The weakness of Ramsey’s theorem under omniscient reductions

Ludovic PATEY

UC Berkeley

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Many **theorems** can be seen as **problems**.

**König’s lemma**  
Every infinite, finitely branching tree admits an infinite path.
Some theorems are more effective than others.

**Intermediate value theorem**
For every continuous function $f$ over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.

**König’s lemma**
Every infinite, finitely branching tree admits an infinite path.
COMPUTABLE REDUCTION

A problem $P$ is computably reducible to a problem $Q$ if for every $P$-instance $X$, there is a $Q$-instance $\hat{X} \leq_T X$ such that for every solution $Y$ to $\hat{X}$, $Y \oplus X$ computes a solution to $X$. 

$$P \leq_c Q$$
COMPUTABLE REDUCTION

“Q is at least as hard as P”
Ramsey’s Theorem

\([X]^n\) is the set of unordered \(n\)-tuples of elements of \(X\)

A \(k\)-coloring of \([X]^n\) is a map \(f : [X]^n \to k\)

A set \(H \subseteq X\) is homogeneous for \(f\) if \(|f([H]^n)| = 1\).

\(\text{RT}_k^n\) Every \(k\)-coloring of \([\mathbb{N}]^n\) admits an infinite homogeneous set.
**PIGEONHOLE PRINCIPLE**

\[ \mathbf{RT}_k^1 \]

Every \( k \)-partition of \( \mathbb{N} \) admits an infinite part.
Ramsey’s theorem for pairs

$RT^2_k$ Every $k$-coloring of the infinite clique admits an infinite monochromatic subclique.
AN EXAMPLE

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is hyperimmune if it is not dominated by any computable function.

A problem $P$ preserves $m$ among $n$ hyperimmunities if for every $n$-tuple of hyperimmune functions $f_0, \ldots, f_{n-1}$ and every computable $P$-instance $X$, there is a solution $Y$ to $X$ such that at least $m$ among the $f$’s are $Y$-hyperimmune.
An example

\[ \text{RT}_k^n \not\leq_c \text{RT}_\ell^n \]

whenever \( k > \ell \geq 2 \) and \( n \geq 2 \).

(P.)

\( \text{RT}_\ell^2 \) preserves 2 among \( k \) hyperimmunities but \( \text{RT}_k^2 \) does not.
\[ \text{RT}_k^1 \cong_c \text{RT}^1_\ell \]

whenever \( k, \ell \geq 1 \).
\[ \text{RT}^1_k =_c \text{RT}^1_\ell \]

whenever \( k, \ell \geq 1 \).
\[ \text{Refining} \leq_c \]

Weihrauch reduction
Consider the uniformity of reductions

Strong computable reduction
Removes access to the instance

\[ \text{RT}_k^1 =_c \text{RT}_\ell^1 \]

whenever \( k, \ell \geq 1 \).
A problem $P$ is **strongly computably reducible** to $Q$ if for every $P$-instance $X$, there is a $Q$-instance $\hat{X} \leq_T X$ such that every solution to $\hat{X}$ computes a solution to $X$. 

$$P \leq_{sc} Q$$
“Q is at least as hard as P”
A function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is hyperimmune if it is not dominated by any computable function.

A problem \( P \) strongly preserves \( m \) among \( n \) hyperimmunities if for every \( n \)-tuple of hyperimmune functions \( f_0, \ldots, f_{n-1} \) and every \( P \)-instance \( X \), there is a solution \( Y \) to \( X \) such that at least \( m \) among the \( f \)'s are \( Y \)-hyperimmune.
$\text{RT}_k^1 \not\leq_{\text{sc}} \text{RT}_\ell^1$

whenever $k > \ell \geq 2$.

(Dzhafarov)

$\text{RT}_\ell^1$ strongly preserves 2 among $k$ hyperimmunities but $\text{RT}_k^1$ does not.
\( \text{RT}_k^1 \not\leq^{sc} \text{RT}_\ell^1 \)

whenever \( k > \ell \geq 2 \).

(Dzhafarov)

The \( \text{RT}_k^1 \)-instance witnessing it defeats all \( \text{RT}_\ell^1 \)-instances.

(Hirschfeldt, Jockusch, P.)
\( \text{RT}^1_k \not\lesssim_{sc} \text{SRT}^2_\ell \)

whenever \( k > \ell \geq 2 \).

(Dzhafarov, P., Solomon, Westrick)

\( \text{SRT}^2_k \) : Restriction of \( \text{RT}^2_k \) to stable colorings.
\[ \mathsf{RT}_k^1 \not\leq_{\text{sc}} \mathsf{SRT}_\ell^2 \]

whenever \( k > \ell \geq 2 \).

(Dzhafarov, P., Solomon, Westrick)

The \( \mathsf{RT}_k^1 \)-instance witnessing it defeats all \( \mathsf{SRT}_\ell^2 \)-instances.
WKL : Restriction of König’s lemma to binary trees.

\[ \text{WKL} \leq_c \text{RT}_k^n \]
whenever \( k \geq 2 \) and \( n \geq 3 \).

(Jockusch)

\[ \text{WKL} \nleq_c \text{RT}_k^2 \]
whenever \( k \geq 1 \).

(Liu)
WKL : Restriction of König’s lemma to binary trees.

\[ \text{WKL} \leq_c \text{RT}^n_k \]
whenever \( k \geq 2 \) and \( n \geq 3 \).

\[ \text{WKL} \not\leq_c \text{RT}^2_k \]
whenever \( k \geq 1 \).

(Jockusich)  
(Liu)
Definition

- A function $f$ is a **modulus** of a set $A$ if every function dominating $f$ computes $A$.

- A set $A$ is **computably encodable** if for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ computing $A$.

$A$ is computably encodable $\iff A$ admits a modulus $\iff A$ is hyperarithmetic

(Solovay, Groszek and Slaman)
WKL \not\leq_{sc} RT^n_k

whenever \( n, k \geq 1 \).

(Hirschfeldt, Jockusch)

The WKL-instance witnessing it defeats all \( RT^n_k \)-instances.
WWKL : Restriction of WKL to trees of positive measure.

\[ \text{WWKL} \leq_c \text{RT}_k^n \]

whenever \( k \geq 2 \) and \( n \geq 3 \).

(Jockusch)

\[ \text{WWKL} \not\leq_c \text{RT}_k^2 \]

whenever \( k \geq 1 \).

(Liu)
Definition

- A function $f$ is a $\Pi^0_1$ modulus of a set $C \subseteq \omega^\omega$ if $C$ has a non-empty $g$-computably bounded $\Pi^0_{1,g}$ subset for every $g \geq f$.

- A set $C \subseteq \omega^\omega$ is $\Pi^0_1$ encodable if for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ such that $C$ admits a non-empty $Y$-computably bounded $\Pi^0_{1,Y}$ subset.

$C$ is $\Pi^0_1$ encodable $\iff C$ admits a $\Pi^0_1$ modulus $\iff C$ has a non-empty $\Sigma^1_1$ subset

(Monin, P.)
\[ \text{WWKL} \not\leq_{sc} \text{RT}^n_k \]

whenever \( n, k \geq 1 \).

(Monin, P.)

The WWKL-instance witnessing it defeats all \( \text{RT}^n_k \)-instances.
STRONG OMNISCIENT COMPUTABLE REDUCTION

\[ P \leq_{soc} Q \]

A problem P is strongly omnisciently computably reducible to Q if for every P-instance \( X \), there is an arbitrary Q-instance \( \hat{X} \) such that every solution to \( \hat{X} \) computes a solution to \( X \).
"Q is at least as hard as P"
The big question

$P \leq_w Q$

$P \leq_{sw} Q$

$P \leq_{sc} Q$

$P \leq_c Q \Rightarrow P \leq_{\omega} Q$. 
Whenever $k > \ell \geq 1$

- $\text{RT}_k^1 \not\leq_{soc} \text{RT}_\ell^1$  
  (Hirschfeldt, Jockusch, P.)

- $\text{RT}_k^1 \not\leq_{soc} \text{SRT}_\ell^2$  
  (Dzhafarov, P., Solomon, Westrick)

- $\text{WKL} \not\leq_{soc} \text{RT}_k^n$  
  (Hirschfeldt, Jockusch)

- $\text{WWKL} \not\leq_{soc} \text{RT}_k^n$  
  (Monin, P.)
OMNISCIENT COMPUTABLE REDUCTIONS

- ACA $\not\leq_{oc} RT^1_k$ (Dzhafarov)
- WWKL $\not\leq_{oc} RT^1_k$ (Liu.)
- WWKL $\not\leq_{oc} FS$ (P.)
- $RT^2_2 \not\leq_{oc} FS$ (P.)
Differences with $\leq_{sc}$

$\text{SRT}_3^2 \not\leq_{sc} \text{RT}_2^2$  \hspace{1cm} $\text{SRT}_{<\infty}^2 \leq_{soc} \text{RT}_2^2$  

(P.)  

(Monin, P.)

Proof sketch: $g(x, y) = 1$ iff $f(x, y) = \lim_s f(y, s)$
Diagram under $\leq_{soc}$

KL $\iff$ WKL

RT $\downarrow$ WWKL

$\vdots$ $\downarrow$

$\vdots$

$\downarrow$

RT$_3^2$ $\iff$ RT$_2^2$

SRT$_2^2$ $\iff$ SRT$_3^2$ $\iff$ SRT$_2^2$

RT$_1^1$ $\iff$ RT$_3^1$ $\iff$ RT$_2^1$
QUESTIONS

Is \( RT \leq_{soc} RT^2_2 \) ?

Is \( RT^n_{k+1} \leq_{soc} RT^n_k \)?  
No for \( n = 1 \).

Is \( RT^{n+1}_k \leq_{soc} RT^n_k \)?  
No for \( n = 1 \).
Revisiting the big question
REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.
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Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

\[ \text{RCA}_0 \vdash A \iff T \]

in a very weak theory \( \text{RCA}_0 \) capturing computable mathematics
Mathematics are computationally very structured.

Almost every theorem is empirically equivalent to one among five big subsystems.

\[
\begin{align*}
\Pi^1_1 \text{CA} \\
\downarrow \\
\text{ATR} \\
\downarrow \\
\text{ACA} \\
\downarrow \\
\text{WKL} \\
\downarrow \\
\text{RCA}_0
\end{align*}
\]
Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey’s theory...
COHESIVE SETS

An infinite set $C$ is **cohesive** for a sequence of sets $R_0, R_1, \ldots$ if for every $i$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

An infinite set $C$ is **p-cohesive** if it is cohesive for the primitive recursive sets.

**COH**  
Every sequence of sets has a cohesive set.
\[ \text{RT}_2^2 \leftrightarrow \text{COH} \, + \, \text{SRT}_2^2 \]

Fix an instance \( f : [\mathbb{N}]^2 \rightarrow 2 \) of RT\(^2_2\).

Define \( R_x = \{ y : f(x, y) = 1 \} \).

Let \( C \) be cohesive for \( R_0, R_1, \ldots \).

\( f : [C]^2 \rightarrow 2 \) is an instance of SRT\(^2_2\).
THE BIG QUESTION

Does $\text{RCA}_0 \vdash \text{SRT}^2_2 \rightarrow \text{COH}$?
THE BIG QUESTION

Does $\text{RCA}_0 \vdash \text{SRT}_2^2 \rightarrow \text{COH}$?

Theorem (Chang, Slaman, Yang)

*Nope.*
Revisiting the big question

Hirschfeldt: “We want a computability-theoretic answer”

An $L_2$-structure $\mathcal{M} = \langle M, S, 0, 1, +, \cdot \rangle$ is an $\omega$-structure if $M$ is the set of standard numbers, equipped with the standard operations.

Does $\text{RCA}_0 \vdash \text{SRT}^2_2 \rightarrow \text{COH}$ on $\omega$-structures?
REVISITING THE BIG QUESTION

Dzhafarov: “One step is already complicated”

Is $\text{COH} \leq_c \text{SRT}_2^2$?
Revisiting the Big Question

P: “This is about the combinatorics of singletons”

Is $\text{COH} \leq_{oc} \text{RT}^1_2$?

Is there a set $X$, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ computes a $p$-cohesive set?
A set $X$ is high if $X' \geq_T \emptyset''$.

Is there a set $X$, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ is high?

If yes, then $\text{COH} \leq_{oc} \text{RT}_2^1$.

If no, well, this is still interesting per se.
A set $S$ is **P-jump-encodable** if there is an instance of P such that the jump of every solution computes $S$.

Are the $RT^1_2$-jump-encodable sets precisely the $\emptyset'$-computable ones?
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