path through $T$ will be a generalized low $\exists^\infty$-$\mathsf{OVW}^0$-solution to $f$, so $P \geq_T U'$.

## 3.1 Notion of forcing

We now use the combinatorics of Section 2 to design a parametric notion of forcing to build solutions to instances of $\exists^\infty$-$\mathsf{OVW}^0$, with a good first-jump control.

**Definition 3.8.** A $\mathbb{P}$-*branch* is a pair $(v, W)$ where $v$ is a finite ordered variable word and $W$ an ordered $\omega$-variable word.

One can think of a $\mathbb{P}$-branch as a Mathias-like condition, where the *stem* $v$ is an initial segment of the ordered variable word we are building, and the *reservoir* $W$ puts some constraints of the future extensions of the stem. We can associate to a $\mathbb{P}$-branch $(v, W)$ its *denotation* $[v, W]$, representing the class of candidates of the solution $G$ we are constructing, namely,

$$[v, W] = \big\{G \in \mathsf{OSub}(v \cdot W) : v \preceq G\big\} \tag{3}$$

Note that, although an $\exists^\infty$-$\mathsf{OVW}^0$-solution to $f$ needs to contain infinitely many variable kinds, and have infinitely many levels of $f$-monochromaticity, this is not necessarily the case for every $G \in [v, W]$. In particular, some $G \in [v, W]$ will only contain the variable kinds of $v$, or will have no level of $f$-monochromaticity. We will therefore need to choose an appropriate $G \in [v, W]$ by satisfying some positive requirements infinitely often (see Lemma 3.21). We also implicitly require that the variable kinds in $G \restriction_{|v|}^{|G|-1}$ and $v$ are disjoint.

We now define the notion of $\mathbb{P}$-branch extension, such that $(\hat{v}, \hat{W})$ extends $(v, W)$ if $(\hat{v}, \hat{W})$ is "more precise" than $(v, W)$. In particular, it will satisfy that if $(\hat{v}, \hat{W})$ extends $(v, W)$, then $[\hat{v}, \hat{W}] \subseteq [v, W]$.

**Definition 3.9** (ℙ-branch extension)**.** A ℙ-branch $(\hat{v}, \hat{W})$ *extends* a ℙ-branch $(v, W)$ (written $(\hat{v}, \hat{W}) \leq (v, W)$) iff: $\hat{v} \succeq v$ and $\hat{v} \cdot \hat{W} \in \mathsf{OSub}^{\omega}(v \cdot W)$. Or equivalently, for some ordered $\omega$-variable word $U \in \mathsf{OSub}^{\omega}(W)$ and some $t \in \omega$, $\hat{v} = v \cdot (U \restriction t)$ and $\hat{W} = U \restriction^{\infty}_{t}$ where $t$ must be in between variables of $U$.

Clearly the extension relation is transitive. We emphasize that it is not necessary the case that $\hat{v} \in \mathsf{OSub}^{*}(v \cdot W)$ (therefore not necessary $\hat{v} \in [v, W]$) since the truncating point $t$ is somewhat more flexible. But $\hat{v}$ must be a prefix of some member in $[v, W]$.

Last, note that a ℙ-branch may not even contain a $\exists^{\infty}$-$\mathsf{OVW}^{0}$ solution to $f$. Therefore, we work with a finite set of ℙ-branches satisfying some universality constraint.

**Definition 3.10** (Forcing condition)**.** A ℙ-*precondition* is a pair $(I, W)$ where

- $I$ is a finite set of ordered variable word that is length-homogeneous. i.e., For some $t \in \omega$ (denoted as $t(I)$) $|v| = t$ for all $v \in I$;
- For every $v \in I$, $(v, W)$ is a ℙ-branch.

A ℙ-precondition $(I, W)$ is $f$-*matching* if for every word $w \in \mathsf{WSub}^{*}(W)$, there exists $v \in I$ such that $\mathsf{WSub}^{=}(v \cdot w)$ is monochromatic for $f$. A ℙ-*condition* is an $f$-matching ℙ-precondition.

One can think of a ℙ-precondition as a finite set of ℙ-branches sharing the same reservoir. Accordingly, any ℙ-precondition $(I, W)$ is given a denotation $[I, W] = \cup_{v \in I}[v, W]$

**Definition 3.11** (ℙ-precondition extension)**.** A ℙ-precondition $(\hat{I}, \hat{W})$ *extends* ℙ-precondition $(I, W)$ iff for every $\hat{v} \in \hat{I}$, there exists a $v \in I$ such that $(\hat{v}, \hat{W}) \leq (v, W)$.

Intuitively, $f$-matching is a universality constraint which ensures that every ℙ-condition contains an $\exists^{\infty}$-$\mathsf{OVW}^{0}$-solution to $f$. To support this intuition, as we did in Lemma 2.3, we shall assume $\mathsf{OVW}^{0}$, and prove the following fact:

**Lemma 3.12** ($\mathsf{RCA}_0 + \mathsf{OVW}^{0}$)**.** *Let $(I, W)$ be a ℙ-condition. Then $[I, W]$ contains an $\exists^{\infty}$-$\mathsf{OVW}^{0}$-solution to $f$.*

*Proof.* Since $(I, W)$ is $f$-matching, let $\bar{f} : A^{<\omega} \to \ell \times I$ be defined as follows:

$$\begin{aligned} &\bar{f} : A^{<\omega} \ni w \mapsto \langle c, v \rangle \text{ such that} \\ &\quad \mathsf{WSub}^{=}(v \cdot W[w]) \text{ is } f\text{-monochromatic for color } c. \end{aligned} \tag{4}$$

Let $U$ be an $\mathsf{OVW}^{0}$-solution to $\bar{f}$, that is, $U$ is an ordered $\omega$-variable word such that

$$\mathsf{WSub}^{\star}(U) \text{ is monochromatic for some color } \langle c, v \rangle \in \ell \times I. \tag{5}$$

Let $\hat{W} = v \cdot W[U]$. Note that $\hat{W} \in [v, W] \subseteq [I, W]$. We claim that $\hat{W}$ is an $\exists^{\infty}$-$\mathsf{OVW}^{0}$-solution to $f$. We shall actually prove the following stronger property:

**Claim 1.** *For every $w \in \mathsf{WSub}^\star(\hat{W})$ with $|w| > |v|$, $f(w) = c$.*

*Proof.* Since $w \in \mathsf{WSub}^\star(\hat{W}) = \mathsf{WSub}^\star(v \cdot W[U])$, there is some $\hat{v} \in \mathsf{WSub}^=(v)$ and $\hat{w} \in \mathsf{WSub}^\star(U)$ such that

$$w = \hat{v} \cdot W[\hat{w}] \tag{6}$$

By (5), $\bar{f}(\hat{w}) = \langle c, v \rangle$, so by (4), $\mathsf{WSub}^=(v \cdot W[\hat{w}])$ is $f$-monochromatic for color $c$. Since $w \overset{(6)}{=} \hat{v} \cdot W[\hat{w}] \in \mathsf{WSub}^=(v \cdot W[\hat{w}])$, we have $f(w) = c$. This completes our claim and the proof of Lemma 3.12. □

□

## 3.2 Effective forcing

It is common in effective forcing to work with a restricted set of $\mathbb{P}$-conditions with computability-theoretic constraints. In our situation, we shall restrict ourselves to $\mathbb{P}$-preconditions $(I, W)$ such that $W$ belongs to a non-empty class of sets $\mathcal{C}$ satisfying the following closure properties:

(1) $\mathcal{C}$ is downward-closed under Turing reducibility: $\forall X \in \mathcal{C} \forall Y \leq_T X, Y \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed under PA degrees: $\forall X \in \mathcal{C} \exists Y \in \mathcal{C}, Y$ is of PA degree over $X$.

Any such class is called a *Scott class*. If one furthermore requires that $\mathcal{C}$ is closed under the effective join, then we obtain the well-known notion of Scott ideal. Any basis theorem for $\Pi^0_1$ classes induces a Scott class. For instance, by the computably dominated and the low basis theorems [32], the classes of all sets of computably dominated degree or of all low degree form Scott classes, respectively. None of the above examples form Scott ideals.

**Definition 3.13.** Given a Scott class $\mathcal{C}$, a $\mathbb{P}^{\mathcal{C}}$-(pre)condition is a $\mathbb{P}$-(pre)condition $(I, W)$ such that $W \in \mathcal{C}$.

To effectivize the construction of a filter, we shall be particularly interested in the class $\mathsf{LOW}$ of all sets of low degree. Any $\mathbb{P}^{\mathsf{LOW}}$-precondition can be represented as a finite object as follows:

**Definition 3.14.** A *lowness index* of a set $X$ of low degree is an integer $e$ such that $\Phi_e^{\emptyset'} = X'$. A *code* for a $\mathbb{P}^{\mathsf{LOW}}$-precondition $(I, W)$ is a pair $\langle I, e \rangle$ where $e$ is a lowness index of $W$.

Note that $\langle I, e \rangle$ is a finitary object, hence can be coded by an integer using any reasonable Gödel numbering. We shall prove the core lemmas of the notion of $\mathbb{P}$-forcing in the general case, but one should keep in mind that they will all hold even when restricting $\mathbb{P}^{\mathcal{C}}$ for any fixed Scott class. Furthermore, when proving a density lemma, we will always specify the amount of computational power needed to transform a code of a $\mathbb{P}^{\mathsf{LOW}}$-precondition into a code of the desired $\mathbb{P}^{\mathsf{LOW}}$-extension.

## 3.3 First-jump control

Given a notion of forcing, there exists a general forcing relation defined as follows: $p$ *forces* a property $\varphi(G)$ if $\varphi(G_{\mathcal{F}})$ holds for every sufficiently generic filter $\mathcal{F}$ containing $p$, where $G_{\mathcal{F}}$ is the set induced by the filter $\mathcal{F}$. Although this forcing relation also admits a simple inductive and syntactical definition, the complexity of the forcing relation often remains too high with respect to the complexity of the forced formula.

For our purpose, it will be sufficient to force $\Sigma^0_1$ and $\Pi^0_1$ formulas. At this level of the arithmetic hierarchy, there is often a stronger forcing relation which does not only hold for any sufficiently generic filter, but for *every* filter containing $p$. We now give a formal definition of a forcing relation for $\Sigma^0_1$ and $\Pi^0_1$-formulas.

By Kleene's normal form theorem, every $\Sigma^0_1$-formula $\varphi(G)$ can be written of the form $\exists x \psi(G\restriction_x)$, where $\psi(\sigma)$ is a *monotonous* $\Delta^0_0$-formula, that is, if $\psi(\sigma)$ holds and $\sigma \preceq \tau$, then $\psi(\tau)$ also holds. We will always assume that a $\Sigma^0_1$-formula is in normal form.

**Definition 3.15** (Forcing relation)**.** Let $(v, W)$ be a $\mathbb{P}$-branch and $\varphi(G) := \exists n \psi(G \restriction n)$ be a $\Sigma^0_1$-formula. Let

(1) $(v, W) \Vdash \varphi(G)$ iff $\psi(v \restriction n)$ holds for some $n$;

(2) $(v, W) \Vdash \neg\varphi(G)$ iff for every $\hat{v} \in [v, W]$, every $n \in \omega$, $\neg\psi(\hat{v} \restriction n)$ holds.

We also say $(v, W)$ *forces* $\varphi(G)$ for $(v, W) \Vdash \varphi(G)$. A $\mathbb{P}$-precondition $(I, W)$ *decides* a $\Sigma^0_1$-formula $\varphi(G)$ iff for every $v \in I$, either $(v, W) \Vdash \varphi(G)$ or $(v, W) \Vdash \neg\varphi(G)$.

Suppose $(v, W) \Vdash \varphi(G)$ for some $\Sigma^0_1$ or $\Pi^0_1$-formula $\varphi(G)$, then $\varphi(G)$ holds for every $G \in [v, W]$. It is also clear that if $(v, W) \Vdash \varphi(G)$, then for every $(\hat{v}, \hat{W}) \leq (v, W)$, we also have $(\hat{v}, \hat{W}) \Vdash \varphi(G)$. Indeed, $[\hat{v}, \hat{W}] \subseteq [v, W]$.

A $\mathbb{P}$-filter $\mathcal{F}$ is *1-generic* if for every $\Sigma^0_1$-formula $\varphi(G)$, there is a condition $p \in \mathcal{F}$ deciding $\varphi(G)$. It is not clear at first sight that every sufficiently generic $\mathbb{P}$-filter is 1-generic. We shall however see that this is the case. For this, we define the so-called *forcing question* $?\vdash$ for $\Sigma^0_1$-formulas, which is a central concept in effective forcing.

In general, a forcing question is a relation $p \mathbin{?\vdash} \varphi(G)$ between a condition $p$ and a formula $\varphi(G)$, such that if $p \mathbin{?\vdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\varphi(G)$, and if $p \mathbin{?\nvdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\neg\varphi(G)$. The computability-theoretic features of a generic set are closely related to the existence of a forcing question with appropriate definability and combinatorial properties [42]. In our case, since we work with a tree-like notion of forcing, we need to adapt the definition.

Recall that $\mathcal{T}_{A,\ell\times I}$ is the Graham-Rothschild tree for $\ell \times I$-colorings over alphabet $A$ (see Section 2).

**Definition 3.16** (Forcing question)**.** Let $(I, W)$ be a $\mathbb{P}$-precondition and $v \in I$. We write $(I, W) \mathbin{?\vdash_v} \varphi(G)$ iff there exists a level $n \in \mathbb{N}$ such that for every $u$ at level $n$ in $\mathcal{T}_{A,\ell\times I}$, there exists some $\hat{u} \in \mathsf{OSub}^{=}(u)$ such that $\varphi(v \cdot W[\hat{u}])$ holds.

The idea is that if $(I, W) ?\vdash_v \varphi(G)$, then there is an extension $(\hat{I}, \hat{W})$ such that $(\hat{v}, \hat{W}) \Vdash \varphi(G)$ for all the stems $\hat{v} \in I$ extending $v$. If on the other hand, $(I, W) ?\nvdash_v \varphi(G)$, then there is an extension $(I, \hat{W})$ with the same set of stems $I$, such that $(v, \hat{W}) \Vdash \neg\varphi(G)$. The issue is that in the positive case, $\hat{I}$ might increase the number of stems incompatible with $v$, and therefore one might never obtain an extension deciding $\varphi(G)$ on all its stems. We shall therefore force $\neg\varphi(G)$ on as many stems as possible, and then force $\varphi(G)$ on all the remaining stems at once.

The easiest case to consider is a negative answer:

**Lemma 3.17** ($\Pi^0_1$-Extension)**.** *Let $(I, W)$ be a $\mathbb{P}$-condition and $v \in I$. If $(I, W) ?\nvdash_v \varphi(G)$, then there exists a $\mathbb{P}$-condition $(I, \hat{W}) \leq (I, W)$ such that $(v, \hat{W}) \Vdash \neg\varphi(G)$. Moreover, we can choose $\hat{W}$ so that $\hat{W}' \leq_T W'$ and the index of $\hat{W}' \leq_T W'$ can be obtained $W'$-computably.*

*Proof.* By definition of $(I, W) ?\nvdash_v \varphi(G)$, for every level $n \in \mathbb{N}$ there is some $u$ at level $n$ in $\mathcal{T}_{A,\ell\times I}$, such that for every $\hat{u} \in \mathsf{OSub}^{=}(u)$, $\varphi(v \cdot W[\hat{u}])$ does not hold. Let $Q$ be the $\Pi^{0,W}_1$ class of all $U \in [\mathcal{T}_{A,\ell\times I}]$ such that

$$\text{for every } u \in \mathsf{OSub}^*(U), \text{ we have } \neg\varphi(v \cdot W[u]). \tag{7}$$

By compactness, the class $Q$ is non-empty. By construction of the Graham-Rothschild tree, every $U \in Q$ is an ordered $\omega$-variable word. By the low basis theorem [32], there is some $U \in Q$ be such that $(W \oplus U)' \leq_T W'$. Clearly,

$$(I, W[U]) \leq (I, W)$$

is a $\mathbb{P}$-precondition. Furthermore, it is $f$-matching since only the reservoir is restricted, so $(I, W[U])$ is a $\mathbb{P}$-condition. It remains to show that $(v, W[U]) \Vdash \neg\varphi(G)$, which follows directly from (7). Thus, we are done.

□

Repeatedly applying Lemma 3.17, we obtain:

**Lemma 3.18** ($\Pi^0_1$-Extension rephrased)**.** *Let $(I, W)$ be a $\mathbb{P}$-condition. There exists a $\mathbb{P}$-condition $(I, U) \leq (I, W)$ and a $J \subseteq I$ such that*

*(1) For every $v \in I \setminus J$, $(v, U) \Vdash \neg\varphi(G)$.*

*(2) For every $v \in J$, $(I, U) ?\vdash_v \varphi(G)$.*

*Moreover, we can choose $U$ so that $U' \leq_T W'$ and the index of $U' \leq_T W'$ and $J$ can be obtained $W'$-computably.*

*Proof.* This is done by applying Lemma 3.17 successively on each $v \in I$ for which the forcing question has a negative answer, until we achieve a $\mathbb{P}$-condition $(I, U) \leq (I, W)$ such that $(I, U) ?\vdash_v \varphi(G)$ on every $v \in I$ such that $(v, U) \nVdash \neg\varphi(G)$. Let $J$ be the set of such stems. □

Now comes the hard case, namely forcing $\varphi(G)$:

**Lemma 3.19** ($\Sigma^0_1$-Extension)**.** *Let $(I, W)$ be a $\mathbb{P}$-condition and $J \subseteq I$. Suppose for every $v \in J$, $(I, W) ?\vdash_v \varphi$. Then there exists a $\mathbb{P}$-condition $(\hat{I}, \hat{W}) \leq (I, W)$ such that for every $\hat{v} \in \hat{I}$ with $\hat{v}$ extending some $v \in J$, we have $(\hat{v}, \hat{W}) \Vdash \varphi$. Moreover, we can choose $(\hat{I}, \hat{W})$ so that $\hat{W} = W \restriction^{\infty}_{t(\hat{I})-t(I)}$ and the index of $\hat{I}$ can be obtained $W'$-computably.*

*Proof.* The main difficulty is to preserve the $f$-matching property while extending stems in $I$. This is possible since each $v \in J$ is extended to sufficiently many $\hat{v}$. Indeed, we concatenate each such $v$ with some ordered variable word generated by nodes in some level of $\mathcal{T}_{A,\ell\times I}$ for *every* node in that level. For the remaining stems $v \in I \setminus J$, we extend $v$ to some $v \cdot w$ where $w$ ranges over a set of words. The concrete proof is as follows.

Unfolding the definition of $(I, W) ?\vdash_v \varphi(G)$, there is a level $n \in \mathbb{N}$ such that for every $v \in J$, every $u$ in level $n$ of $\mathcal{T}_{A,\ell\times I}$,

$$\begin{aligned} &\text{there exists some } \hat{u} \in \mathsf{OSub}^{=}(u) \\ &\text{such that } \varphi(v \cdot W[\hat{u}]) \text{ holds.} \end{aligned} \tag{8}$$

In this proof, let $witness(v)$ denote the set of all $\hat{u}$ satisfying (8). Recall that by construction of the Graham-Rothschild tree, all the nodes at level $n$ in $\mathcal{T}_{A,\ell\times I}$ are of length $t_n$. Let

$$\begin{aligned} \hat{I} &:= \left\{ v \cdot W[\hat{u}] : \begin{array}{l} \text{either } v \in J \text{ and } \hat{u} \in witness(v) \\ \text{or } v \in I \setminus J \text{ and } \hat{u} \in A^{t_n} \end{array} \right\} \\ \hat{W} &:= W \restriction^{\omega}_{t(\hat{I})-t(I)} . \end{aligned} \tag{9}$$

We show that $(\hat{I}, \hat{W})$ is the desired extension. It is obviously a $\mathbb{P}$-precondition. By definition of $\hat{I}$ (see also (8)), for every $\hat{v} \in \hat{I}$ with $\hat{v}$ extending some member of $J$, $\varphi(\hat{v})$ holds (so $(\hat{v}, \hat{W}) \Vdash \varphi(G)$). Thus, it remains to show that

**Claim 1.** *$(\hat{I}, \hat{W})$ is $f$-matching.*

*Proof.* Let

$$w \in \mathsf{WSub}^*(W \restriction^{\infty}_{t(\hat{I})-t(I)}).$$

It suffices to find $\hat{v} \in \hat{I}$ such that

$$\mathsf{WSub}^{=}(\hat{v} \cdot w) \text{ is monochromatic for } f. \tag{10}$$

Since $(I, W)$ is $f$-matching, let $\bar{f} : A^{<\omega} \to \ell \times I$ be defined as follows:

$$\begin{aligned} &\bar{f} : A^{<\omega} \ni w \mapsto \langle c, v \rangle \text{ such that} \\ &\quad \mathsf{WSub}^{=}(v \cdot W[w]) \text{ is } f\text{-monochromatic for color } c. \end{aligned} \tag{11}$$

Let $\hat{w} \in A^{<\omega}$ be such that

$$w = W \restriction^{\omega}_{t(\hat{I})-t(I)} [\hat{w}].$$

By the fundamental property of the Graham-Rothschild tree $\mathcal{T}_{A,\ell\times I}$ (Lemma 2.5) applied to the coloring $\bar{f}$ and the word $\hat{w}$, there is some $u$ at level $n$ in $\mathcal{T}_{A,\ell\times I}$ such that

$$\mathsf{WSub}^{=}(u\cdot\hat{w}) \text{ is } \bar{f}\text{-monochromatic for some color } \langle c, v\rangle \in \ell\times I \tag{12}$$

By (11),

$$\mathsf{WSub}^{=}(v\cdot W[u\cdot\hat{w}]) \text{ is } f\text{-monochromatic for color } c \tag{13}$$

Note that

$$W[u\cdot\hat{w}] = W[u]\cdot W\restriction^{\omega}_{t(\hat{I})-t(I)}[\hat{w}] = W[u]\cdot w$$

So by (13), we have

$$\mathsf{WSub}^{=}(v\cdot W[u]\cdot w) \text{ is } f\text{-monochromatic for color } c \tag{14}$$

Suppose first that $v\in I\setminus J$. In particular, letting $\hat{u}\in\mathsf{Sub}^{=}(u)$, we have in particular $\hat{u}\in A^{t_n}$, so by (9), $v\cdot W[\hat{u}]\in\hat{I}$. Moreover, by (14), we have

$$\mathsf{WSub}^{=}(v\cdot W[\hat{u}]\cdot w) \text{ is } f\text{-monochromatic for color } c.$$

Suppose now that $v\in J$. By (8), there is some $\hat{u}\in \mathit{witness}(v)$ such that $\hat{u}\in\mathsf{OSub}^{=}(u)$. By (9), $v\cdot W[\hat{u}]\in\hat{I}$. Moreover, by (14), we have

$$\mathsf{WSub}^{=}(v\cdot W[\hat{u}]\cdot w) \text{ is } f\text{-monochromatic for color } c.$$

In both cases we are done. This completes the proof of Claim 1.

□

The "Moreover" part follows by analyzing the effectiveness of above proof. This completes the proof of Lemma 3.19. □

In summary of $\Pi^0_1$ extension Lemma 3.18 and $\Sigma^0_1$ extension Lemma 3.19, we have

**Lemma 3.20.** *Let $(I,W)$ be a $\mathbb{P}$-condition and $\varphi(G)$ a $\Sigma^0_1$ formula. There exists a $\mathbb{P}$-condition $(\hat{I},\hat{W})\leq(I,W)$ deciding $\varphi(G)$. Moreover, we can choose $(\hat{I},\hat{W})$ so that $\hat{W}'\leq_T W'$ and the index of $\hat{W}'\leq_T W'$ and $\hat{I}$ can be obtained $W'$-computably.*

*Proof.* Immediate by Lemma 3.18 and Lemma 3.19. The "Moreover" part follows by also noting that the predicate $(I,W)\mathbin{?\vdash_v}\varphi(G)$ is $W'$-decidable (uniformly in $I, v$). □

## 3.4 Positive requirements

Recall that the notion of forcing does not structurally ensures that the resulting set $G$ will contain infinitely many variable kinds, and have infinitely many levels of $f$-monochromaticity. We therefore need to force these requirements explicitly. For this, given $t \in \mathbb{N}$, consider the following $\Sigma^0_1$-formula $\varphi_t(G)$:

$$\text{for some } n \text{ with } t \leq |G[0^n]| < |G|,\ G[A^n] \text{ is } f\text{-monochromatic.} \tag{15}$$

Note that $\varphi_t(G)$ implies that $G$ contains more variable kinds than $G \restriction t$. In order to ensure that $G$ is an $\exists^\infty$-$\mathsf{OVW}^0$-solution to $f$, it suffices to force $\varphi_t(G)$ for infinitely many $t$. In case $(I, W) \mathrel{?}\vdash_v \varphi_{t(I)}$ for all $v \in I$, we can simply apply $\Sigma^0_1$-extension Lemma 3.19 to extend $(I, W)$ to force $\varphi_{t(I)}(G)$. If on the other hand, $(I, W) \mathrel{?}\nvdash_v \varphi_{t(I)}$, we will show that we can delete $v$ without sabotaging the $f$-matching property.

**Lemma 3.21** (Positive requirement)**.** *Let $(I, W)$ be a $\mathbb{P}$-condition and $v \in I$. If $(v, W) \Vdash \neg\varphi_{t(I)}(G)$, then $(I \setminus \{v\}, W)$ is a $\mathbb{P}$-condition.*

*Proof.* $(I \setminus \{v\}, W)$ is obviously a $\mathbb{P}$-precondition. To verify it is $f$-matching, let $w \in \mathsf{WSub}^*(W)$. By $f$-matching of $(I, W)$, there exists $\tilde{v} \in I$ such that

$$\mathsf{WSub}^=(\tilde{v} \cdot w) \text{ is monochromatic for } f. \tag{16}$$

It suffices to show that $\tilde{v} \neq v$. Suppose otherwise. Substitute $\tilde{v}$ by $v$ in (16), we get

$$\mathsf{WSub}^=(v \cdot w) \text{ is monochromatic for } f. \tag{17}$$

But obviously, we can extend $v \cdot w$ to some $v_1 \in (v, W)$ with

$$v_1 \text{ containing more variable kind (than } v) \text{ and} \tag{18}$$

$|v \cdot w|$ is the first occurrence of a new variable kind in $v_1$, which means: let $n$ be the number of variables in $v$

$$v_1[A^n] = \mathsf{WSub}^=(v \cdot w) \text{ (by (17)) is monochromatic for } f. \tag{19}$$

Clearly

$$t(I) = |v| \leq |v \cdot w| = |v_1[A^n]| \overset{(18)}{<} |v_1|.$$

Thus, combining with (19), we have $\varphi_{t(I)}(v_1)$ holds. This is a contradiction to $(v, W) \Vdash \neg\varphi_{t(I)}(G)$. Thus, we are done. □

Repeatedly applying Lemma 3.21, we have:

**Lemma 3.22** (Positive requirement rephrased)**.** *Let $(I, W)$ be a $\mathbb{P}$-condition. There is a $\mathbb{P}$-condition $(\hat{I}, \hat{W}) \leq (I, W)$ such that for every $\hat{v} \in \hat{I}$, $\varphi_{t(I)}(\hat{v})$ holds. Moreover, we can choose $(\hat{I}, \hat{W})$ so that $\hat{W} = W \restriction^\omega_m$ for some $m$ and the index of $\hat{I}$ can be obtained $W'$-computably.*

*Proof.* By $\Pi^0_1$-extension Lemma 3.18, there is a set $J \subseteq I$ and a $\mathbb{P}$-condition $(I, U) \leq (I, W)$ such that

$$\text{for every } v \in I \setminus J,\ (v, U) \Vdash \neg\varphi_{t(I)}(G);$$
$$\text{for every } v \in J,\ (I, U)\ ?\vdash_v \varphi_{t(I)}(G).$$

By Lemma 3.21, $(J, U)$ is a $\mathbb{P}$-condition. By the $\Sigma^0_1$-extension Lemma 3.19, there is a $\mathbb{P}$-condition $(\hat{I}, \hat{W}) \leq (J, U)$ such that

$$\text{for every } \hat{v} \in \hat{I},\ \varphi_{t(I)}(v) \text{ holds.}$$

Thus, we are done. □

Now we are ready to prove Theorem 3.3.

## 3.5 Proof of Theorem 3.3

Recall that a $\mathbb{P}^{\mathsf{LOW}}$-condition is a $\mathbb{P}$-condition $(I, W)$ such that $W$ is low. Start with the following $\mathbb{P}^{\mathsf{LOW}}$-condition $(I_0, W_0)$ where $I_0 := \{\varepsilon\}$, $W_0 := x_0x_1x_2\cdots$ By Lemma 3.20 and Lemma 3.21, we can $\emptyset'$-compute a sequence of codes of $\mathbb{P}^{\mathsf{LOW}}$-conditions

$$(I_0, W_0) \geq (I_1, W_1) \geq \cdots$$

such that

(1) for every $\Sigma^0_1$ formula $\varphi$, there is some $s \in \omega$ such that $(I_s, W_s)$ decides $\varphi$;

(2) for infinitely many $t \in \omega$, there exists $s \in \omega$ such that $t(I_s) \geq t$ and for every $v \in I_s$, $\varphi_t(v)$ holds (recall (15) for definition of $\varphi_t$).

Note that the set of stems $I = \cup_s I_s$ forms an infinite, finitely branching tree. To see that $I$ is infinite, note that by Lemma 3.12, $I_s \neq \emptyset$ for every $s \in \mathbb{N}$. Since this tree is $\emptyset'$-computable, for any set $P$ of PA degree over $\emptyset'$, $P$ computes an infinite path through this tree, say $G = \cup_s v_s$ where $v_s \in I_s$.

By item (2),

$$G \text{ is a } \exists^\infty\text{-}\mathsf{OVW}^0 \text{ solution to } f.$$

To see $P$ computes $G'$, for each $\Sigma^0_1$ formula $\varphi$, $\emptyset'$-computably find $s$ so that $(I_s, W_s)$ decides $\varphi(G)$ and $P$-computably decide whether $(v_s, W_s) \Vdash \varphi(G)$ or $(v_s, W_s) \Vdash \neg\varphi(G)$. Note that we need $P$ to know which stem in $I_s$ is a prefix of $G$. Also note that $G \in [v_s, W_s]$ for all $s$. Thus, $(v_s, W_s) \Vdash \varphi(G)$ ($(v_s, W_s) \Vdash \neg\varphi(G)$ respectively) implies $\varphi(G)$ ($\neg\varphi(G)$ respectively) holds. i.e.,

$$\text{whether } \varphi(G) \text{ holds is } P\text{-decidable (so } G' \leq_T P).$$

Thus, $G$ is as desired and we are done. This completes the proof of Theorem 3.3.

# 4 Preservation of hyperimmunities

The halting set being $\Sigma^0_1$-definable, every $\omega$-model of $\mathsf{ACA}_0$ contains $\emptyset'$. It follows that to prove that a $\Pi^1_2$-problem $\mathsf{P}$ does not imply $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$, it suffices to show the existence of an $\omega$-model of $\mathsf{RCA}_0 + \mathsf{P}$ which does not contain $\emptyset'$. Actually, most statements $\mathsf{P}$ such that $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{ACA}_0$ are shown to satisfy a stronger property, called *cone avoidance*.

**Definition 4.1.** A $\Pi^1_2$-problem $\mathsf{P}$ admits *cone avoidance* if for every set $Z$, every set $C \not\leq_T Z$ and every $Z$-computable $\mathsf{P}$-instance $X$, there is a $\mathsf{P}$-solution $Y$ such that $C \not\leq_T Y \oplus Z$.

If a $\Pi^1_2$-problem $\mathsf{P}$ admits cone avoidance, then for every non-computable set $C$, there is an $\omega$-model of $\mathsf{RCA}_0 + \mathsf{P}$ which does not contain $C$. This technique was used, for instance, by Seetapun and Slaman [47] to prove that Ramsey's theorem for pairs is strictly weaker than $\mathsf{ACA}_0$.

Later, the author [41] introduced the notion of *preservation of hyperimmunities*, as a refinement of cone avoidance to prove separations from other statements such as $\mathsf{RT}^2_2$, where $\mathsf{RT}^n_k$ is the restriction of Ramsey's theorem for $k$-colorings of the $n$-tuples. A function $h : \mathbb{N} \to \mathbb{N}$ is *$Z$-hyperimmune* if it is not dominated by any $Z$-computable function, where $g$ *dominates* $h$ means that $\forall x g(x) \geq h(x)$.

**Definition 4.2.** Let $k \leq \omega$. A $\Pi^1_2$-problem $\mathsf{P}$ admits *preservation of $k$ hyperimmunities* if for every set $Z$, every sequence $\langle h_i : i < k \rangle$ of $Z$-hyperimmune functions and every $Z$-computable $\mathsf{P}$-instance $X$, there is a $\mathsf{P}$-solution $Y$ such that for every $i < k$, $h_i$ is $Y \oplus Z$-hyperimmune.

Both notions are related: Downey et al [15] proved that cone avoidance and preservation of 1 hyperimmunity coincide. On the other hand, for every $k$, there is a $\Pi^1_2$-problem which admits preservation of $k$, but not $k+1$ hyperimmunities. In particular, $\mathsf{RT}^2_2$ admits preservation of 1 hyperimmunity, but the following classical argument shows that the bound is optimal:

**Lemma 4.3** ([31])**.** *$\mathsf{RT}^2_2$ does not admit preservation of 2 hyperimmunities.*

*Proof.* The *principal function* $p_X$ of an infinite set $X = \{x_0 < x_1 \dots\}$ is the map $n \mapsto x_n$. By [30, Theorem 5.2], there exists a $\Delta^0_2$ set $A$ such that $p_A$ and $p_{\overline{A}}$ are both hyperimmune. By Shoenfield's limit lemma [49, Theorem 2], there exists a computable coloring $f : [\mathbb{N}]^2 \to 2$ such that for every $x$, $\lim_y f(x, y)$ exists and equals $A(x)$. Seeing $f$ as a computable instance of $\mathsf{RT}^2_2$, every infinite $f$-homogeneous set $H$ is either a subset of $A$ or of $\overline{A}$, hence $p_H$ dominates either $p_A$ or $p_{\overline{A}}$. □

Preservation of $k$ hyperimmunities is a particular case of a general notion of preservation of a weakness property. One can show that if a $\Pi^1_2$-problem $\mathsf{P}$ preserves some weakness property, but another $\Pi^1_2$-problem $\mathsf{Q}$ does not, then $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{Q}$, as witnessed by an $\omega$-model (see [40, Section 3.4]). We reprove the separation in the case of preservation of hyperimmunities for the sake of containment:

**Lemma 4.4** ([41])**.** *Suppose that* $\mathsf{P}$ *admits preservation of* $k$ *hyperimmunities, but* $\mathsf{Q}$ *does not. Then there is an* $\omega$*-model of* $\mathsf{RCA}_0 + \mathsf{P}$ *which is not a model of* $\mathsf{Q}$

*Proof.* Since $\mathsf{Q}$ does not admit preservation of $k$ hyperimmunities, there is some set $Z$, some sequence $(h_i : i < k)$ of $Z$-hyperimmune functions and a $Z$-computable $\mathsf{Q}$-instance $X$ such that for every $\mathsf{Q}$-solution $Y$ to $X$, at least one $h_i$ is not $Y \oplus Z$-hyperimmune. Since $\mathsf{P}$ admits preservation of $k$ hyperimmunities, one can define an infinite sequence of sets $Z_0, Z_1, \dots$ such that

(1) for every $n \in \omega$ and every $X_0 \oplus \cdots \oplus X_n$-computable $\mathsf{P}$-instance $I$, $Z_m$ is a $\mathsf{P}$-solution to $I$ for some $m \in \omega$.

(2) for every $i < k$ and $n \in \omega$, $h_i$ is $Z_0 \oplus \cdots \oplus Z_n$-hyperimmune.

Let $\mathcal{M}$ be the $\omega$-structure whose second-order part is the Turing ideal $\mathcal{I} = \{A : \exists n(A \leq_T Z_0 \oplus \cdots \oplus Z_n)\}$. Since $\mathcal{I}$ is a Turing ideal, then by Friedman [21], $\mathcal{M} \models \mathsf{RCA}_0$. By (1), $\mathcal{M} \models \mathsf{P}$, and by (2), $\mathcal{M} \not\models \mathsf{Q}$. □

It follows in particular from Lemma 4.4 that if $\mathsf{P}$ admits preservation of $k$ hyperimmunities, and $\mathsf{P}$ implies $\mathsf{Q}$ over $\omega$-models, then so does $\mathsf{Q}$. The goal of this section is to prove the following theorem:

**Theorem 4.5.** $\mathsf{OVW}^0$ *admits preservation of* $\omega$ *hyperimmunities.*

Before proving Theorem 4.5, we derive multiple consequences: Miller and Solomon [37, Problem 1] asked whether $\mathsf{OVW}^0$ implies $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$. We answer negatively:

**Main Theorem 1.4.** $\mathsf{OVW}^0$ implies neither $\mathsf{RT}^2_2$ nor $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$.

*Proof.* By Theorem 4.5, $\mathsf{OVW}^0$ admits preservation of $\omega$ (and a fortiori 2) hyperimmunities, while by Lemma 4.3, $\mathsf{RT}^2_2$ does not. By Lemma 4.4, it follows that $\mathsf{RCA}_0 + \mathsf{OVW}^0 \nvdash \mathsf{RT}^2_2$. Since $\mathsf{ACA}_0 \vdash \mathsf{RT}^2_2$ (see Simpson [50, Lemma III.7.4]), then $\mathsf{RCA}_0 + \mathsf{OVW}^0 \nvdash \mathsf{RT}^2_2$. □

Weaker consequences of $\mathsf{RT}^2_2$ are known not to admit preservation of 2 hyperimmunities. For instance, the *Ascending Descending Sequence* for linear orders of order type $\omega+\omega^*$ ($\mathsf{SADS}$) states that every such linear orders admit an infinite ascending or descending sequence (see Hirschfeldt and Shore [27]).

**Lemma 4.6** ([28])**.** $\mathsf{SADS}$ *does not admit preservation of 2 hyperimmunities.*

*Proof.* Tennenbaum (see Rosenstein [44]) and Denisov (see Goncharov and Nurtazin [23]) constructed a computable linear ordering $\mathcal{L} = (\omega, <_{\mathcal{L}})$ of order type $\omega+\omega^*$ with no computable infinite ascending or descending sequence. Let $U$ and $V$ be the $\omega$ and $\omega^*$ parts, respectively. By assumption, $U \sqcup V = \omega$. We claim that their principal functions $p_U$ and $p_V$ are hyperimmune (see Lemma 4.3 for the notion of principal function). Indeed, suppose that a computable function $f : \omega \to \omega$ dominates $p_U$. Then, one can compute an infinite $\mathcal{L}$-ascending

sequence by letting $x_0$ be the $<_{\mathcal{L}}$-least element $<_\omega$-smaller than $f(0)$, and having defined $x_n$, letting $x_{n+1} >_{\mathcal{L}} x_n$ be the $<_{\mathcal{L}}$-least element $<_\omega$-smaller than $f(n+1)$. Similarly, if $f$ dominates $p_V$, one can compute an infinite $\mathcal{L}$-descending sequence. On the other hand, any infinite ascending or descending sequence $H$ is a subset of $U$ or $V$, in which case $p_H$ dominates $p_U$ or $p_V$. □

**Corollary 4.7.** $\mathsf{OVW}^0$ *does not imply* $\mathsf{SADS}$ *over* $\mathsf{RCA}_0$.

*Proof.* Immediate by Theorem 4.5, Lemma 4.6 and Lemma 4.4. □

We now turn to the proof of Theorem 4.5. Again, we will prove the theorem in its unrelativized form. Fix a computable coloring $f : A^{<\omega} \to \ell$ and a countable sequence $\vec{h} = (h_i)_{i\in\mathbb{N}}$ of hyperimmune functions.

A set $X$ is of *computably dominated* (or *hyperimmune-free*) degree if it does not compute any hyperimmune function. In other words, $X$ is of computably dominated degree if every $X$-computable function is dominated by a computable function. Note that any hyperimmune function is $X$-hyperimmune for every set $X$ of computably dominated degree. By the computably dominated basis theorem [32, Theorem 2.4], the class $\mathsf{HIF}$ of all computably dominated degrees is a Scott class. Recall that a $\mathbb{P}^{\mathsf{HIF}}$-condition is a $\mathbb{P}$-condition $(v, W)$ such that $W$ is of hyperimmune-free degree.

## 4.1 Diagonalization requirements

In order to force preservation of $\omega$ hyperimmunities, one needs to satisfy the following requirements, for every hyperimmune function $h \in \vec{h}$ and Turing functional $\Psi$:

$\mathcal{R}^h_\Psi$: Either $\Psi^G$ is non-total or $\Psi^G$ does not dominate $h$.

The following density lemma shows how to satisfy the $\mathcal{R}^h_\Psi$-requirements. Since $\mathsf{HIF}$ forms a Scott class, all the abstract lemmas from Section 3.3 and Section 3.4 hold when restricting to $\mathbb{P}^{\mathsf{HIF}}$-conditions.

**Lemma 4.8** (Diagonalization requirement)**.** *Let $h : \mathbb{N} \to \mathbb{N}$ be a hyperimmune function, $(I, W)$ be a $\mathbb{P}^{\mathsf{HIF}}$-condition and $\Psi$ be a Turing functional. There is a $\mathbb{P}^{\mathsf{HIF}}$-condition $(\hat{I}, \hat{W})$ of $(I, W)$ such that for every $v \in \hat{I}$,*

*(1) either $(v, \hat{W}) \Vdash \Psi^G(n)\!\uparrow$ for some $n \in \mathbb{N}$;*

*(2) or $(v, \hat{W}) \Vdash \Psi^G(n) < h(n)$ for some $n \in \mathbb{N}$.*

*Proof.* By Lemma 3.18, there is a $\mathbb{P}^{\mathsf{HIF}}$-condition $(I, U) \leq (I, W)$ and $J \subseteq I$ such that

(1) for every $v \in J$ and every $n \in \mathbb{N}$, $(I, U)\, ?\vdash_v \Psi^G(n)\!\downarrow$;

(2) for every $v \in I \setminus J$, there is some $n \in \mathbb{N}$ such that $(v, U) \Vdash \Psi^G(n)\!\uparrow$.

Note that, unfolding the definition of the forcing question, by item (1), for every $n \in \mathbb{N}$, there exists $b_n \in \mathbb{N}$ such that

$$\text{for every } v \in J,\ (I, U) \mathbin{?\vdash_v} \Psi^G(n) \downarrow < b_n. \tag{20}$$

Moreover, since the forcing question is $\Sigma^0_1(U)$, the sequence $(b_n)_{n\in\mathbb{N}}$ can be chosen to be $U$-computable. Since $U$ is of hyperimmune-free degree and $h$ is hyperimmune, $h$ is $U$-hyperimmune, so there is some $n \in \mathbb{N}$ such that $b_n < h(n)$. By Lemma 3.19, there is a $\mathbb{P}^{\mathsf{HIF}}$-condition $(\hat{I}, \hat{W}) \leq (I, U)$ such that

$$\text{for every } \hat{v} \in \hat{I} \text{ extending some stem in } J,\ (\hat{v}, \hat{W}) \Vdash \Psi^G(n) \downarrow < b_n.$$

This completes the proof of Lemma 4.8. □

We are now ready to prove Theorem 4.5.

## 4.2 Proof of Theorem 4.5

Let $f : A^{<\omega} \to \ell$ be a computable coloring, for a finite alphabet $A$ and some $\ell \geq 1$. Let $\vec{h} = (h_i)_{i\in\mathbb{N}}$ be a countable sequence of hyperimmune functions. Let

$$(I_0, W_0) \geq (I_1, W_1) \geq \cdots$$

be a sufficiently generic decreasing sequence of $\mathbb{P}^{\mathsf{HIF}}$-conditions. By Lemma 4.8, for every $h \in \vec{h}$ and every Turing functional $\Psi$, there is some $s \in \mathbb{N}$ such that for every $v \in I_s$,

(1) either $(v, W_s) \Vdash \Psi^G(n) \uparrow$ for some $n$;

(2) or $(v, W_s) \Vdash \Psi^G(n) \downarrow < h(n)$ for some $n$.

As in the proof of Theorem 3.3, let $G \in \cap_s [I_s, W_s]$. By Lemma 3.21, $G[A^n]$ is $f$-monochromatic for infinitely many $n$. By items (1,2), $h$ is $G$-hyperimmune for each $h \in \vec{h}$. This completes the proof of Theorem 4.5.

# 5 Strong weakness of $\exists^\infty$-IHJ

Preservation of 1 hyperimmunity shows the inability, for a $\Pi^1_2$-problem $\mathsf{P}$, to encode in its computable instances sufficiently fast-growing functions that dominate some fixed hyperimmune function. There are some $\Pi^1_2$-problems that are so weak that even arbitrary instances have this inability. This yields the notion of strong preservation of 1 hyperimmunity.

**Definition 5.1.** Let $k \leq \omega$. A $\Pi^1_2$-problem $\mathsf{P}$ admits *strong preservation of $k$ hyperimmunities* if for every set $Z$, every sequence $\langle h_i : i < k\rangle$ of $Z$-hyperimmune functions and every *arbitrary* $\mathsf{P}$-instance $X$, there is a $\mathsf{P}$-solution $Y$ such that for every $i < k$, $h_i$ is $Y \oplus Z$-hyperimmune.

The equivalence from Downey et al [15] between preservation of 1 hyperimmunity and cone avoidance extends to their strong counterparts. The most famous example of $\Pi^1_2$-problem admitting strong cone avoidance is Ramsey's theorem for singletons ($\mathsf{RT}^1$), also known as the infinite pigeonhole principle (see Dzhafarov and Jockusch [17]). For many hierarchies of partition theorems $(\mathsf{P}^n)_{n\in\mathbb{N}}$, there exists a deep link between strong preservation of properties for $\mathsf{P}^n$ and preservation of properties for $\mathsf{P}^{n+1}$. This will be exemplified in Section 8 to show that $\mathsf{CSL}^1$ admits preservation of 1 hyperimmunity, and therefore does not imply $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$.

It is natural to ask whether $\mathsf{OVW}^0$ admits strong preservation of 1 hyperimmunity. While it is the case, the proof happens to be surprisingly intricate, and will be decomposed into two important steps. In this section, we will prove that an infinite version of the Hales-Jewett theorem admits strong preservation of $\omega$ hyperimmunities. Then, in Section 6, this fact will be used to prove that $\mathsf{OVW}^0$ admits strong preservation of 1 hyperimmunity.

## 5.1 Infinite Hales-Jewett theorem

A *left-variable word* over an alphabet $A$ is a 1-variable word $v$ over $A$ such that $v(0)$ is the first occurrence of the unique variable kind. $\mathsf{OVW}^0$ can be reformulated as a theorem about left-variable words [25] as follows:

**Theorem 5.2** (Ordered Variable Word theorem, rephrased)**.** *Fix an alphabet $A$ and some $\ell \in \mathbb{N}$. For every $\ell$-coloring of $A^{<\omega}$, there exists an infinite sequence $(v_n)_{n\in\mathbb{N}}$ of 1-variable words over $A$ such that for every $n > 0$, $v_n$ is a left-variable word, and the following set is monochromatic:*

$$\{v_0[a_0] \cdot v_1[a_1] \cdots v_s[a_s] : s \in \mathbb{N}, a_0, \dots, a_s \in A\}$$

Under this formulation, it is a natural strengthening the infinite version of the Hales-Jewett theorem where none of $v_n$ are required to be left-variable words. We shall actually consider an even weaker statement, which can be thought of as a Hales-Jewett counterpart of $\exists^\infty$-$\mathsf{OVW}$:

**Definition 5.3.** Let $\exists^\infty$-$\mathsf{IHJ}$ be the $\Pi^1_2$-problem whose instances are colorings $f : A^{<\omega} \to \ell$ for a finite alphabet $A$ and some $\ell \geq 1$. A $\exists^\infty$-$\mathsf{IHJ}^0$-*solution* to $f$ is an ordered $\omega$-variable word $W$ such that for infinitely many $n \in \mathbb{N}$ in between variable kinds of $W$, $\mathsf{WSub}^=(W\restriction_n)$ is $f$-monochromatic.

Here, an integer $n \in \mathbb{N}$ is *in between variable kinds of $W$* if no variable kind occurs both in $W\restriction_n$ and in $W\restriction^\omega_n$. Note that an $\exists^\infty$-$\mathsf{IHJ}^0$-solution gives no information about the positions of monochromaticity. The remainder of this section is devoted to the proof of the following theorem:

**Theorem 5.4.** $\exists^\infty$-$\mathsf{IHJ}$ *admits strong preservation of $\omega$ hyperimmunities.*

The study of this seemingly artificial statement is justified by its use in the proof of strong preservation of 1 hyperimmunity of $\mathsf{OVW}^0$ in Section 6.

## 5.2 Notion of forcing

Fix a finite alphabet $A$ and an arbitrary coloring $f : A^{<\omega} \to \ell$ for some number of colors $\ell \geq 1$. We shall use the same notion of forcing as in Section 4, that is, we shall work with $\mathbb{P}^{\mathsf{HIF}}$-conditions, where $\mathsf{HIF}$ denotes the class of all sets of hyperimmune-free degree.

The only difficulty comes from the fact that, the coloring $f$ being arbitrary, it might be of hyperimmune degree, and therefore too complex to be "directly accessed". Thankfully, the core lemmas about first-jump control rely on the so-called Graham-Rotschild tree (see Section 2), whose combinatorial features are universal, in the sense that they hold for every coloring, and therefore do not need to access $f$ in particular. It follows that Lemma 3.18 and Lemma 3.19 about the forcing question still hold, except for the "Moreover" part about uniformity. The main diagonalization lemma of Section 4 uses abstractly the lemmas about the forcing question, so it also holds:

**Lemma 4.8 (Diagonalization requirement).** Let $h : \mathbb{N} \to \mathbb{N}$ be a hyperimmune function, $(I, W)$ be a $\mathbb{P}^{\mathsf{HIF}}$-condition and $\Psi$ be a Turing functional. There is an extension $(\hat{I}, \hat{W})$ of $(I, W)$ such that for every $v \in \hat{I}$,

(1) either $(v, \hat{W}) \Vdash \Psi^G(n) \uparrow$ for some $n \in \mathbb{N}$;

(2) or $(v, \hat{W}) \Vdash \Psi^G(n) < h(n)$ for some $n \in \mathbb{N}$.

## 5.3 Positive requirements

For positive requirements, the formula $\varphi_t$ being $\Sigma^{0,f}_1$, it is no longer $\Sigma^0_1$ since $f$ is not computable, so the $\Pi^0_1$ extension Lemma 3.17 and Lemma 3.21 do not hold. We therefore split $\varphi_t$ into two formulas, one focusing on adding more variable kinds, and the other on adding more monochromatic levels. For the "more variable" part, it is still the same as Lemma 3.21 (since the "more variable" predicate is still $\Sigma^0_1$).

**Lemma 5.5** (Positive requirement: variable)**.** *Let $(I, W)$ be a $\mathbb{P}^{\mathsf{HIF}}$-condition. There exists a $\mathbb{P}^{\mathsf{HIF}}$-condition $(\hat{I}, \hat{W}) \leq (I, W)$ such that for every $\hat{v} \in \hat{I}$ and $v \in I$ with $\hat{v} \succeq v$, $\hat{v}$ contains more variable kinds than $v$.*

*Proof.* Same as Lemma 3.21. □

For the "more monochromatic level" part, for an ordered variable word $v$, let $p_f(v)$ denotes the number of $t \leq |v|$ in between variable kinds of $v$ such that $\mathsf{WSub}^{=}(v \upharpoonright t)$ is $f$-monochromatic.

**Lemma 5.6** (Positive requirement: monochromaticity)**.** *Let $(I, W)$ be a $\mathbb{P}^{\mathsf{HIF}}$-condition and $v \in I$. There exists a $\mathbb{P}^{\mathsf{HIF}}$-condition $(J, U) \leq (I, W)$ such that for every $\hat{v} \in J$ with $\hat{v} \succeq v$, we have $p_f(\hat{v}) > p_f(v)$. Moreover, every $v \in I$ admits at most one extension in $J$.*

*Proof.* We first try to find a $w \in \mathsf{WSub}^*(W)$ such that

$$\mathsf{WSub}^{=}(v \cdot w) \text{ is monochromatic for } f. \tag{21}$$

We divide into two cases depending on whether such a $w$ exists.

**Case 1:** The word $w$ as (21) does not exist. We show that $(I \setminus \{v\}, W)$ is the desired extension. In this case, there is no $\hat{v} \in I \setminus \{v\}$ such that $\hat{v} \succeq v$. Therefore, it remains to verify that $(I \setminus \{v\}, W)$ is a $\mathbb{P}^{\mathsf{HIF}}$-condition. It is obviously a $\mathbb{P}^{\mathsf{HIF}}$-precondition. To see that $(I \setminus \{v\}, W)$ is $f$-matching, let $w \in \mathsf{WSub}^*(W)$. By $f$-matching of $(I, W)$, there exists $\tilde{v} \in I$ such that

$$\mathsf{WSub}^{=}(\tilde{v} \cdot w) \text{ is monochromatic for } f. \tag{22}$$

Since (21) does not hold, $\tilde{v} \in I \setminus \{v\}$. The "Moreover" part follows obviously. Thus we are done for Case 1.

**Case 2:** The word $w$ as (21) exists. Consider the set

$$J := \{\hat{v} \cdot w : \hat{v} \in I\} \text{ and ordered variable word } U := W \restriction^{\infty}_{|w|} .$$

We show that $(J, U)$ is the desired extension. Clearly $(J, U)$ is a $\mathbb{P}^{\mathsf{HIF}}$-precondition extending $(I, W)$ and (by (21))

$$p_f(v \cdot w) > p_f(v).$$

We need to verify that $(J, U)$ is $f$-matching. Let $u \in \mathsf{WSub}^*(U)$, clearly

$$w \cdot u \in \mathsf{WSub}^*(W).$$

Therefore, by $f$-matching of $(I, W)$, there exists $v \in I$ such that

$$\mathsf{WSub}^{=}(v \cdot w \cdot u) \text{ is monochromatic for } f. \tag{23}$$

But $v \cdot w \in J$. Thus, combine with (23), we have $(J, U)$ is $f$-matching. The "Moreover" part follows obviously. Thus we are done for Case 2.

□

We are now ready to prove Theorem 5.4.

## 5.4 Proof of Theorem 5.4

The proof is the same as Theorem 3.3, except that the positive requirements are forced separately: Let $\vec{h} = (h_n)_{n \in \mathbb{N}}$ be a countable sequence of hyperimmune functions. Let

$$(I_0, W_0) \geq (I_1, W_1) \geq \cdots$$

be a sufficiently generic decreasing sequence of $\mathbb{P}^{\mathsf{HIF}}$-conditions. By Lemma 4.8, for every hyperimmune function $h \in \vec{h}$ and Turing functional $\Psi$, there is a $s \in \mathbb{N}$ such that for every $v \in I_s$,

(1) either $(v, W_s) \Vdash \Psi^G(n) \uparrow$ for some $n$;

(2) or $(v, W_s) \Vdash \Psi^G(n) \downarrow < h(n)$ for some $n$.

As in the proof of Theorem 3.3, let $G \in \cap_s [I_s, W_s]$. By items (1,2), each $h \in \vec{h}$ is $G$-hyperimmune. By Lemma 5.5, $G$ contains infinitely many variable kinds, and by Lemma 5.6, there are infinitely many $n \in \mathbb{N}$ in between variable kinds of $G$ such that $\mathsf{WSub}^{=}(G \restriction_n)$ is $f$-monochromatic. This completes the proof of Theorem 5.4.

# 6 Strong weakness of $\mathsf{OVW}^0$

The goal of this section is to prove the following theorem:

**Theorem 6.1.** $\mathsf{OVW}^0$ *admits strong preservation of 1 hyperimmunity.*

Actually, given a coloring $f : A^{<\omega} \to \ell$, we shall construct an ordered $\omega$-variable word $G$ with slightly weaker properties: each level of $G$ will be $f$-monochromatic, but the color of monochromaticity will depend on the level:

**Definition 6.2.** An ordered variable word $v$ is *level-homogeneous* for a coloring $f : A^{<\omega} \to \ell$ iff for every $u_0, u_1 \in \mathsf{WSub}^{\star}(v)$ such that $|u_0| = |u_1|$, we have $f(u_0) = f(u_1)$.

This weaker notion of monochromaticity yields the following statement, that we shall refer to as *Level Ordered Variable Word* theorem:

**Definition 6.3.** Let $\mathsf{LOVW}^0$ be the $\Pi^1_2$-problem whose instances are colorings $f : A^{<\omega} \to \ell$ for a finite alphabet $A$ and some number of colors $\ell \geq 1$. A $\mathsf{LOVW}^0$-*solution* to $f$ is an ordered $\omega$-variable word $W$ which is level-homogeneous for $f$.

$\mathsf{LOVW}^0$ can be thought of as $\mathsf{OVW}^0$ without one last application of the infinite pigeonhole principle ($\mathsf{RT}^1$). Because of this, it satisfies a stronger notion of preservation:

**Theorem 6.4.** $\mathsf{LOVW}^0$ *admits strong preservation of $k$ hyperimmunities for every $k \in \mathbb{N}$.*

Note that $\mathsf{OVW}^0$ does not admit strong preservation of 2 hyperimmunities. Indeed, $\mathsf{RT}^1$ is strongly computably reducible to $\mathsf{OVW}^0$, and it is known that $\mathsf{RT}^1$ even for 2-colorings does not admit strong preservation of 2 hyperimmunities (by [30, Theorem 5.2], there exists a coloring $f : \mathbb{N} \to 2$ such that $f^{-1}(0)$ and $f^{-1}(1)$ are both hyperimmune). Before proving Theorem 6.4, let us assume it holds, and deduce Theorem 6.1.

*Proof of Theorem 6.1.* As usual, we prove it in its unrelativized form. Let $f : A^{<\omega} \to \ell$ be an arbitrary coloring for some finite alphabet $A$ and $\ell \geq 1$. Let $h : \mathbb{N} \to \mathbb{N}$ be a hyperimmune function. By Theorem 6.4, there is an ordered

$\omega$-variable word $W$ which is level-homogeneous for $f$, and such that $h$ is $W$-hyperimmune. Let $X = \{t_0 < t_1 < \dots\}$ be such that $t_n$ is the position of the first occurrence of $x_n$ in $W$. Let $g : \mathbb{N} \to \ell$ be such that $g(n)$ is the color of $f$-monochromaticity of $W[A^n]$. By Patey [41, Theorem 28], $\mathsf{RT}^1$ admits strong preservation of 1 hyperimmunity, so there is an infinite $g$-homogeneous set $H \subseteq \mathbb{N}$ for some color $c < \ell$ such that $h$ is $H \oplus W$-hyperimmune. Let $\hat{W} \leq_T H \oplus W$ be obtained from $W$ by merging variable kinds so that for every $u \in \mathsf{WSub}^\star(\hat{W})$, $|u| \in \{t_n : n \in H\}$. In particular, $f(u) = g(n)$ for some $n \in H$ such that $|u| = t_n$, so $f(u) = c$. □

Through this section, fix a coloring $f : A^{<\omega} \to \ell$ and a family $\vec{h} = (h_i)_{i<k}$ of hyperimmune functions for some $k \in \mathbb{N}$. We shall construct $G$ through the following notion of forcing:

**Definition 6.5.** A $\mathbb{Q}$-*condition* is a pair $(v, W)$ such that

(1) $v$ is a finite ordered variable word and $W$ an ordered $\omega$-variable word;

(2) $v$ is level-homogeneous for $f$;

(3) each $h \in \vec{h}$ is $W$-hyperimmune.

In other words, a $\mathbb{Q}$-condition corresponds with a $\mathbb{P}$-branch with some color constraints on $v$ and some computability-theoretic weakness on $W$. Note that this notion of forcing is parameterized by both the coloring $f$, and the family of hyperimmune function $\vec{h}$. The denotation $[v, W]$ of a $\mathbb{Q}$-condition is exactly the same as for a $\mathbb{P}$-branch. In particular, do not impose any $G \in [v, W]$ to be level-homogeneous for $f$. The notion of $\mathbb{Q}$-extension is also the same as $\mathbb{P}$-branch extension. We state it again for the sake of completeness:

**Definition 6.6** ($\mathbb{Q}$-condition extension)**.** A $\mathbb{Q}$-condition $(\hat{v}, \hat{W})$ *extends* a $\mathbb{Q}$-condition $(v, W)$ (written $(\hat{v}, \hat{W}) \leq (v, W)$) iff: $\hat{v} \succeq v$ and $\hat{v} \cdot \hat{W} \in \mathsf{OSub}^\omega(v \cdot W)$.

## 6.1 Requirement and progress

For a Turing functional $\Psi$ and some hyperimmune function $h \in \vec{h}$, let $\mathcal{R}^h_\Psi$ be the requirement:

$\mathcal{R}^h_\Psi$: Either $\Psi^G$ is non-total or $\Psi^G$ does not dominate $h$.

We now define what it means for a $\mathbb{Q}$-condition to satisfy a requirement $\mathcal{R}^h_\Psi$. Instead of asking $\mathcal{R}^h_\Psi$ to hold for every $G \in [v, W]$, we ask for a stronger property:

**Definition 6.7.** A $\mathbb{Q}$-condition $(v, W)$ *satisfies* $\mathcal{R}^h_\Psi$ iff:

- either for every $G \in [v, W]$, $\Psi^G$ is non total
- or $\Psi^v(n) \downarrow < h(n)$ for some $n$.

This notion of satisfaction is more uniform, in the sense that every $G \in [v, W]$ satisfies $\mathcal{R}^h_\Psi$ by the same disjunct. This stronger definition will be used in the proof of Theorem 6.1. In general, even for some fixed $h \in \vec{h}$ and $\Psi$, the set of $\mathbb{Q}$-conditions satisfying an $\mathcal{R}^h_\Psi$-requirement is not dense.

**Definition 6.8** (Non-progressable)**.** A $\mathbb{Q}$-condition $(v, W)$ is *non-progressable for $h$* iff for some Turing functional $\Psi$, there is no extension $(\hat{v}, \hat{W})$ of $(v, W)$ such that $(\hat{v}, \hat{W})$ satisfies $\mathcal{R}^h_\Psi$. Otherwise, we say $(v, W)$ is *progressable for $h$* (i.e., it can be extended to satisfy $\mathcal{R}^h_\Psi$ for every $\Psi$).

The whole difficulty of the proof will be to show the existence of a $\mathbb{Q}$-condition $(\epsilon, W^*)$ below which every $\mathbb{Q}$-extension is progressable for each $h \in \vec{h}$. In other words, the set of $\mathbb{Q}$-conditions satisfying any $\mathcal{R}^h_\Psi$-requirement is dense below $(\epsilon, W^*)$.

## 6.2 Witness for progress

We now introduce the core concept of witness for progress which will be used to construct the initial condition $(\epsilon, W^*)$.

**Definition 6.9.** Let $W$ be an ordered $\omega$-variable word such that each $h \in \vec{h}$ is $W$-hyperimmune. A *$W$-witness for progress for $h$* is a coloring $g : A^{<\omega} \to \ell$ such that for every finite ordered variable word $v$ such that $\mathsf{WSub}^=(v)$ is $g$-monochromatic and level-homogeneous for $f$, the $\mathbb{Q}$-condition $(v, W \restriction^\omega_{|v|})$ is progressable for $h$.

Given such an ordered $\omega$-variable word $W$ and $h \in \vec{h}$, we write $WitCol_h(W)$ for the class of all $W$-witnesses for progress for $h$. It is of course not clear at all that $WitCol_h(W) \neq \emptyset$. Before proving the existence of such witnesses for progress, we sketch how they will be used in the proof.

**Remark 6.10** (How to use witnesses for progress)**.** Suppose we found some family of colorings $\vec{g} = (g_i)_{i<k}$ of type $A^{<\omega} \to \ell$ such that $g_i$ is not only a $W$-witness for progress for $h_i$, but also simultaneously a $U$-witness for progress for $h_i$ for every $U \in \mathsf{OSub}^\omega(W)$ such that each $h \in \vec{h}$ is $U$-hyperimmune.

Then, by letting $v$ staying in the monochromatic-for-$g$ world for each $g \in \vec{g}$, $(v, U)$ will always be progressable for each $h \in \vec{h}$. Thus, one may want to construct an ordered $\omega$-variable word $W^* \in \mathsf{WSub}^=(W)$ that is somewhat $g$-monochromatic for each $g \in \vec{g}$, and consider $\mathbb{Q}$-conditions of the form $(v, U)$ where $(v, U) \leq (\varepsilon, W^*)$. Here, by "somewhat $g$-monochromatic", we shall see that it is sufficient for $W^*$ to be a $\exists^\infty$-$\mathsf{IHJ}$ solution to $g$. We will therefore use strong preservation of $\omega$-hyperimmunity of $\exists^\infty$-$\mathsf{IHJ}$ proven in Section 5.

The remainder of Section 6.2 is devoted to the proof of existence of such witnesses. Consider the following notion of matching:

**Definition 6.11** ($A^{<\omega} \to \ell$-matching)**.** A set $I$ of ordered variable words, each of length $t$, is *$A^{<\omega} \to \ell$-matching* iff for every coloring $g : A^t \to \ell$, there is some $v \in I$ such that $\mathsf{WSub}^=(v)$ is monochromatic for $g$.

A starting point but also a key observation is that the set of initial segments which are non-progressable for $h$ cannot form a $A^{<\omega} \to \ell$-matching set.

**Lemma 6.12** (Set of non-progressable initial segment is not $A^{<\omega} \to \ell$-matching)**.** *Let $W$ be an ordered $\omega$-variable word such that each $h \in \vec{h}$ is $W$-hyperimmune and fix some $h \in \vec{h}$. Let $I$ be a $A^{<\omega} \to \ell$-matching set of ordered variable words, with each element of $I$ of length $t$, such that $\mathsf{WSub}^*(v)$ is level-homogeneous for $f$ for all $v \in I$. Then there is some $v \in I$ such that $(v, W)$ is a $\mathbb{Q}$-condition which is progressable for $h$.*

*Proof.* Suppose otherwise. For each $v \in I$, let $\Psi_v$ be the witness of $(v, W)$ being non-progressable for $h$. To derive a contradiction, we firstly try to entail $\Psi_v^G$ to be non-total. To this end, for every $n \in \mathbb{N}$, consider the set $Q_n$ of colorings $g : A^{<\omega} \to \ell$ with $g \upharpoonright A^{\leq t} = f \upharpoonright A^{\leq t}$ such that

$$\begin{aligned} &\text{for every } v \in I, \text{ every } \hat{v} \in [v, W] \text{ with} \\ &\hat{v} \text{ level-homogeneous for } g, \text{ we have } \Psi^{\hat{v}}(n) \uparrow . \end{aligned} \tag{24}$$

Intuitively, $Q_n \neq \emptyset$ is seen as it is possible to entail $\Psi_v^G(n) \uparrow$ for some $v \in I$. Clearly $Q_n$ is a $\Pi_1^{0,W}$ class uniformly in $n$. We divide into two cases and will show that in each case there is some $v \in I$ so that $(v, W)$ can be extended to satisfy $\mathcal{R}_{\Psi_v}$, thus a contradiction to the otherwise hypothesis.

**Case 1:** For every $n \in \mathbb{N}$, $Q_n = \emptyset$.

By compactness, for every $n \in \mathbb{N}$, there exists a finite set $I_n$ of ordered variable words witnessing $Q_n = \emptyset$. That is: for every coloring $g : A^{<\omega} \to \ell$ with $g \upharpoonright A^{\leq t} = f \upharpoonright A^{\leq t}$, there exists $\hat{v} \in I_n$ such that:

$$\begin{aligned} &\text{for some } v \in I, \text{ we have } \hat{v} \in [v, W]; \\ &\hat{v} \text{ level-homogeneous for } g \text{ and } \Psi^{\hat{n}}(n) \downarrow. \end{aligned} \tag{25}$$

Moreover, $(I_n : n \in \mathbb{N})$ could be chosen so that the map $n \mapsto I_n$ is $W$-computable. We may also assume $I_n$ is not redundant in the sense that for each $\hat{v} \in I_n$, there exists a $v \in I$ and coloring $g$ so that $\hat{v}, v, g$ together satisfies (25), in particular $\Psi_v^{\hat{v}}(n) \downarrow$. Therefore, consider the following function:

$$\hat{h} : \mathbb{N} \ni n \mapsto \max\{\Psi_v^{\hat{v}}(n) : \hat{v} \in I_n, v \in I, \hat{v} \succeq v\}. \tag{26}$$

Note that $\hat{h}$ is $W$-computable, but $h$ is $W$-hyperimmune, therefore there is some $n^* \in \mathbb{N}$ such that

$$\hat{h}(n^*) < h(n^*). \tag{27}$$

Since $Q_{n^*} = \emptyset$, in particular, $f \notin Q_{n^*}$, by replacing $g$ with $f$ in (25), there exists a $v \in I$ and $\hat{v} \in [v, W]$ such that:

$$\hat{v} \text{ level-homogeneous for } f \text{ and } \Psi^{\hat{n}}(n^*) \downarrow < h(n^*).$$

The last inequality follows since we could chose $\hat{v}$ in $I_{n^*}$ (see (26)(27)). But now, $(\hat{v}, W\restriction^{\omega}_{|\hat{v}|-|v|})$ is a $\mathbb{Q}$-condition extending $(v, W)$ that satisfies $\mathcal{R}^h_{\Psi_v}$. This is a contradiction to the otherwise hypothesis.

**Case 2:** For some $n$, $Q_n \neq \emptyset$.

Since $Q_n$ is a $\Pi^{0,W}_1$ class and $h$ is $W$-hyperimmune, by the computably dominated basis theorem ([32, Theorem 2.4]), there exists a $g \in Q_n$ such that each $h \in \vec{h}$ is $g \oplus W$-hyperimmune. Let $g_w$ denotes the $w$-shift of $g$, namely the coloring $g_w(\sigma) := g(\sigma \cdot w)$. Note that $g$ induces a coloring $\hat{g} : A^{<\omega} \to A^t \to \ell$ defined by

$$\hat{g} : A^{<\omega} \ni \sigma \mapsto g_{W[\sigma]} \restriction A^t.$$

By preservation of $\omega$ hyperimmunities of $\mathsf{OVW}^0$ (Theorem 4.5), let $U$ be an $\mathsf{OVW}^0$ solution to $\hat{g}$ such that $h$ is $g \oplus U \oplus W$-hyperimmune; and suppose

$$\mathsf{WSub}^*(U) \text{ is monochromatic for } \hat{g} \text{ for color } g^*. \tag{28}$$

Note that $g^*$ is an $\ell$-coloring on $A^t$. By $A^{<\omega} \to \ell$-matching of $I$, there exists $v^* \in I$ such that

$$\mathsf{WSub}^{=}(v^*) \text{ is monochromatic for } g^*, \text{ say for color } \ell^*. \tag{29}$$

Unfolding (28),

$$\text{for every } u \in \mathsf{WSub}^*(U),\ g_{W[u]} \restriction A^t = g^*\ . \tag{30}$$

Combine with (29), for every $u \in \mathsf{WSub}^*(U)$ and $w \in \mathsf{WSub}^{=}(v^*)$, we have:

$$\begin{aligned} g(w \cdot W[u]) &= g_{W[u]}(w) \\ &\overset{(30)}{=} g^*(w) \overset{(29)}{=} \ell^*. \end{aligned} \tag{31}$$

In particular (and combining with $v^*$ level-homogeneous for $f$ and therefore level-homogeneous for $g$ since $Q_n$ requires $g \restriction A^{\leq t} = f \restriction A^{\leq t}$), for every ordered variable word $\hat{v} \in [v^*, W[U]]$,

$$\hat{v} \text{ is level-homogeneous for } g.$$

By definition (24) of $Q_n$ and the fact $g \in Q_n$, and since $[v^*, W[U]] \subseteq [v^*, W])$, we have:

$$\text{for every ordered variable word } \hat{v} \in [v^*, W[U]],\ \Psi^{\hat{v}}_{v^*}(n) \uparrow\ . \tag{32}$$

Clearly this means $(v^*, W[U])$ satisfies $\mathcal{R}^h_{\Psi_{v^*}}$, a contradiction to the otherwise hypothesis: non-progression of $(v^*, W)$ for $h$.

In either case, we derived a contradiction. This completes the proof of Theorem 6.12. □

Since the set of initial segments which are non-progressable for $h$ are not $A^{<\omega} \to \ell$-matching, there is a witnessing coloring by unfolding the definition of $A^{<\omega} \to \ell$-matching:

**Lemma 6.13** (Lemma 6.12 rephrased)**.** *Let $W$ be an ordered $\omega$-variable word such that each $h \in \vec{h}$ is $W$-hyperimmune and fix some $h \in \vec{h}$. For every $t \in \mathbb{N}$, there is a coloring $g : A^t \to \ell$ such that for every $\mathbb{Q}$-condition $(v, W)$ with $|v| = t$ and with $\mathsf{WSub}^{=}(v)$ monochromatic for $g$, $(v, W)$ is progressable for $h$.*

Rephrasing Lemma 6.13: Let $W$ be an ordered $\omega$-variable word $W$ such that each $h \in \vec{h}$ is $W$-hyperimmune. For every $h \in \vec{h}$, we have

$$WitCol_h(W) \neq \emptyset. \tag{33}$$

## 6.3 Maximizing non-progressable initial segments

In Remark 6.10, we assumed the existence of a $W$-witness for progress in a very strong sense: a simultaneous witness for every $U \in \mathsf{OSub}^{\omega}(W)$ such that $h$ is $U$-hyperimmune. It is not clear that such witnesses exist. If we simply take $g \in WitCol(W)$, then even if $v$ stays in the monochromatic-for-$g$ world, we only have that the $\mathbb{Q}$-condition $(v, W^* \restriction^{\omega}_{|v|})$ is progressable. But its extension $(v, U)$ may not be progressable. To settle this, we make sure by shrinking the reservoir that the set of initial segment $v$, with $|v| = t$ and $(v, W^* \restriction^{\omega}_{|v|})$ being a non-progressable $\mathbb{Q}$-condition, is somewhat stabilized at infinitely many levels $t$, in the sense that this set can no longer be changed via shrinking the reservoir $W^* \restriction^{\omega}_{|v|}$.

**Definition 6.14** (Maximizing the set of non-progressable initial segments)**.** Given an ordered $\omega$-variable word $W$ such that each $h \in \vec{h}$ is $W$-hyperimmune, we say $W$ *maximizes the set of non-progressable initial segments at level $t$* iff for every $h \in \vec{h}$, and every $\mathbb{Q}$-condition of the form $(v, W \restriction^{\omega}_{t})$ with $|v| = t$, if some extension $(v, U)$ of $(v, W \restriction^{\omega}_{t})$ is non-progressable for $h$, then $(v, W \restriction^{\omega}_{t})$ is already non-progressable for $h$.

In particular, if an ordered $\omega$-variable word $W$ maximizes the set of non-progressable initial segments at level $t$, then for any $h \in \vec{h}$ if $(v, W)$ is a $\mathbb{Q}$-condition progressable for $h$ with $|v| = t$, then for any extension $(v, U)$ of $(v, W)$, $(v, U)$ is also progressable for $h$.

Next we note that there does exist an ordered $\omega$-variable word that maximizes the set of non-progressable initial segments at infinitely many levels.

**Proposition 6.15.** *There exists an ordered $\omega$-variable word $W$ with each $h \in \vec{h}$ being $W$-hyperimmune and an infinite set $T \subseteq \omega$ with each $t \in T$ in between variables of $W$ such that $W$ maximizes the set of non-progressable initial segments at every level $t \in T$.*

*Proof.* This is done by a finite extension argument. In this proof, we will use a variant of $\mathbb{Q}$-forcing, where the stems are not required to be level-homogeneous for $f$. In other words, a condition is a pair $(v, U)$ such that

(1) $v$ is a finite ordered variable word and $U$ is an ordered $\omega$-variable word;

(2) each $h \in \vec{h}$ is $U$-hyperimmune.

To make sure each $h \in \vec{h}$ is $W$-hyperimmune, given a condition $(v, U)$ and a requirement $\mathcal{R}^h_\Psi$, we simply find an arbitrary extension $(\hat{v}, \hat{U})$ satisfying $\mathcal{R}^h_\Psi$ Unlike $\mathbb{Q}$-forcing, this is now easy since the initial segment need not be level-homogeneous for a given coloring. One can similarly ensure that $W$ contains infinitely many variable kinds.

Also note that for any condition $(v, U)$ there is an extension $(v, \hat{U})$ so that for each $h \in \vec{h}$, we can no longer increase the set of ordered variable word of length $|v|$ that are, together with a reservoir in $\mathsf{OSub}^\omega(\hat{U})$, non-progressable for $h$ via shrinking $\hat{U}$. In other words, $v \cdot \hat{U}$ maximizes the set of non-progressable initial segments at level $|v|$. This implies that for every ordered $\omega$-variable word $\tilde{U} \in [v, \hat{U}]$, $\tilde{U}$ maximizes the set of non-progressable initial segments at level $|v|$ as well. We can therefore build a sequence of conditions

$$(v_0, U_0) \geq (v_1, U_1) \geq \cdots$$

such that $W := \cup_n v_n$ maximizes the set of non-progressable initial segments at infinitely many levels and each $h \in \vec{h}$ is $W$-hyperimmune. This completes the proof of Proposition 6.15. □

We are now ready to prove Theorem 6.1.

## 6.4 Proof of Theorem 6.1

Let $f : A^{<\omega} \to \ell$ be an arbitrary coloring, for some finite alphabet $A$ and some $\ell \geq 1$. Let $\vec{h} = (h_i)_{i<k}$ be hyperimmune functions for some $k \in \mathbb{N}$. By Proposition 6.15, there is an $\omega$-variable word $W$ (with variable set $(x_n)_{n\in\mathbb{N}}$) such that each $h \in \vec{h}$ is $W$-hyperimmune and

$$\begin{aligned} &W \text{ maximizes the set of non-progressable initial segments} \\ &\text{at levels in an infinite set } T \subseteq \mathbb{N} \text{ where each } t \in T \text{ is} \\ &\text{in between variables of } W. \end{aligned} \tag{34}$$

By (33), for each $i < k$, there is some $g_i \in WitCol_{h_i}(W)$. Let $g : A^{<\omega} \to \ell^k$ be the product of $g_i$. To stay in the monochromatic-for-$g$ world, we use strong preservation of $\omega$-hyperimmunity for $\exists^\infty$-$\mathsf{IHJ}$.

**Claim 1.** *There exists an ordered $\omega$-variable word $W^* \in \mathsf{OSub}^=(W)$ with each $h \in \vec{h}$ being $W^*$-hyperimmune such that for infinitely many $t \in T$, $\mathsf{WSub}^=(W^* \restriction t)$ is monochromatic for $g$.*

*Proof.* Let $t_\sigma$ be the largest number in $T$ such that $t_\sigma \leq |W[\sigma]|$, if it exists. Otherwise, $t_\sigma = 0$. Let $\hat{g} : A^{<\omega} \to \ell^k$ be defined by

$$\hat{g}(\sigma) := g(W[\sigma] \restriction t_\sigma).$$

By strong preservation of $\omega$ hyperimmunity for $\exists^\infty$-$\mathsf{IHJ}$ (see Theorem 5.4), there exists an ordered $\omega$-variable word $U$ with each $h \in \vec{h}$ being $U \oplus W$-hyperimmune such that for infinitely many $t \in \mathbb{N}$,

$$\mathsf{WSub}^=(U \upharpoonright t) \text{ is monochromatic for } \hat{g}$$

Let $W^\star = W[U]$. Clearly $W^* \leq_T W \oplus U$ so $h$ is $W^*$-hyperimmune. It remains to find arbitrary large $t \in T$ such that

$$\mathsf{WSub}^=(W^* \upharpoonright t) \text{ is monochromatic for } g. \tag{35}$$

Let $\hat{t} \in \mathbb{N}$ be such that there exists $t \in T$ large enough such that

$$t \leq \left|W[A^{\hat{t}}]\right| \text{ and } \mathsf{WSub}^=(U \upharpoonright \hat{t}) \text{ is monochromatic for } \hat{g} \tag{36}$$

Fix such $\hat{t}$ and assume $t$ is the largest number in $T$ satisfying $t \leq |W[A^{\hat{t}}]|$. By definition of $\hat{g}$,

$$\hat{g} : A^{\hat{t}} \ni \sigma \mapsto g(W[\sigma] \upharpoonright t).$$

Combine with (36) we have

$$\mathsf{WSub}^=(W[U \upharpoonright \hat{t}]) \upharpoonright t \text{ is monochromatic for } g.$$

Since $W^* = W[U]$, one can equivalently say

$$\mathsf{WSub}^=(W^* \upharpoonright t) \text{ is monochromatic for } g.$$

This verifies (35). Thus, the proof of Claim 1 is completed. □

As long as the initial segment stay in the somewhat monochromatic-for-$g$ world, the $\mathbb{Q}$-condition is always progressable for all $h \in \vec{h}$:

**Claim 2** (Key Claim)**.** *For every $\mathbb{Q}$-condition $(v, U) \leq (\varepsilon, W^*)$, every $h \in \vec{h}$, $(v, U)$ is progressable for $h$.*

*Proof.* Without loss of generality, suppose $h$ is $h_0$. By definition of a $\mathbb{Q}$-extension, $v \cdot U \in \mathsf{OSub}^=(W^*)$. By Claim 1, there exists $t \in T$ with $t \geq |v|$ such that

$$\mathsf{WSub}^=((v \cdot U) \upharpoonright t) \text{ is monochromatic for } g.$$

Note that we do not require $t$ to be in between variables of $U$. Since $g$ is the product of $(g_i : i < k)$, we have

$$\mathsf{WSub}^=((v \cdot U) \upharpoonright t) \text{ is monochromatic for } g_0.$$

In particular, choose $\sigma \in A^{<\omega}$ long enough so that $|v \cdot U[\sigma]| \geq t$, we have

$$\mathsf{WSub}^=((v \cdot U[\sigma]) \upharpoonright t) \text{ is monochromatic for } g_0.$$

Combining with $g_0 \in WitCol_{h_0}(W)$, and since $(v, U)$ is $\mathbb{Q}$-condition, we have

$$\big((v \cdot U[\sigma]) \restriction t, W \restriction^\omega_t\big) \text{ is a } \mathbb{Q}\text{-condition progressable for } h_0. \tag{37}$$

Say that the variable set of $v$ is $\{x_i : i < r\}$. Let

$$U_\sigma := U[\sigma \cdot x_r x_{r+1} \cdots].$$

In particular, $U_\sigma$ is an ordered $\omega$-variable word over $A$ with variable set $\{x_i : i \geq r\}$ so $v \cdot U_\sigma$ is an ordered $\omega$-variable word over $A$ with variable set $\{x_i : i \in \mathbb{N}\}$. Note that $(v \cdot U_\sigma) \restriction t = (v \cdot U[\sigma]) \restriction t$.

**Subclaim 2.1.** $\big((v \cdot U_\sigma) \restriction t, U_\sigma \restriction^\omega_{t-|v|}\big) \leq \big((v \cdot U_\sigma) \restriction t, W \restriction^\omega_t\big)$.

*Proof.* By Definition 6.6 of $\mathbb{Q}$-condition extension, it suffices to show that

$$U_\sigma \restriction^\omega_{t-|v|} \in \mathsf{OSub}^=(W \restriction^\omega_t). \tag{38}$$

Because

$$v \cdot U \in \mathsf{OSub}^=(W^*) \subseteq \mathsf{OSub}^=(W).$$

Which means

$$v \cdot U_\sigma \in \mathsf{OSub}^=(W).$$

Which means

$$\begin{aligned} U_\sigma \restriction^\omega_{t-|v|} &= (v \cdot U_\sigma) \restriction^\omega_t \\ &\in \mathsf{OSub}^=(W) \restriction^\omega_t \\ &= \mathsf{OSub}^=(W \restriction^\omega_t). \end{aligned}$$

Thus, verifying (38) and we are done.

□

But $W$ maximizes the set of non-progressable initial segments at level $t$ by (34), therefore, combining with Subclaim 2.1 and (37), we have

$$\big((v \cdot U_\sigma) \restriction t, U_\sigma \restriction^\omega_{t-|v|}\big) \text{ is a } \mathbb{Q}\text{-condition progressable for } h_0.$$

But

$$\big((v \cdot U_\sigma) \restriction t, U_\sigma \restriction^\omega_{t-|v|}\big) \leq (v, U),$$

thus, $(v, U)$ is progressable for $h_0$. Thus, we are done proving Claim 2.

□

By Claim 2, we can build a sequence of $\mathbb{Q}$-conditions

$$(\varepsilon, W^*) \geq (v_0, W_0) \geq (v_1, W_1) \geq \cdots$$

so that every requirement is satisfied by some $\mathbb{Q}$-condition in the sequence. Therefore, $G := \cup_n v_n$ is an ordered variable word satisfying all requirements, meaning each $h \in \vec{h}$ is not dominated by $\Psi^G$ for all Turing functionals $\Psi$. In

other words, each $h \in \vec{h}$ is $G$-hyperimmune. Clearly $G$ is level-homogeneous for $f$ since it is the case for $v_n$ for each $n \in \mathbb{N}$.

To see $G$ contains infinitely many variable kinds, note that for every $n \in \mathbb{N}$, there is such Turing functional $\Psi$ so that $\Psi^G(m) \downarrow$ (for all $m$) iff $G$ contains more than $n$ variable kinds. Clearly, for any $\mathbb{Q}$-condition $(v, W)$, it cannot be the case $\Psi^G$ is non total for all $G \in [v, W]$. Thus, by definition of $(v, W)$ satisfying $\mathcal{R}^h_\Psi$, suppose $(v_N, W_N)$ satisfies $\mathcal{R}^h_\Psi$, then $\Psi^{v_N}(m) \downarrow$. Which means $v_N$ contains more than $n$ many variable kinds. Thus $G$ is an $\mathsf{LOVW}^0$-solution to $f$ such that each $h \in \vec{h}$ is $G$-hyperimmune. This completes the proof of Theorem 6.1.

# 7 Carlson-Simpson Lemma in $\mathsf{ACA}_0$

Carlson and Simpson [6] used the Ordered Variable Word theorem as a pigeonhole principle to prove inductively what is now known as the Carlson-Simpson Lemma. As for Ramsey's theorem, the proof consists in reducing the dimension $n+1$ to the dimension $n$ through the notion of *pre-homogeneous* space, which is a sub-space over which the coloring does not depend on the last dimension. In the case of the Carlson-Simpson Lemma, one can obtain a pre-homogeneous unordered $\omega$-variable word by sequentially applying the Ordered Variable Word. The previous best known upper bound for $\mathsf{OVW}^0$ being any PA over $\emptyset'$, a pre-homogeneous $\omega$-variable word was computed using $\emptyset^{(\omega)}$ and the proof was formalizable over $\mathsf{ACA}^+_0$. In our case, thanks to the $\text{low}_2$ basis theorem for $\mathsf{OVW}^0$, pre-homogeneous $\omega$-variable words can be obtained over $\mathsf{ACA}_0$.

The following proof is in the spirit of the original one [6] and uses the same definitions as [1, Section 5]. Only the meta-analysis based on Theorem 3.5 differs.

## 7.1 Notation

Given two $\omega$-variable words $W$ and $\hat{W}$, and $m \in \mathbb{N}$, we write $\hat{W} \leq_m W$ iff $\hat{W} = W[V]$ for some $\omega$-variable word $V$ such that $x_0 \cdots x_{m-1} \prec V$.

Given $\beta \leq \alpha \leq \omega$ and some $\alpha$-variable word $w$, we write $w \Vdash_\beta$ for the $\beta$-variable word obtained by cutting $w$ at the first occurrence of the variable kind $x_\beta$ (or the $\beta$th variable kind if the variable set is different).

Recall that given $\beta \leq \alpha \leq \omega$, we write $A^{\alpha,\beta}$ and $A^{\leq\alpha,\beta}$ for the sets of all unordered $\beta$-variable word of length $\alpha$, or at most $\alpha$, respectively. One defines the unordered counterpart $\mathsf{USub}$ of $\mathsf{OSub}$ in a straightforward way. For instance, given $\gamma \leq \beta \leq \omega$ and an unordered $\beta$-variable word $w$, we let $\mathsf{USub}^\gamma_A(w)$ be the set of all unordered $\gamma$-variable words generated by $w$, that is,

$$\mathsf{USub}^\gamma_A(w) = \{w[u] : u \in A^{\leq\beta,\gamma}\}$$

The notations $\mathsf{USub}_A(w)$, $\mathsf{USub}^=_A(w)$ and $\mathsf{USub}^\star_A(w)$ are defined accordingly.

## 7.2 Inductive proof of CSL

**Definition 7.1.** Fix a finite alphabet $A$ and some $\ell \geq 1$. An unordered $\omega$-variable word $W$ over $A$ is *pre-homogeneous* for a coloring $f : A^{<\omega,n+1} \to \ell$ if for every $u \in \mathsf{USub}_A^{n,\star}(W)$, there is a color $c < \ell$ such that for every $v \in \mathsf{USub}_A^{n+1,\star}(W)$, if $v \Vdash_n = u$, then $f(v) = c$.

Every pre-homogeneous set $W$ induces a coloring $\hat{f} : \mathsf{USub}_A^{n,\star}(W) \to \ell$ defined by $\hat{f}(u) = c$ for the witness color $c$. Note that for any unordered $\omega$-variable word $U \in \mathsf{USub}_A^{\omega}(W)$ such that $\mathsf{USub}_A^{n,\star}(U)$ is $\hat{f}$-monochromatic, $\mathsf{USub}_A^{n+1,\star}(U)$ is $f$-monochromatic.

The novelty will be to prove the existence of pre-homogeneous unordered $\omega$-variable words over $\mathsf{ACA}_0$. Such $\omega$-variable words can be obtained by iterating the following lemma, which corresponds to [1, Lemma 5.3], except that the $\omega$-variable word is provably of $\mathrm{low}_2$ degree.

**Lemma 7.2** ($\mathsf{ACA}_0$)**.** *Fix a set $Z$, a finite alphabet $A$ and $\ell \geq 1$. Let $W$ be a $Z$-computable $\omega$-variable word, $f : A^{<\omega,n+1} \to \ell$ be a $Z$-computable coloring and $u \in A^{<\omega,n}$. There exists an $\omega$-variable word $\hat{W} \leq_{|u|+1} W$ and a color $c < \ell$ such that*

*(1) for every $v \in A^{<\omega,n+1}$ with $v \Vdash_n = u$, we have $f(\hat{W}[v]) = c$.*

*(2) $\hat{W}$ is of low$_2$ degree over $Z$, that is, $(\hat{W} \oplus Z)'' \leq_T Z''$.*

*Proof.* Let $A_{n+1} = A \sqcup \{x_0, \dots, x_n\}$. Let $g : A_{n+1}^{<\omega} \to \ell$ be defined as follows: for every $w \in A_{n+1}^{<\omega}$, $g(w) = f(W[u \cdot x_n \cdot w])$. Note that the variables $x_0, \dots, x_n$ are part of the alphabet $A_{n+1}$, and therefore considered as constants from the viewpoint of $g$. In particular, $g$ is an instance of $\mathsf{OVW}^0$ for $\ell$-coloring of the alphabet $A_{n+1}$.

By Theorem 3.5, there is an (ordered) $\omega$-variable word $U$ over $A_{n+1}$ such that the set $\mathsf{WSub}^{\star}_{A_{n+1}}(U) = \{U[w] : w \in A_{n+1}^{<\omega}\}$ is $g$-monochromatic for some color $c < \ell$, and $(U \oplus Z)'' \leq_T Z''$. For simplicity, we can assume that the variable set of $U$ is $\{x_{|u|+1}, x_{|u|+2}, \dots\}$. Then, $x_0 \cdots x_{|s|} \cdot U$ forms an unordered $\omega$-variable over $A$. Indeed, $U$ may contain infinitely many occurrences of $x_s$ for some $s \leq n$.

Let $\hat{W} = W[x_0 \cdots x_n \cdot U]$. By construction, $\hat{W} \leq_{|u|+1} W$. Moreover, $\hat{W} \leq_T U$, so $(\hat{W} \oplus Z)'' \leq_T Z''$. We claim that (1) holds. Fix some $v \in A^{<\omega,n+1}$ with $v \Vdash_n = u$. In particular, $u \cdot x_n \preceq v$. Let $w \in A_{n+1}^{<\omega}$ be such that $v = u \cdot x_n \cdot w$. Then $f(\hat{W}[v]) = f(\hat{W}[u \cdot x_n \cdot w]) = f(W[u \cdot x_n \cdot U[w]]) = g(U[w]) = c$. This completes the proof of Lemma 7.2. □

Iterating the previous lemma, we obtain the existence of a pre-homogeneous unordered $\omega$-variable word over $\mathsf{ACA}_0$.

**Lemma 7.3** ($\mathsf{ACA}_0$)**.** *Fix a finite alphabet $A$ and $\ell \geq 1$. For every coloring $f : A^{<\omega,n+1} \to \ell$, there is an unordered $\omega$-variable word $W$ pre-homogeneous for $f$.*

*Proof.* Let $u_0, u_1, \dots$ be an enumeration of $A^{<\omega,n}$ in shortlex order. In particular, $|u_n| \leq |u_{n+1}|$. Apply successively Lemma 7.2 to obtain an sequence of unordered $\omega$-variable words $W_0, W_1, \dots$ with $W_0 = x_0 x_1 \cdots$, such that for every $n \in \mathbb{N}$,

(1) $W_{n+1} \leq_{|u_n|+1} W_n$;

(2) there is a color $c_n < \ell$ such that for every $v \in A^{<\omega,n+1}$ with $v \Vdash_n = u_n$, we have $f(W_{n+1}[t]) = c_n$;

(3) $W_n$ is $\text{low}_2$ over $f$, that is, $(W_n \oplus f)'' \leq_T f$.

By (3), the sequence $W_0, W_1, \dots$ is arithmetic in $f$, so exists by the arithmetic comprehension axiom. By (1) and the fact that the enumeration $(u_n)_{n\in\mathbb{N}}$ is of non-decreasing length order, for every $n \in \mathbb{N}$ and every $m \geq n$, $W_m \leq_{|u_n|+1} W_n$. It follows in particular that: $\lim_n W_n$ exists, call it $W$. In particular, for every $n \in \mathbb{N}$, $W \leq_{|u_n|+1} W_n$. By (2) and the previous fact, $W$ is pre-homogeneous for $f$. □

We are now ready to prove over $\mathsf{ACA}_0$ that the base case and the induction step hold for the statement $\forall n \mathsf{CSL}^n$. However, note that $\mathsf{ACA}_0$ does not prove $\Pi^1_2$-induction, so we cannot deduce that $\forall n \mathsf{CSL}^n$ holds over $\mathsf{ACA}_0$. We shall see in Theorem 7.7 that the statement $\forall n \mathsf{CSL}^n$ is equivalent to $\mathsf{ACA}'_0$ over $\mathsf{RCA}_0$, where $\mathsf{ACA}'_0$ is $\mathsf{RCA}_0$ augmented with the statement $\forall n \forall X \exists Y (Y = X^{(n)})$.

**Proposition 7.4.** $\mathsf{ACA}_0 \vdash \mathsf{CSL}^0 \wedge \forall n (\mathsf{CSL}^n \to \mathsf{CSL}^{n+1})$.

*Proof.* For the base case, Anglès d'Auriac, Mignoty, Liu and Patey [1] proved that $\mathsf{ACA}_0 \vdash \mathsf{OVW}^0$ (see also Section 2 for a simple proof). Since $\mathsf{CSL}^0$ is a weakening of $\mathsf{OVW}^0$, $\mathsf{ACA}_0 \vdash \mathsf{CSL}^0$. Suppose now that $\mathsf{CSL}^n$ holds for some $n \in \mathbb{N}$. Let $f : A^{<\omega,n+1} \to \ell$ be a coloring. By Lemma 7.3, there is an unordered $\omega$-variable word $W$ which is pre-homogeneous for $f$. Let $\hat{f} : \mathsf{USub}^{n,\star}_A(W) \to \ell$ be the coloring induced by $W$ (see after Theorem 7.1), and let $g : A^{<\omega,n} \to \ell$ be defined by $g(u) = \hat{f}(W[u])$. By $\mathsf{CSL}^n$, there is an unordered $\omega$-variable word $U$ such that $\mathsf{USub}^{n,\star}_A(U)$ is $g$-monochromatic. In particular, $\mathsf{USub}^{n,\star}_A(W[U])$ is $\hat{f}$-monochromatic, so $\mathsf{USub}^{n+1,\star}_A(W[U])$ is $f$-monochromatic. So $\mathsf{CSL}^{n+1}$ holds. □

## 7.3 Open and Combinatorial Dual Ramsey theorem

Given $n, k \in \mathbb{N}$, we write $\mathsf{CSL}^n(k)$ for the restriction of $\mathsf{CSL}^n$ to alphabets of size $k$. Dzhafarov, Flood, Solomon and Westrick [16] introduced a the *Combinatorial Dual Ramsey theorem* ($\mathsf{CDRT}^n$) representing the combinatorial core of the Open Dual Ramsey theorem ($\mathsf{ODRT}^n$). Formulated in terms of unordered variable words theorems, $\mathsf{CDRT}^n$ corresponds to the statement $\mathsf{CSL}^{n-1}(0)$. They proved that $\mathsf{RCA}_0 \vdash \forall n (\mathsf{ODRT}^n \leftrightarrow \mathsf{CDRT}^n)$. Anglès d'Auriac, Mignoty, Liu and Patey [1] proved that the statements $\forall n \mathsf{CDRT}^n$ and $\forall n \mathsf{CSL}^n$ are equivalent over $\mathsf{RCA}_0$. We give a slight improvement:

**Lemma 7.5** ($\mathsf{RCA}_0$)**.** *For every $n \in \mathbb{N}$, $\mathsf{CSL}^n(1)$ implies both $\mathsf{CDRT}^{n+2}$ and $\mathsf{ODRT}^{n+2}$ over $\mathsf{RCA}_0$.*

*Proof.* Fix $n \in \mathbb{N}$. By Dzhafarov, Flood, Solomon and Westrick [16, Theorem 3.9], $\mathsf{CDRT}^{n+2}$ and $\mathsf{ODRT}^{n+2}$ are equivalent over $\mathsf{RCA}_0$. Furthermore, $\mathsf{CDRT}^{n+2}$ is $\mathsf{CSL}^{n+1}(0)$, so it suffices to prove $\mathsf{RCA}_0 \vdash \mathsf{CSL}^n(1) \to \mathsf{CSL}^{n+1}(0)$.

Let $f : \emptyset^{<\omega,n+1} \to \ell$ be an instance of $\mathsf{CSL}^{n+1}(0)$. It induces an instance $g : \{0\}^{<\omega,n} \to \ell$ of $\mathsf{CSL}^n(1)$ defined by $g(u) = f(x_0 \cdot u[x_i \mapsto x_{i+1}, 0 \mapsto x_0])$, where $u[x_i \mapsto x_{i+1}, 0 \mapsto x_0]$ means that all the occurrences of $x_i$ are replaced with $x_{i+1}$ and all the occurrences of the unique alphabet symbol 0 are replaced with $x_0$. By $\mathsf{CSL}^n$, there is an unordered $\omega$-variable word $U$ over $\{0\}$ such that $\mathsf{USub}^{n,\star}_{\{0\}}(U)$ is $g$-monochromatic for some color $c < \ell$. Let $W = x_0 \cdot U[x_i \mapsto x_{i+1}, 0 \mapsto x_0]$. Then $W$ is an $\omega$-variable word over $\emptyset$ such that $\mathsf{USub}^{n+1,\star}_{\emptyset}(W)$ is $f$-monochromatic for color $c$. Indeed, given $v \in \mathsf{USub}^{n+1,\star}_{\emptyset}(W)$, since the alphabet is empty, $v = x_0 \cdot v'$ for some $v' \in \{x_0\}^{<\omega,n}$ where the variable set is $\{x_1, x_2, \dots\}$. Then, letting $u = v'[x_0 \mapsto 0, x_{i+1} \mapsto x_i]$, we have $f(v) = g(u) = c$. □

We can now prove our main theorem:

**Main Theorem 1.5.** For every $n \in \omega$, $\mathsf{ACA}_0 \vdash \mathsf{CSL}^n$ and $\mathsf{ACA}_0 \vdash \mathsf{ODRT}^{n+2}$. Moreover, a reversal holds for every $n \geq 2$.

*Proof.* By external induction on $n$, using Proposition 7.4, $\mathsf{ACA}_0 \vdash \mathsf{CSL}^n$ for every $n \in \omega$. By Lemma 7.5, it follows that $\mathsf{ACA}_0 \vdash \mathsf{ODRT}^{n+2}$ for every $n \in \omega$. Conversely, Miller and Solomon [37, Theorem 2.3] proved that $\mathsf{RCA}_0 \vdash \forall n(\mathsf{ODRT}^{n+1} \to \mathsf{RT}^n)$, hence that $\mathsf{RCA}_0 \vdash \mathsf{ODRT}^4 \to \mathsf{ACA}_0$. By Lemma 7.5, $\mathsf{RCA}_0 \vdash \mathsf{CSL}^2 \to \mathsf{ODRT}^4$, so $\mathsf{RCA}_0 \vdash \mathsf{CSL}^2 \to \mathsf{ACA}_0$. □

We conclude this section by proving that the full statement $\forall n \mathsf{CSL}^n$ is equivalent to $\mathsf{ACA}'_0$ over $\mathsf{RCA}_0$. Kołodziejczyk (personal communication) noticed the following consequence of Yokoyama [54]. Note that $\mathsf{ACA}'_0$ is not strong enough to prove $\Pi^1_2$-induction, so we cannot replace $\mathsf{ACA}_0$ with $\mathsf{ACA}'_0$ in the following proposition.

**Proposition 7.6.** *Let $\mathsf{P}(n)$ be a $\Pi^1_2$-formula with free integer variable $n$. If $\mathsf{ACA}_0 \vdash \mathsf{P}(0) \wedge \forall n(\mathsf{P}(n) \to \mathsf{P}(n+1))$, then $\mathsf{ACA}'_0 \vdash \forall n \mathsf{P}(n)$.*

*Proof by Kołodziejczyk.* Since provability over $\mathsf{ACA}_0$ is provable over $\mathsf{ACA}_0$, $\mathsf{ACA}_0 \vdash$ "$\mathsf{ACA}_0 \vdash \mathsf{P}(0) \wedge \forall n(\mathsf{P}(n) \to \mathsf{P}(n+1))$". Therefore, by $\Sigma^0_1$-induction, $\mathsf{ACA}_0 \vdash$ "$\forall n \mathsf{ACA}_0 \vdash \mathsf{P}(n)$". By Yokoyama [54] (see also [39]), $\mathsf{ACA}'_0$ is equivalent to the $\Pi^1_2$-reflection principle for $\mathsf{ACA}_0$, so $\mathsf{ACA}'_0 \vdash \forall n \mathsf{P}(n)$. □

We now have all the necessary tools to prove the following equivalences:

**Theorem 7.7.** *The following are equivalent over $\mathsf{RCA}_0$:*

*(1)* $\mathsf{ACA}'_0$
*(2)* $\forall n \mathsf{CSL}^n$
*(3)* $\forall n \mathsf{CDRT}^n$
*(4)* $\forall n \mathsf{ODRT}^n$
*(5)* $\forall n \mathsf{RT}^n$

*Proof.* (1) $\to$ (2) is Proposition 7.4 combined with Proposition 7.6. (2) $\leftrightarrow$ (3) is Anglès d'Auriac, Mignoty, Liu and Patey [1, Theorem 6.3]. (3) $\leftrightarrow$ (4) is Dzhafarov, Flood, Solomon and Westrick [16, Theorem 3.9]. (4) $\to$ (5) is Miller and Solomon [37, Theorem 2.3]. (1) $\leftrightarrow$ (5) is McAloon [36] (see [26, Theorem 6.27] or [18, Theorem 8.1.6]). □

# 8 The weakness of $\mathsf{CSL}^1$

By Main Theorem 1.5, $\mathsf{CSL}^n$ is equivalent to $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$ for every $n \geq 2$. On the other hand, by Theorem 4.5, $\mathsf{OVW}^0$ (and a fortiori $\mathsf{CSL}^0$) admits preservation of $\omega$ hyperimmunities, and therefore does not even imply $\mathsf{RT}^2$ over $\mathsf{RCA}_0$. The goal of this section is to close the gap, and prove the following theorem:

**Theorem 8.1.** $\mathsf{CSL}^1$ *admits preservation of 1 hyperimmunity.*

Main Theorem 1.6 is an immediate consequence of Theorem 8.1:

**Main Theorem 1.6.** Neither $\mathsf{CSL}^1$, nor $\mathsf{ODRT}^3$ implies $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$.

*Proof.* There exists an arithmetical hyperimmune function, so $\mathsf{ACA}_0$ does not admit preservation of 1 hyperimmunity. By Theorem 8.1 and Lemma 4.4, $\mathsf{CSL}^1$ does not imply $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$. By Lemma 7.5, $\mathsf{RCA}_0 \vdash \mathsf{CSL}^1 \to \mathsf{ODRT}^3$, so $\mathsf{ODRT}^3$ does not imply $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$ either. □

As mentioned, by Downey et al [15], Theorem 8.1 is equivalent to saying that $\mathsf{CSL}^1$ admits cone avoidance. First, note that Theorem 8.1 is optimal, in the following sense:

**Lemma 8.2.** $\mathsf{CSL}^1(1)$ *does not admit preservation of 2 hyperimmunities.*

*Proof.* By Lemma 7.5, $\mathsf{RCA}_0 \vdash \mathsf{CSL}^1(1) \to \mathsf{ODRT}^3$. By Miller and Solomon [37, Theorem 2.3], $\mathsf{RCA}_0 \vdash \mathsf{ODRT}^3 \to \mathsf{RT}^2$, and by Lemma 4.3, $\mathsf{RT}^2_2$ does not admit preservation of 2 hyperimmunities. □

## 8.1 Level Carlson-Simpson lemma

As in Section 6, instead of directly proving that $\mathsf{CSL}^1$ admits preservation of 1 hyperimmunity, we shall actually consider a level version of the Carlson-Simpson lemma and prove that it admits preservation of $k$ hyperimmunities simultaneously. Note that one could have proven Theorem 8.1 without this level version, but it actually gives more information about the relation between the combinatorial features of $\mathsf{CSL}^1$ and its computability-theoretic strength.

**Definition 8.3.** The *projection* $\pi(w)$ of an unordered $n$-variable-word $w$ over an alphabet $A$, is the set $\{t_0 < \cdots t_n\}$ such that for $i < n$, $t_i$ is the position of the first occurrence of the variable $x_i$ in $w$, and $t_n = |w|$. Given a coloring $f : A^{<\omega,n} \to \ell$, an unordered $\omega$-variable word $W$ is *level-homogeneous* for $f$ if for every $u_0, u_1 \in \mathsf{USub}^{n,\star}_A(W)$ such that $\pi(u_0) = \pi(u_1)$, we have $f(u_0) = f(u_1)$.

This notion of level-homogeneity is a natural generalization of Definition 6.2 for unordered variable words. It induces the following Level Carlson-Simpson lemma:

**Definition 8.4.** $\mathsf{LCSL}^n$ is the $\Pi^1_2$-problem whose instances are colorings $f : A^{<\omega,n} \to \ell$ for a finite alphabet $A$ and some $\ell \geq 1$. A $\mathsf{LCSL}^n$*-solution* to $f$ is an unordered $\omega$-variable word $W$ which is level-homogeneous for $f$.

It is useful to think of $\mathsf{LCSL}^n$ as a "Ramsey-free" version of $\mathsf{CSL}^n$, in the sense that $\mathsf{CSL}^n$ is equivalent to $\mathsf{RT}^{n+1} \wedge \mathsf{LCSL}^n$ over $\mathsf{RCA}_0$, and we shall see that $\mathsf{LCSL}^1$ does not imply $\mathsf{RT}^2$, even for 2-colorings. In what follows, given an unordered $\omega$-variable word $W$ over $A$, we write $\pi(W)$ for the set of all $t_i \in \mathbb{N}$ such that $t_i$ is the position of the first occurrence of $x_i$ in $W$.

**Lemma 8.5.** $\mathsf{RCA}_0 \vdash \forall n(\mathsf{CSL}^n \leftrightarrow \mathsf{LCSL}^n \wedge \mathsf{RT}^{n+1})$.

*Proof.* Fix $n$. Suppose first $\mathsf{CSL}^n$ holds. Then $\mathsf{LCSL}^n$ clearly also holds as it is a weakening of $\mathsf{CSL}^n$. Moreover, by Lemma 7.5 $\mathsf{ODRT}^{n+2}$ holds, hence by Miller and Solomon [37, Theorem 2.3], $\mathsf{RT}^{n+1}$ holds. Suppose now that $\mathsf{LCSL}^n$ and $\mathsf{RT}^{n+1}$ both hold. Let $f : A^{<\omega,n} \to \ell$ be a coloring, for some alphabet $A$ and some $\ell \geq 1$. By $\mathsf{LCSL}^n$, there is an infinite $\omega$-variable word $W$ such that for every $u_0, u_1 \in \mathsf{USub}^{n,\star}_A(W)$, if $\pi(u_0) = \pi(u_1)$, then $f(u_0) = f(u_1)$. Let $X = \pi(W)$ and $g : [X]^{n+1} \to \ell$ be defined by $g(F) = f(u)$, for any $u \in \mathsf{USub}^{n,\star}_A(W)$ such that $\pi(u) = F$. Such a coloring is well-defined by choice of $W$. By $\mathsf{RT}^{n+1}$, there is an infinite $g$-homogeneous subset $Y \subseteq X$, for some color $c < \ell$. Let $\hat{W}$ be an unordered $\omega$-variable word in $\mathsf{USub}^{\omega}_A(W)$ such that $\pi(\hat{W}) = Y$. Note that $\hat{W}$ can be obtained from $W$ by replacing any variable $x_n$ for $n \in Y - X$ with any fixed constant in $A$. Then, for any $u \in \mathsf{USub}^{n,\star}_A(\hat{W})$, $\pi(u) \in [Y]^{n+1}$, so $f(u) = g(\pi(u)) = c$. □

We will prove the following theorem:

**Theorem 8.6.** $\mathsf{LCSL}^1$ *admits preservation of* $k$ *hyperimmunities for every* $k \in \mathbb{N}$.

Before proving Theorem 8.6, we assume it holds, and deduce Theorem 8.1:

*Proof of Theorem 8.1.* By Theorem 8.6, $\mathsf{LCSL}^1$ admits preservation of 1 hyperimmunity. By Patey [41, Theorem 23], so does $\mathsf{RT}^2$. It follows that $\mathsf{LCSL}^1 \wedge \mathsf{RT}^2$ admits preservation of 1 hyperimmunity. By Lemma 8.5, $\mathsf{LCSL}^1 \wedge \mathsf{RT}^2$ implies $\mathsf{CSL}^1$ over $\mathsf{RCA}_0$, and a fortiori over $\omega$-models, so by Lemma 4.4, $\mathsf{CSL}^1$ admits preservation of 1 hyperimmunity. □

## 8.2 Cohesive Carlson-Simpson lemma

In order to prove Theorem 8.6, we shall define a *cohesive* version of $\mathsf{CSL}^1$. Recall that given $\beta \leq \alpha \leq \omega$ and some $\alpha$-variable word $w$, we write $w \restriction_\beta$ for the $\beta$-variable word obtained by cutting $w$ at the first occurrence of the variable kind $x_\beta$ (or the $\beta$th variable kind if the variable set is different).

**Definition 8.7.** Fix $A$ and $\ell \geq 1$. Let $W$ be an unordered $\omega$-variable word over $A$. A coloring $f : \mathsf{USub}^{1,\star}_A(W) \to \ell$ is *stable* if for every $u \in \mathsf{WSub}^{\star}_A(W)$, there is a threshold $t \in \mathbb{N}$ and a color $c < \ell$ such that for every $v \in \mathsf{USub}^{1,\star}_A(W)$, if $v \Vdash_0 = u$ and $|v| \geq t$, then $f(v) = c$.

Note that any stable coloring $f : \mathsf{USub}^{1,\star}_A(W) \to \ell$ induces a coloring $\hat{f} : \mathsf{WSub}^{\star}_A(W) \to \ell$ defined by

$$\hat{f}(u) = \lim_{v \in \mathsf{USub}^{1,\star}_A(W), v \Vdash_0 = u} f(v)$$

Such a coloring is called the *limit coloring* of $f$. Let $\mathsf{CCSL}^1$, standing for *Cohesive* $\mathsf{CSL}^1$, be the statement "For every finite alphabet $A$, every $\ell \geq 1$ and every coloring $f : A^{<\omega,1} \to \ell$, there is an unordered $\omega$-variable word over which $f$ is stable".

**Theorem 8.8.** $\mathsf{CCSL}^1$ *admits preservation of* $\omega$ *hyperimmunities.*

Once again, before proving Theorem 8.8, let us suppose it holds, and deduce Theorem 8.6.

*Proof of Theorem 8.6.* As usual, we prove the unrelativized version. Let $(h_i)_{i<k}$ be a family of hyperimmune functions for some $k \in \mathbb{N}$. Let $f : A^{<\omega,1} \to \ell$ be a computable coloring, for some finite alphabet $A$ and some $\ell \geq 1$. Since $\mathsf{CCSL}^1$ admits preservation of $\omega$ hyperimmunities (Theorem 8.8), there is an unordered $\omega$-variable word $W$ on which $f$ is stable, and such that each $h_i$ is $W$-hyperimmune. Let $\hat{f} : \mathsf{WSub}^{\star}_A(W) \to \ell$ be the limit coloring of $f$. Since $\mathsf{LOVW}^0$ admits strong preservation of $k$ hyperimmunities (Theorem 6.4), there is some ordered $\omega$-variable word $U$ such that $W[U]$ is level-homogeneous for $\hat{f}$, and such that each $h_i$ is $U \oplus W$-hyperimmune. By a greedy algorithm, $U \oplus W$ computes an unordered $\omega$-variable word $\hat{W} \in \mathsf{USub}^{\omega}_A(W[U])$ which is level-homogeneous for $f$. □

The remainder of the section is devoted to the proof of Theorem 8.8.

## 8.3 Notion of forcing

In what follows, fix a computable coloring $f : A^{<\omega,1} \to \ell$ for a finite alphabet $A$ and some $\ell \geq 1$. Let $\vec{h} = (h_i)_{i \in \mathbb{N}}$ be a countable collection of hyperimmune functions. For every $n \in \mathbb{N}$, let $A_n = A \sqcup \{x_0, \cdots, x_{n-1}\}$.

**Remark 8.9.** Fix $n \in \mathbb{N}$ and $\alpha \leq \omega$. Given an unordered $n$-variable word $w$ over $A$ with variable set $\{x_i : i < n\}$ and an unordered $\alpha$-variable $u$ over $A_n$ with variable set $\{x_i : n \leq i < \alpha + n\}$, the concatenation $w \cdot u$ forms an unordered $(\alpha + n)$-variable word over $A$ with variable set $\{x_i : i < \alpha + n\}$. In this case, any element of $\mathsf{WSub}^{=}_A(w \cdot u)$ is of the form

$$w[v_0] \cdot u[v_1][x_i \mapsto v_0(i)]$$

for some $v_0 \in A^n$ and $v_1 \in A^{\alpha}_n$. Here, we write $u[x_i \mapsto v_0(i)]$ for the word over $A$ obtained from $u$ by replacing every occurrence of $x_i$ for $i < n$ with $v_0(i)$.

We now define the notion of forcing for $\mathsf{CCSL}^1$:

**Definition 8.10.** An $\mathbb{R}$-*precondition* is a pair $(w, W)$ such that, for some $n \in \mathbb{N}$,

(1) $w$ is an unordered $n$-variable word over $A$ with variable set $\{x_i : i < n\}$;

(2) $W$ is an unordered $\omega$-variable word over $A_n = A \sqcup \{x_i : i < n\}$, with variable set $\{x_i : i \geq n\}$;

(3) each $h \in \vec{h}$ is $W$-hyperimmune.

An $\mathbb{R}$-precondition $(w, W)$ is $f$-*stabilizing* if there is a function $\hat{f} : \mathsf{WSub}^\star_A(w) \to \ell$ such that for every $v \in \mathsf{USub}^{1,\star}_A(w \cdot W)$, if $v \Vdash_0 \in \mathsf{WSub}^\star_A(w)$ and $|v| \geq |w|$, then $f(v) = \hat{f}(v \Vdash_0)$. An $\mathbb{R}$-*condition* is an $f$-stabilizing $\mathbb{R}$-precondition.

Note that $x_i$ for $i < n$ plays the role of a variable in $w$, but of an alphabet symbol in $W$. By Remark 8.9, $w \cdot W$ is seen as an unordered $\omega$-variable word over $A$, in which case all occurrences of $x_i$ in $w \cdot W$ are now considered as variables. Given an $\mathbb{R}$-precondition $(w, W)$, $w$ is called the *stem*, and $W$ is the *reservoir*. The set of $\mathbb{R}$-conditions is non-empty, as witnessed by the pair $(\epsilon, W)$, where $W = x_0 x_1 x_2 \cdots$

**Definition 8.11.** An $\mathbb{R}$-precondition $(w', U)$ *extends* $(w, W)$ (written $(w', U) \leq (w, W)$) if $w \preceq w'$ and $w' \cdot U \in \mathsf{USub}^\omega_A(w \cdot W)$.

It is useful to think of an $\mathbb{R}$-precondition $(w, W)$ as the compact collection of its candidate solutions

$$[w, W] = \{U \in \mathsf{USub}_A(w \cdot W) : w \preceq U\}$$

The following lemma states that, as expected, the class of candidate solutions is refined when considering extensions.

**Lemma 8.12.** *Let $(w, W)$ be an $\mathbb{R}$-precondition. For every $(w', U) \leq (w, W)$, $[w', U] \subseteq [w, W]$.*

*Proof.* Let $v \in [w', U]$. First, $v \in \mathsf{USub}_A(w' \cdot U)$, and $w' \cdot U \in \mathsf{USub}^\omega_A(w \cdot W)$, so $v \in \mathsf{USub}_A(w \cdot W)$. Second, $w' \preceq v$, and $w \preceq w'$, so $w \preceq v$. It follows that $v \in [w, W]$. □

## 8.4 Stability tree

As in Section 3, a condition is defined as a precondition satisfying some universality property ($f$-matching for $\mathbb{P}$-forcing and $f$-stabilizing for $\mathbb{R}$-forcing). Given a condition, the difficulty is to find a precondition extension while maintaining this universality property. This is usually done by considering "sufficiently many" extensions of the stem. In Section 3, the appropriate combinatorial object is the Graham-Rothschild tree which, at every level, contains sufficiently many variable words to maintain the $f$-matching property. We now

define another tree, called the *stability tree*, which will play the same role for the $f$-stabilizing property.

For every $n \in \mathbb{N}$, let $B_n = A \sqcup \{s_0, \cdots, s_{n-1}\}$, where $s_i$ are fresh constant symbols. Let $(t_n)_{n\in\mathbb{N}}$ be inductively defined by

$$t_0 = 0 \; ; \; t_1 = 1 \text{ and for } n \geq 1, \text{ let } t_{n+1} = t_n + HJ(B_{t_n}, \ell^{\operatorname{card} A^{t_n,1}})$$

The *stability tree for $\ell$-colorings of $A^{<\omega,1}$* is the tree $\mathcal{S}_{A,\ell}$ whose level $n$ consists of all unordered $n$-variable words $w$ of length $t_n$, such that for every $s < n$, the first occurrence of the variable $x_s$ appears within the positions $[t_s, t_{s+1})$. This tree is partially ordered by the prefix relation. Note that this tree is computably bounded, and every infinite path is an unordered $\omega$-variable word over $A$. The fundamental property of this stability tree is the following:

**Lemma 8.13** ($\mathsf{RCA}_0$)**.** *Fix $A$ and $\ell \geq 1$. For every $n \in \mathbb{N}$ and every coloring $f : A^{t_n,1} \to \ell$, there is some unordered $n$-variable word $w$ at level $n$ in $\mathcal{S}_{A,\ell}$ and some function $\hat{f} : \mathsf{WSub}^{\star}_A(w) \to \ell$ such that for every $v \in \mathsf{USub}^{1,=}_A(w)$, $f(v) = \hat{f}(v \restriction_0)$.*

*Proof.* By $\Delta^0_0$-induction over $n$. The case $n = 0$ is vacuously true as $\mathsf{USub}^{1,=}_A(\epsilon) = \emptyset$. The case $n = 1$ is trivially true as $\mathsf{USub}^{1,=}_A(x_0) = \{x_0\}$. Suppose it holds for $n \geq 1$, we now show that it holds for $n + 1$. Fix a coloring $f : A^{t_{n+1},1} \to \ell$, and let $b = HJ(B_{t_n}, \ell^{\operatorname{card} A^{t_n,1}})$.

Let $g : B^b_{t_n} \to \ell^{A^{t_n,1}}$ be defined by

$$g(v) := A^{t_n,1} \ni u \mapsto f(u \cdot v[s_i \mapsto u(i)])$$

In other words, the color of $g(v)$ is a function of type $A^{t_n,1} \to \ell$. By choice of $b = HJ(B_{t_n}, \ell^{\operatorname{card} A^{t_n,1}})$ applied to $g$, there is some 1-variable word $w_n \in B^{b,1}_{t_n}$ (with some fresh variable kind, say $\star$, to avoid name clash with $x_0$) and some color $h : A^{t_n,1} \to \ell$ such that for every $a \in B_{t_n}$, $g(w_n[a]) = h$. In other words, for every $u \in A^{t_n,1}$ and every $a \in B_{t_n}$,

$$f(u \cdot (w_n[a])[s_i \mapsto u(i)]) = h(u). \tag{39}$$

By induction hypothesis applied to $h$, there is some unordered $n$-variable word $w$ at level $n$ in $\mathcal{S}_{A,\ell}$ and some coloring $\hat{h} : \mathsf{WSub}^{\star}_A(w) \to \ell$ such that for every $u \in \mathsf{USub}^{1,=}_A(w)$, $h(u) = \hat{h}(u \restriction_0)$. Let $w' = w \cdot w_n[s_i \mapsto w(i)]$. First, note that $w'$ is an unordered $(n+1)$-variable word at level $n+1$ in $\mathcal{S}_{A,\ell}$. Let $\hat{f} : \mathsf{WSub}^{\star}_A(w') \to \ell$ be defined by $\hat{f}(u) = \hat{h}(u)$ if $|u| < t_n$, and $\hat{f}(u) = f(v)$ for the unique $v \in \mathsf{USub}^{1,=}_A(w')$ such that $v \restriction_0 = u$ otherwise.

**Claim 1**: For every $v \in \mathsf{USub}^{1,=}_A(w)$, $f(v) = \hat{f}(v \restriction_0)$. If $|v \restriction_0| \geq t_n$, then $v$ is the unique 1-variable word in $\mathsf{USub}^{1,=}_A(w')$ extending $v \restriction_0$, so by definition of $\hat{f}$, $f(v) = \hat{f}(v \restriction_0)$. Suppose now that $|v \restriction_0| < t_n$. Let $u = v \restriction_{t_n}$, that is, the initial segment of $v$ of length $t_n$ (which is different from $v \restriction_{t_n}$). In particular, $u \in A^{t_n,1}$ and $u \restriction_0 = v \restriction_0$. Say that the unique variable kind of $v$

is $\star$. Since $u \in A^{t_n,1}$, there is at least one occurrence of $\star$ in $u$, say at some position $j < t_n$. Let $v'$ be such that $v = u \cdot v'$. Note that $v'$ is a word over the alphabet $B_{t_n} \sqcup \{\star\}$. Let $v''$ be the word over the alphabet $B_{t_n}$ obtained from $v'$ by replacing every occurrence of $\star$ with the symbol $s_j$ from the alphabet $B_{t_n}$. We then have $v' = v''[s_i \mapsto u(i)]$. In particular, $v'' \in \mathsf{WSub}^{=}_{B_{t_n}}(w_n)$, so there is some $a \in B_{t_n}$ such that $v'' = w_n[a]$. Then

$$\begin{aligned} f(v) &= f(u \cdot v') \\ &= f(u \cdot v''[s_i \mapsto u(i)]) \\ &= f(u \cdot (w_n[a])[s_i \mapsto u(i)]) \\ &\overset{(39)}{=} h(u) = \hat{h}(u \Vdash_0) = \hat{h}(v \Vdash_0) \end{aligned}$$

This completes the proof of Claim 1, and Lemma 8.13. □

## 8.5 First-jump control

We now turn to the definition of the forcing relation for $\Sigma^0_1$ and $\Pi^0_1$-formulas, and of a forcing question with the good definitional properties.

**Definition 8.14.** Let $(w, W)$ be an $\mathbb{R}$-precondition and let $\varphi(G) \equiv \exists x \psi(G \upharpoonright_x)$ be a $\Sigma^0_1$-formula.

(1) $(w, W) \Vdash \varphi(G)$ if $\psi(w)$ holds.

(2) $(w, W) \Vdash \neg\varphi(G)$ if for every $\hat{v} \in [v, W]$, every $n \in \omega$, $\neg\psi(\hat{v} \upharpoonright n)$ holds.

As usual, it is easy to check that if $(w, W) \Vdash \varphi(G)$ for a $\Sigma^0_1$ or $\Pi^0_1$-formula $\varphi(G)$, then for every $U \in [w, W]$, $\varphi(U)$ holds. On can also prove that the forcing relation for $\Sigma^0_1$-formulas and $\Pi^0_1$-formulas is closed under extensions.

**Definition 8.15.** Let $(w, W)$ be an $\mathbb{R}$-precondition with variable kinds of $w$ being $\{x_i : i < n\}$; and let $\varphi(G) \equiv \exists x \psi(G \upharpoonright_x)$ be a $\Sigma^0_1$-formula. We define $(w, W) \mathrel{?\!\vdash} \varphi(G)$ to hold if there is a level $r \in \mathbb{N}$ in the stability tree $\mathcal{S}_{A,\ell^{A^n}}$ for $\ell^{A^n}$-colorings of $A^{<\omega,1}$ such that for every node $v$ at level $r$ in $\mathcal{S}_{A,\ell^{A^n}}$, there is some $v_1 \in \mathsf{USub}^{=}_A(v)$ such that $\psi(w \cdot W[v_1])$ holds.

We now prove that the forcing question satisfies its abstract specifications, that is, if $(w, W) \mathrel{?\!\vdash} \varphi(G)$ for some $\mathbb{R}$-condition $(w, W)$, then there is an extension forcing $\varphi(G)$, and if $(w, W) \mathrel{?\!\nvdash} \varphi(G)$, there is an extension forcing $\neg\varphi(G)$. As in Section 3.3, we start with the negative answer, which is the simplest case.

**Lemma 8.16** ($\Pi^0_1$-extension)**.** *Let $(w, W)$ be an $\mathbb{R}$-condition, and let $\varphi(G)$ be a $\Sigma^0_1$-formula. If $(w, W) \mathrel{?\!\nvdash} \varphi(G)$, then there is an $\mathbb{R}$-condition $(w, U) \leq (w, W)$ such that $(w, U) \Vdash \neg\varphi(G)$.*

*Proof.* Say $\varphi(G) \equiv \exists x \psi(G \upharpoonright_x)$ and that $(w, W)$ is an $\mathbb{R}$-condition with variable kinds of $w$ being $\{x_i : i < n\}$. Suppose $(w, W) \mathrel{?\!\nvdash} \varphi(G)$ holds. Then, for every level $r$ there is some node $v$ at level $r$ in $\mathcal{S}_{A,\ell^{A^n}}$ such that for every $v_1 \in$

$\mathsf{USub}^{=}_A(v)$, $\psi(w \cdot W[v_1])$ does not hold. Let $\mathcal{S}$ be the infinite sub-tree of $\mathcal{S}_{A,\ell^{A^n}}$ of all such nodes. Note that $\mathcal{S} \leq_T f \oplus W$. By the computably dominated basis theorem ([32, Theorem 2.4]), there is an infinite path $U$ through $\mathcal{S}$ such that each $h \in \vec{h}$ is $U \oplus W$-hyperimmune. Then for every $v_1 \in \mathsf{USub}^{\star}_A(U)$, $\psi(w \cdot W[v_1])$ does not hold. It follows that $(w, W[U])$ is a valid $\mathbb{R}$-condition extending $(w, W)$ such that $(w, W[U]) \Vdash \neg\varphi(G)$. □

**Lemma 8.17** ($\Sigma^0_1$-extension)**.** *Let $(w, W)$ be an $\mathbb{R}$-condition, and let $\varphi(G)$ be a $\Sigma^0_1$-formula. If $(w, W) \mathrel{?}\vdash \varphi(G)$, then there is an $\mathbb{R}$-condition $(\hat{w}, \hat{W}) \leq (w, W)$ such that $(\hat{w}, \hat{W}) \Vdash \varphi(G)$.*

*Proof.* Say $\varphi(G) \equiv \exists x \psi(G \upharpoonright_x)$ and that $(w, W)$ is an $\mathbb{R}$-condition with variable kinds of $w$ being $\{x_i : i < n\}$. Note that for every $u' \in A^{<\omega,1}$ with variable kind $\{y\}$, $w \cdot W[u']$ is an unordered variable word (over $A$) with variable kinds $\{x_i : i < n\} \cup \{y\}$. Thus, for any $u \in A^n$, $(w \cdot W[u'])[x_i \mapsto u(i)]$ is a 1-variable word over $A$ with variable kind $y$. Let $\bar{f} : A^{<\omega,1} \to \ell^{A^n}$ be defined for each $u' \in A^{<\omega,1}$ by

$$\bar{f}(u') := A^n \ni u \mapsto f\Big((w \cdot W[u'])[x_i \mapsto u(i)]\Big) \tag{40}$$

Note that that the colors of $f$ are themselves colorings: $f(u')$ is a coloring of type $A^n \to \ell$. We shall therefore write $f(u')(u)$ for $f(u')$ applied to $u \in A^n$. Also note that if $(w, W)$ is the initial $\mathbb{R}$-condition $(\epsilon, x_0 x_1 \cdots)$, then $\bar{f} = f$. One can think of $\bar{f}$ as the instance of $\mathsf{CCSL}^1$ obtained by transforming the $\mathbb{R}$-condition $(w, W)$ into the canonical $\mathbb{R}$-condition $(\epsilon, x_0 x_1 \cdots)$.

Suppose $(w, W) \mathrel{?}\vdash \varphi(G)$ holds. Let $r$ be the level in $\mathcal{S}_{A,\ell^{A^n}}$ witnessing that it holds. By definition, all the variable words at level $r$ in $\mathcal{S}_{A,\ell^{A^n}}$ have length $t_r$. We first stabilize $\bar{f}$ above level $t_r$ with respect to all 1-variable word of length $t_r$ [1] via shrinking $W$ $\Vdash^{\omega}_{t_r}$. To this end, let $B_r = A \sqcup \{s_i : i < t_r\}$ where $s_i$ are fresh constant symbols; let

$$g : B_r^{<\omega} \to (A^{t_r,1} \to \ell^{A^n})$$

be defined by

$$g(\sigma) := A^{t_r,1} \ni v \mapsto \bar{f}(v \cdot \sigma[s_i \mapsto v(i)]) \tag{41}$$

Here, we see $A^{t_r,1} \to \ell^{A^n}$ as the space of $\ell^{A^n}$-colorings of $A^{t_r,1}$, and therefore $g$ maps any $\sigma$ to such a coloring. Note that $A^{t_r,1} \to \ell^{A^n}$ is finite as the domain and the codomain are finite. One can therefore see $g$ as a $W$-computable instance of $\mathsf{CSL}^0$ (and therefore of $\mathsf{OVW}^0$). Since $\mathsf{OVW}^0$ admits preservation of $\omega$ hyperimmunities (Theorem 4.5), there is an (ordered, and a fortiori unordered) $\omega$-variable word $U$ over alphabet $B_r$ such that

$$\mathsf{WSub}^{\star}_{B_r}(U) \text{ is } g\text{-monochromatic for some color } \mu : A^{t_r,1} \to \ell^{A^n} \tag{42}$$

[1] We actually only need to stabilize it with respect to a part of such variable words. But it is notationally convenient to do this for all such variable words.

and such that each $h \in \vec{h}$ is $U \oplus W$-hyperimmune. Here, $\mu$ is a function, but it is seen as a color of $g$.

By the fundamental property of the stability tree (Lemma 8.13) applied to $\mu : A^{t_r,1} \to \ell^{A^n}$, there is some unordered $r$-variable word $\bar{w}$ at level $r$ in $\mathcal{S}_{A,\ell^{A^n}}$ and some function $\hat{\mu} : \mathsf{WSub}^\star_A(\bar{w}) \to \ell^{A^n}$ such that

$$\text{for every } \hat{v} \in \mathsf{USub}^{1,=}_A(\bar{w}),\ \mu(\hat{v}) = \hat{\mu}(\hat{v} \Vdash_0) \tag{43}$$

By definition of the forcing question, there is some $w_1 \in \mathsf{USub}^{=}_A(\bar{w})$ such that $\psi(w \cdot W[w_1])$ holds. Since $w_1$ is generated by $\bar{w}$, $w_1$ also satisfy (43). i.e.,

$$\text{for every } \hat{v} \in \mathsf{USub}^{1,=}_A(w_1),\ \mu(\hat{v}) = \hat{\mu}(\hat{v} \Vdash_0) \tag{44}$$

Let $r_1 \leq r$ be the number of variable kinds of $w_1$. We can assume without loss of generality that the variable set of $w_1$ is $\{x_i : n \leq i < n + r_1\}$ and that the variable set of $U$ is $\{x_i : i \geq n + r_1\}$. Let

$$\hat{W} = (W \Vdash^\omega_{t_r})[U][s_i \mapsto w_1(i)]$$

and $\hat{w} = w \cdot W[w_1]$.

**Claim 1.** *$(\hat{w}, \hat{W})$ is an $\mathbb{R}$-precondition extending $(w, W)$.*

*Proof.* Note that $\hat{W} \leq_T U \oplus W$, hence for every $h \in \vec{h}$, $h$ is $\hat{W}$-hyperimmune. Moreover, $\hat{w} = w \cdot W[w_1]$ is an unordered $(n + r_1)$-variable word over $A$ with variable set $\{x_i : i < n + r_1\}$ and $\hat{W} = (W \Vdash^\omega_{t_r})[U][s_i \mapsto w_1(i)]$ is an unordered $\omega$-variable word over $A_{n+r_1} = A \sqcup \{x_i : i < n + r_1\}$ with variable set $\{x_i : i \geq n + r_1\}$. It follows that $(\hat{w}, \hat{W})$ is an $\mathbb{R}$-precondition.

We now show that $(\hat{w}, \hat{W}) \leq (w, W)$. We have

$$\begin{aligned} \hat{w} \cdot \hat{W} &= w \cdot W[w_1] \cdot (W \Vdash^\omega_{t_r})[U][s_i \mapsto w_1(i)] \\ &= w \cdot W\big[w_1 \cdot U[s_i \mapsto w_1(i)]\big] \end{aligned}$$

It follows that $\hat{w} \cdot \hat{W} \in \mathsf{USub}^\omega_A(w \cdot W)$. This proves Claim 1. □

Let $\hat{f} : \mathsf{WSub}^\star_A(w) \to \ell$ witness $f$-stabilization of $(w, W)$, that is, for every $v \in \mathsf{USub}^{1,\star}_A(w \cdot W)$, if $v \Vdash_0 \in \mathsf{WSub}^\star_A(w)$ and $|v| \geq |w|$, then $f(v) = \hat{f}(v \Vdash_0)$ Let $\hat{f}_1 : \mathsf{WSub}^\star_A(\hat{w}) \to \ell$ be defined for $u \in \mathsf{WSub}^\star_A(\hat{w})$ by

$$\hat{f}_1(u) = \begin{cases} \hat{f}(u) & \text{if } |u| < |w| \\ \hat{\mu}(u_1)(u_0) & \text{if } u_0 \in A^n, u_1 \in A^{<t_r}, u = w[u_0] \cdot (W[x_i \mapsto u_0(i)])[u_1] \end{cases} \tag{45}$$

Note that any $u \in \mathsf{WSub}^\star_A(\hat{w})$ falls into one of the two cases above.

**Claim 2.** *$(\hat{w}, \hat{W})$ is $f$-stabilizing, with witness $\hat{f}_1$.*

*Proof.* Fix some $v \in \mathsf{USub}^{1,\star}_A(\hat{w} \cdot \hat{W})$ such that $v \Vdash_0 \in \mathsf{WSub}^\star_A(\hat{w})$ and $|v| \geq |\hat{w}|$. Suppose first $|v \Vdash_0| < |w|$. Then

$$f(v) = \hat{f}(v \Vdash_0) \overset{(45)}{=} \hat{f}_1(v \Vdash_0)$$

The first equality holds since $\hat{f}$ witnesses $f$-stabilization of $(w, W)$.

Suppose now that $|v \Vdash_0| \geq |w|$. Since $|v| \geq |\hat{w}|$, we can write $v$ of the form

$$v = w[u_0] \cdot W[x_i \mapsto u_0(i)][v_0] \cdot (W \Vdash^\omega_{t_r})\big[\sigma[s_i \mapsto v_0(i)]\big]\big[x_i \mapsto u_0(i)\big]$$

for some $u_0 \in A^n$, $v_0 \in \mathsf{USub}^{1,=}(W[w_1])$ and $\sigma \in \mathsf{WSub}^\star_{B_r}(U)$. We firstly show that $f(v)$ does not depend on $\sigma$. This is due to $\mathsf{WSub}^\star_{B_r}(U)$ being monochromatic for $g$ (see (42)). More precisely, by (42), $g(\sigma) = \mu$, so

$$\begin{aligned}
\mu(v_0)(u_0) &\overset{(42)}{=} g(\sigma)(v_0)(u_0) \\
&\overset{(41)}{=} \bar{f}(v_0 \cdot \sigma[s_i \mapsto v_0(i)])(u_0) \\
&\overset{(40)}{=} f\Big(w[u_0] \cdot W\big[v_0 \cdot \sigma[s_i \mapsto v_0(i)]\big][x_i \mapsto u_0(i)]\Big) \\
&= f(w[u_0] \cdot W[x_i \mapsto u_0(i)][v_0] \cdot (W \Vdash^\omega_{t_r})[\sigma[s_i \mapsto v_0(i)]][x_i \mapsto u_0(i)]) \\
&= f(v)
\end{aligned}$$

i.e., $\mu$ captures $f$ on a prefix of $v$. Moreover, by choice of $w_1$ (and some segment of $v$ is generated by $w_1$), $\hat{\mu}$ (thus $\hat{f}_1$) captures $\mu$ on the word prefix of $v$. More precisely, we have

$$v \Vdash_0 = w[u_0] \cdot W[x_i \mapsto u_0(i)][v_0 \Vdash_0]$$

so

$$\hat{f}_1(v \Vdash_0) \overset{(45)}{=} \hat{\mu}(v_0 \Vdash_0)(u_0) \overset{(44)}{=} \mu(v_0)(u_0)$$

Putting all together, we have $f(v) = \mu(v_0)(u_0) = \hat{f}_1(v \Vdash_0)$. This proves our claim. □

Last, note that, by choice of $w_1$, $\psi(w \cdot W[w_1])$ holds, so $\psi(\hat{w})$ holds, hence $(\hat{w}, \hat{W}) \Vdash \varphi(G)$. This completes the proof of Lemma 8.17. □

## 8.6 Forcing requirements

We need to force two families of requirements. The first one is structural, and ensures that the resulting set will have infinitely many variable kinds. For this, given $t \in \mathbb{N}$, consider the following $\Sigma^0_1$-formula $\varphi_t(G)$:

$$G \text{ contains at least } t \text{ variable kinds.} \tag{46}$$

**Lemma 8.18.** *For every $\mathbb{R}$-condition $(w, W)$ and $t \in \mathbb{N}$, there is an $\mathbb{R}$-condition $(\hat{w}, \hat{W}) \leq (w, W)$ such that $(\hat{w}, \hat{W}) \Vdash \varphi_t(G)$.*

*Proof.* By definition 8.15 of forcing question, we obviously have $(w, W) ?\vdash \varphi_t(G)$. Then by the $\Sigma^0_1$-Extension Lemma 8.17, there is an $\mathbb{R}$-condition $(\hat{w}, \hat{W}) \leq (w, W)$ such that $(\hat{w}, \hat{W}) \Vdash \varphi_t(G)$ and we are done. □

The second family of requirements concern preservation of $\omega$ hyperimmunities. Given a function $h \in \vec{h}$ and some Turing functional $\Psi$, let $\mathcal{R}^h_\Psi$ be the requirement

Either $\Psi^G$ is non-total or $\Psi^G$ does not dominate $h$.

**Lemma 8.19.** *Fix $h \in \vec{h}$ and a Turing functional $\Psi$. For every $\mathbb{R}$-condition $(w, W)$, there is an $\mathbb{R}$-condition $(\hat{w}, \hat{W}) \leq (w, W)$ such that*

*(1) either $(v, \hat{W}) \Vdash \Psi^G(n) \uparrow$ for some $n \in \mathbb{N}$;*

*(2) or $(v, \hat{W}) \Vdash \Psi^G(n) < h(n)$ for some $n \in \mathbb{N}$.*

*Proof.* Suppose first that there is some $n \in \mathbb{N}$ such that $(w, W) ?\nvdash \Psi^G(n) \downarrow$. Then by the $\Pi^0_1$-Extension Lemma 8.16, there is an $\mathbb{R}$-condition $(\hat{w}, \hat{W}) \leq (w, W)$ such that $(\hat{w}, \hat{W}) \Vdash \Phi^G(n) \uparrow$, and we are done.

Suppose now that for every $n \in \mathbb{N}$, $(w, W) ?\vdash \Psi^G(n) \downarrow$. Unfolding the definition of the forcing question, for every $n \in \mathbb{N}$, there exists some $b_n \in \mathbb{N}$ such that $(\hat{w}, \hat{W}) \Vdash \Phi^G(n) \uparrow < b_n$. Moreover, since the forcing question is $\Sigma^0_1(W)$, the sequence $(b_n)_{n\in\mathbb{N}}$ can be chosen to be $W$-computable. Since $h$ is $W$-hyperimmune, there is some $n \in \omega$ such that $b_n < h(n)$. By the $\Sigma^0_1$-Extension Lemma 8.17, ther eis an $\mathbb{R}$-condition $(\hat{w}, \hat{W}) \leq (w, W)$ such that $(\hat{w}, \hat{W}) \Vdash \Psi^G(n) \downarrow < b_n$. Either case, we are done. □

We are now ready to prove Theorem 8.8.

## 8.7 Proof of Theorem 8.8

By Lemma 8.18 and Lemma 8.19, there is an infinite decreasing sequence of $\mathbb{R}$-conditions $(w_0, W_0) \geq (w_1, W_1) \geq \cdots$ such that for every $s \in \mathbb{N}$, $(w_s, W_s) \Vdash \varphi_s(G)$, and for every $h \in \vec{h}$ and every Turing functional $\Psi$, there is some $s \in \mathbb{N}$ such that

(1) either $(w_s, W_s) \Vdash \Psi^G(n) \uparrow$ for some $n \in \mathbb{N}$;

(2) or $(w_s, W_s) \Vdash \Psi^G(n) \downarrow < h(n)$ for some $n \in \mathbb{N}$.

Let $G = \bigcup_s w_s$. Note that $G \in \bigcap_s [w_s, W_s]$. In particular, for every $s \in \mathbb{N}$, $\varphi_s(G)$ holds, so $G$ is an unordered $\omega$-variable word. Moreover, by item (1-2), every $h \in \vec{h}$ is $G$-hyperimmune. Last, by the $f$-stabilizing property of each $\mathbb{R}$-condition, $f$ is stable on $G$. This completes the proof of Theorem 8.8.

# 9 Applications and conclusion

Besides the proof of the Dual Ramsey theorem, the main application of infinite variable word theorems comes from Structural Ramsey Theory, which studies partition theorems of mathematical structures. Given two structures $\mathbf{F}, \mathbf{G}$, we write $\binom{\mathbf{G}}{\mathbf{F}}$ for the set of all embeddings from $\mathbf{F}$ to $\mathbf{G}$. We write $\mathbf{F} \cong \mathbf{G}$ iff $\mathbf{F}$ is isomorphic to $\mathbf{G}$. Given integers $k, \ell \in \omega$, structures $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ we write $\mathbf{C} \longrightarrow (\mathbf{A})^{\mathbf{B}}_{k,\ell}$ for the following statement:

**Definition 9.1.** $\mathbf{C} \longrightarrow (\mathbf{A})^{\mathbf{B}}_{<\omega,\ell}$: For any $k \geq 1$ and any coloring $\chi : \binom{\mathbf{A}}{\mathbf{B}} \to k$, there is an embedding $g \in \binom{\mathbf{A}}{\mathbf{C}}$ such that $\binom{g(\mathbf{C})}{\mathbf{B}}$ uses at most $\ell$ many colors.

For instance, Ramsey's theorem can be reformulated as

$$(\mathbb{N}, <) \to (\mathbb{N}, <)^{\mathbf{n}}_{<\omega,1}$$

for every $n \in \mathbb{N}$, where $\mathbf{n}$ is the ordered set $(\{0, \dots, n-1\}, <)$.

**Definition 9.2.** An infinite structure $\mathbf{A}$ has the *Ramsey property* if for every finite substructure $\mathbf{B}$, the relation $\mathbf{A} \to (\mathbf{A})^{\mathbf{B}}_{<\omega,1}$ holds.

All the infinite structures $\mathbf{A}$ that we shall consider in this section have a unique sub-structure $\mathbf{1}$ of size 1, in which case any coloring of $\binom{\mathbf{A}}{\mathbf{1}}$ corresponds to a coloring of the elements of $\mathbf{A}$. The Ramsey property of $\mathbf{A}$ restricted to $\mathbf{1}$ is called *indivisibility* of $\mathbf{A}$.

**Definition 9.3.** An infinite structure $\mathbf{A}$ is *indivisible* if $\mathbf{A} \to (\mathbf{A})^{\mathbf{1}}_{<\omega,1}$ holds, where $\mathbf{1}$ is its unique sub-structure of size 1.

Many infinite structures do not admit the Ramsey property, but still enjoy a weaker partition theorem:

**Definition 9.4.** Let $\mathbf{A}$ be an infinite structure. A finite sub-structure $\mathbf{B}$ has *big Ramsey degree* $\ell \in \mathbb{N}$ if $\ell$ is the least number such that the relation $\mathbf{A} \to (\mathbf{A})^{\mathbf{B}}_{<\omega,\ell}$ holds, if it exists. We say that $\mathbf{A}$ has *finite big Ramsey degree* if every finite sub-structure admits a big Ramsey degree.

Structural Ramsey theory gives a particular focus on Fraissé limits, such as the dense linear order with no endpoints $(\mathbb{Q}, <)$, the Rado (or random) graph $\mathcal{R}$, and in our case, the universal $k$-clique-free Henson graph $\mathbb{H}_k$. Each of the previous examples are known to admit a finite big Ramsey degree. Such partition theorems are usually proven by coding the mathematical structures into trees, and applying a tree partition theorem. For instance, Devlin [10] and Sauer [45] used Milliken's tree partition theorem to prove that $(\mathbb{Q}, <)$ and $\mathcal{R}$ admit finite Big Ramsey degree, respectively. The exact big Ramsey degrees for the Rado graph were then characterized by Laflamme, Sauer and Vuksanovic [34]

## 9.1 Big Ramsey degree of the triangle-free Henson graph

The big Ramsey degree of the Henson graphs were a longstanding open question. In 1986, Komjáth and Rödl [33] proved that $\mathbb{H}_3$ is indivisible and in 1989, El-Zahar and Sauer [19] generalized it to all Henson graphs. In 1998, Sauer [46] proved that the edges of $\mathbb{H}_3$ have big Ramsey degree 2. In 2017, Dobrinen [11] proved that $\mathbb{H}_3$ admits finite big Ramsey degree using the method of strong coding trees. She then generalized it to all Henson graphs [13] with an elaboration of the technique, and characterized [12, 3] the exact big Ramsey degrees of $\mathbb{H}_3$. Hubička [29] gave an arguably simpler proof of the finite big Ramsey degree of $\mathbb{H}_3$ using the Carlson-Simpson lemma. However, his proof is less precise than the coding tree method, in that it only yields upper bounds to the Ramsey degrees. More recently, in a collective work, Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena and Zucker [2] characterized the big Ramsey degrees of free amalgamation classes in finite binary languages defined by finitely many forbidden irreducible substructures.

From a computability-theoretic and reverse-mathematical viewpoint, Anglès d'Auriac et al. [1] formalized the proof of Hubička [29] and noticed that for every $n \in \mathbb{N}$, $\mathsf{RCA}_0 + \mathsf{CSL}^{2^n+n-1}(1)$ proves that every sub-structure of $\mathbb{H}_3$ of size $n$ admits finite big Ramsey degree, and deduced that over $\mathsf{ACA}_0^+$, $\mathbb{H}_3$ has finite big Ramsey degree for $\mathbb{H}_3$. Gill [22] proved that the indivisibility of $\mathbb{H}_n$ is not computably true for $n \geq 3$. More recently, Cholak, Dobrinen and McCoy [7] proved that the indivisibility of $\mathbb{H}_3$ holds over $\mathsf{ACA}_0$ and that the existence of big Ramsey degrees for sub-structures of $\mathbb{H}_3$ of size 2 implies $\mathsf{ACA}_0$. Finally, Cholak, Dobrinen, and Towsner [8] proved that for every standard $n, m$, the existence of big Ramsey degrees for sub-structures of $\mathbb{H}_m$ of size $n$ holds over $\mathsf{ACA}_0$ using a Milliken-style proof for coding trees. Note that their proof over $\mathsf{ACA}_0$ yields exact bounds for finite big Ramsey degree.

We now combine the new bounds on the Carlson-Simpson lemma with the analysis of Anglès d'Auriac et al. [1] of the proof of Hubička [29] to obtain similar bounds as Cholak, Dobrinen, and Towsner.

**Definition 9.5.** Let $\mathbb{G} = (\{0\}^{<\omega,1}, E)$ be the graph where $E$ is the symmetric binary relation defined as follows: for $v, w \in \{0\}^{<\omega,1}$, $vEw$ if $|v| \neq |w|$ and, assuming $|v| < |w|$, the following holds:

(1) Passing number property: $w(|v|) = x_0$;

(2) Triangle-freeness condition: there is no $i < |v|$ with $v(i) = w(i) = x_0$.

Hubička [29] proved that for every unordered $\omega$-variable word $W$ over $\{0\}$, $(\mathsf{USub}^{1,\star}(W), E)$ is computably isomorphic to the triangle-free Henson graph $\mathbb{H}_3$. In particular, $\mathbb{G} \cong \mathbb{H}_3$. Moreover, the proof holds over $\mathsf{RCA}_0$. Combined with Theorem 8.1, we obtain

**Theorem 9.6.** *The indivisibility of $\mathbb{H}_3$ admits preservation of 1 hyperimmunity.*

Since by Downey et al [15], preservation of 1 hyperimmunity and cone avoidance coincide, it follows that the indivisibility of $\mathbb{H}_3$ admits cone avoidance

and does not imply $\mathsf{ACA}_0$ over $\omega$-models, answering a question of Gill [22] and Cholak, Dobrinen and McCoy [7, Question 48].

Combining Main Theorem 1.5 with Anglès d'Auriac et al. [1], we obtain:

**Theorem 9.7** ([8])**.** *For every $n \in \omega$, $\mathsf{ACA}_0$ proves that every finite triangle-free graph of size $n$ has finite big Ramsey degree. Moreover, $\mathsf{ACA}'_0$ proves that $\mathbb{H}_3$ admits finite big Ramsey degree.*

## 9.2 Summary diagram

We conclude with a summary diagram (Figure 1):

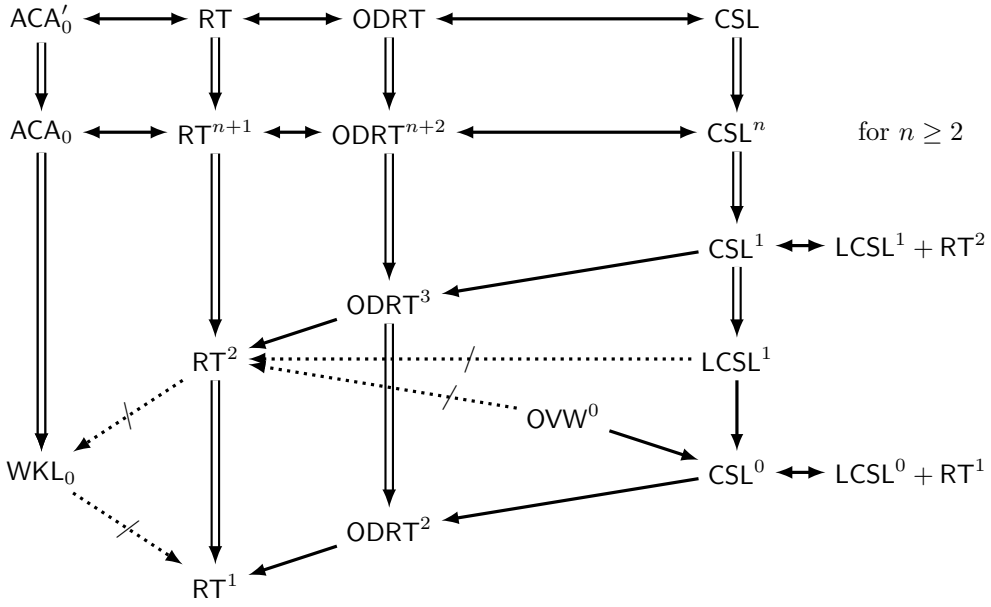



Figure 1: Summary diagram of implications and separations over $\mathsf{RCA}_0$. A double arrow denotes a strict implication, and a dotted arrow is a non-implication. Some arrows which can be deduced by the transitive closure of the implication are missing.

**Acknowledgement**

The authors are thankful to Natasha Dobrinen, Leszek Kołodziejczyk and Keita Yokoyama for insightful comments and discussions.

# References


[1] Paul-Elliot Angles d'Auriac, Lu Liu, Bastien Mignoty, and Ludovic Patey. Carlson-Simpson's lemma and applications in reverse mathematics. *Ann. Pure Appl. Logic*, 174(9):Paper No. 103287, 16, 2023.

[2] Martin Balko, David Chodounskỳ, Natasha Dobrinen, Jan Hubička, Matěj Konečnỳ, Lluis Vena, and Andy Zucker. Exact big ramsey degrees via coding trees. *arXiv preprint arXiv:2110.08409*, page 97, 2021.

[3] Martin Balko, David Chodounskỳ, Natasha Dobrinen, Jan Hubicka, Matej Konecnỳ, Lluís Vena, and Andy Zucker. Exact big ramsey degrees for finitely constrained binary free amalgamation classes. *Journal of the European Mathematical Society*, 53, 2024.

[4] Tristan Bompard, Lu Liu, and Ludovic Patey. The reverse mathematics of Carlson's theorem for located words. *arXiv preprint arXiv:2208.03152*, 2022.

[5] Timothy J. Carlson. Some unifying principles in Ramsey theory. *Discrete Math.*, 68(2-3):117–169, 1988.

[6] Timothy J. Carlson and Stephen G. Simpson. A dual form of Ramsey's theorem. *Adv. in Math.*, 53(3):265–290, 1984.

[7] Peter Cholak, Natasha Dobrinen, and Charlie McCoy. The henson graphs: colorings and codings, 2026.

[8] Peter Cholak, Natasha Dobrinen, and Henry Towsner. The finite big ramsey degrees of henson graphs are provable in $\mathrm{ACA}_0$, 2026.

[9] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1–55, 2001.

[10] Denis Campau Devlin. *SOME PARTITION THEOREMS AND ULTRAFILTERS ON OMEGA*. ProQuest LLC, Ann Arbor, MI, 1980. Thesis (Ph.D.)–Dartmouth College.

[11] Natasha Dobrinen. The Ramsey theory of the universal homogeneous triangle-free graph. *Journal of Mathematical Logic*, 20(2):2050012, 75, 2020. tex.fjournal: Journal of Mathematical Logic tex.mrclass: 03E02 (03C15 03E05 03E40 03E75 05C05 05C55) tex.mrnumber: 4128725 tex.mrreviewer: G.Cherlin.

[12] Natasha Dobrinen. The ramsey theory of the universal homogeneous triangle-free graph part ii: exact big ramsey degrees. *arXiv preprint arXiv:2009.01985*, 2020.

[13] Natasha Dobrinen. The Ramsey theory of Henson graphs. *J. Math. Log.*, 23(1):Paper No. 2250018, 88, 2023.

[14] Pandelis Dodos, Vassilis Kanellopoulos, and Konstantinos Tyros. A density version of the Carlson-Simpson theorem. *J. Eur. Math. Soc. (JEMS)*, 16(10):2097–2164, 2014.

[15] Rod Downey, Noam Greenberg, Matthew Harrison-Trainor, Ludovic Patey, and Dan Turetsky. Relationships between computability-theoretic properties of problems. *J. Symb. Log.*, 87(1):47–71, 2022.

[16] Damir Dzhafarov, Stephen Flood, Reed Solomon, and Linda Westrick. Effectiveness for the dual Ramsey theorem. *Notre Dame J. Form. Log.*, 62(3):455–490, 2021.

[17] Damir D. Dzhafarov and Carl G. Jockusch. Ramsey's theorem and cone avoidance. *Journal of Symbolic Logic*, 74(2):557–578, 2009.

[18] Damir D. Dzhafarov and Carl Mummert. *Reverse mathematics—problems, reductions, and proofs*. Theory and Applications of Computability. Springer, Cham, [2022] ©2022.

[19] M. El-Zahar and N. Sauer. The indivisibility of the homogeneous $K_n$-free graphs. *J. Combin. Theory Ser. B*, 47(2):162–170, 1989.

[20] Julia Christina Erhard. *The Carlson-Simpson Lemma in Reverse Mathematics*. PhD thesis, UC Berkeley, 2013.

[21] Harvey Friedman. Some systems of second order arithmetic and their use. In *in Proceedings of the International Congress of Mathematicians, Vancouver 1974*, pages 235–242, 1975.

[22] Kenneth Gill. A note on the indivisibility of the henson graphs, 2023.

[23] S. S. Gončarov and A. T. Nurtazin. Constructive models of complete decidable theories. *Algebra i Logika*, 12:125–142, 243, 1973.

[24] A. W. Hales and R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.*, 106:222–229, 1963.

[25] Neil Hindman and Randall McCutcheon. Partition theorems for left and right variable words. *Combinatorica*, 24(2):271–286, 2004.

[26] Denis R. Hirschfeldt. *Slicing the truth*, volume 28 of *Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. On the computable and reverse mathematics of combinatorial principles, Edited and with a foreword by Chitat Chong, Qi Feng, Theodore A. Slaman, W. Hugh Woodin and Yue Yang.

[27] Denis R. Hirschfeldt and Richard A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. *J. Symbolic Logic*, 72(1):171–206, 2007.

[28] Denis R. Hirschfeldt, Richard A. Shore, and Theodore A. Slaman. The atomic model theorem and type omitting. *Trans. Amer. Math. Soc.*, 361(11):5805–5837, 2009.

[29] Jan Hubička. Big ramsey degrees using parameter spaces. *arXiv preprint arXiv:2009.00967*, page 20, 2020.

[30] Carl G. Jockusch, Jr. Semirecursive sets and positive reducibility. *Trans. Amer. Math. Soc.*, 131:420–436, 1968.

[31] Carl G. Jockusch, Jr. Ramsey's theorem and recursion theory. *J. Symbolic Logic*, 37:268–280, 1972.

[32] Carl G. Jockusch, Jr. and Robert I. Soare. $\Pi^0_1$ classes and degrees of theories. *Trans. Amer. Math. Soc.*, 173:33–56, 1972.

[33] Péter Komjáth and Vojtěch Rödl. Coloring of universal graphs. *Graphs Combin.*, 2(1):55–60, 1986.

[34] C. Laflamme, N. W. Sauer, and V. Vuksanovic. Canonical partitions of universal structures. *Combinatorica*, 26(2):183–205, 2006.

[35] Lu Liu, Benoit Monin, and Ludovic Patey. A computable analysis of variable words theorems. *Proc. Amer. Math. Soc.*, 147(2):823–834, 2019.

[36] Kenneth McAloon. Paris-Harrington incompleteness and progressions of theories. In *Recursion theory (Ithaca, N.Y., 1982)*, volume 42 of *Proc. Sympos. Pure Math.*, pages 447–460. Amer. Math. Soc., Providence, RI, 1985.

[37] Joseph S. Miller and Reed Solomon. Effectiveness for infinite variable words and the dual Ramsey theorem. *Arch. Math. Logic*, 43(4):543–555, 2004.

[38] Antonio Montalbán. Open questions in reverse mathematics. *Bull. Symbolic Logic*, 17(3):431–454, 2011.

[39] Leonardo Pacheco and Keita Yokoyama. Determinacy and reflection principles in second-order arithmetic. *arXiv preprint arXiv:2209.04082*, 2022.

[40] Ludovic Patey. *The reverse mathematics of Ramsey-type theorems*. PhD thesis, Université Paris Diderot (Paris 7) Sorbonne Paris Cité, 2016.

[41] Ludovic Patey. Iterative forcing and hyperimmunity in reverse mathematics. *Computability*, 6(3):209–221, 2017.

[42] Ludovic Levy Patey. Lowness and avoidance. Available at https://hal.science/hal-05343144/, 2025.

[43] F. P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc. (2)*, 30(4):264–286, 1929.

[44] Joseph G. Rosenstein. *Linear orderings*, volume 98 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.

[45] N. W. Sauer. Coloring subgraphs of the Rado graph. *Combinatorica*, 26(2):231–253, 2006.

[46] Norbert Sauer. Edge partitions of the countable triangle free homogeneous graph. *Discrete Math.*, 185(1-3):137–181, 1998.

[47] David Seetapun and Theodore A. Slaman. On the strength of Ramsey's theorem. *Notre Dame J. Formal Logic*, 36(4):570–582, 1995. Special Issue: Models of arithmetic.

[48] Saharon Shelah. Primitive recursive bounds for van der Waerden numbers. *J. Amer. Math. Soc.*, 1(3):683–697, 1988.

[49] J. R. Shoenfield. On degrees of unsolvability. *Ann. of Math. (2)*, 69:644–653, 1959.

[50] Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edition, 2009.

[51] Theodore A. Slaman. A note on dual Ramsey theorem. January 1997.

[52] Robert I. Soare. *Turing Computability*. Theory and Applications of Computability. Springer Berlin Heidelberg, Berlin, Heidelberg, 2016.

[53] Henry Towsner. A simple proof and some difficult examples for Hindman's theorem. *Notre Dame J. Form. Log.*, 53(1):53–65, 2012.

[54] Keita Yokoyama. The Paris-Harrington principle and second-order arithmetic-bridging the finite and infinite Ramsey theorem. In *ICM—International Congress of Mathematicians. Vol. 3. Sections 1–4*, pages 1504–1528. EMS Press, Berlin, [2023] ©2023.