Partial conservation of Ramsey's theorem for pairs

Ludovic LEVY PATEY

Joint work with Quentin Le Houérou and Keita Yokoyama

Introduction

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

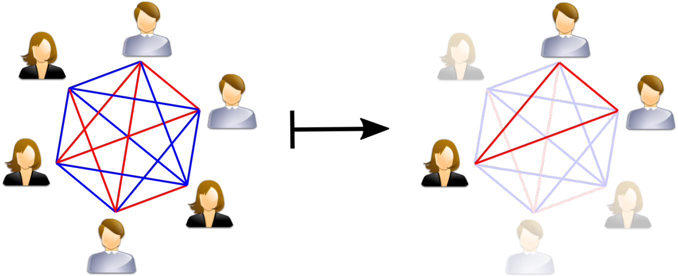
A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



RCA_0

Robinson arithmetics (Q)

$$m + 1 \neq 0$$
$$m + 1 = n + 1 \rightarrow m = n$$
$$\neg(m < 0)$$
$$m < n + 1 \leftrightarrow (m < n \vee m = n)$$

$$m + 0 = m$$
$$m + (n + 1) = (m + n) + 1$$
$$m \times 0 = 0$$
$$m \times (n + 1) = (m \times n) + m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n + 1)) \rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is a Σ^0_1 formula

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula where X appears freely, and ψ is a Π^0_1 formula.

Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$\Pi^1_1\mathsf{CA}$

$\downarrow$

ATR

$\downarrow$

ACA

$\downarrow$

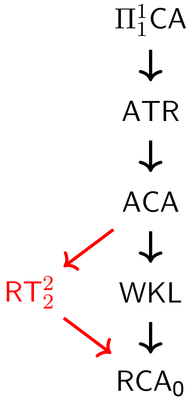
WKL

$\downarrow$

RCA_0

Reverse mathematics

Mathematics are computationally very structured



Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

The first order-part of a theory T is the set of its theorems in the language of first-order arithmetic.

What is the first-order part of Ramsey's theorem for pairs?

Weak arithmetic 101

Induction scheme

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall y \varphi(y)$$

for every formula $\varphi(x)$

Collection scheme

$$(\forall x < a)(\exists y)\varphi(x, y) \to (\exists b)(\forall x < a)(\exists y < b)\varphi(x, y)$$

for every $a \in \mathbb{N}$ and every formula $\varphi(x, y)$

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

- exp: totality of the exponential
- A set *X* is *M*-regular if every initial segment of *X* is *M*-coded
- Least principle: every non-empty set admits a minimum element

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

$$\mathsf{RCA}_0 \equiv \mathsf{Q} + \Delta^0_1\text{-comprehension} + \mathsf{I}\Sigma^0_1$$

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

$$\mathsf{RCA}^*_0 \equiv \mathsf{Q} + \Delta^0_1\text{-comprehension} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$$

First-order parts

Induction	System	First-order part
$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$	$\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$	$\mathsf{Q} + \mathsf{I}\Sigma_2$
$\mathsf{I}\Delta^0_2$	$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$	$\mathsf{Q} + \mathsf{I}\Delta_2$
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$	RCA_0	$\mathsf{Q} + \mathsf{I}\Sigma_1$
$\mathsf{I}\Delta^0_1 + \mathsf{exp}$	RCA^*_0	$\mathsf{Q} + \mathsf{I}\Delta_1 + \mathsf{exp}$

Failure of induction

$\equiv$

Existence of proper cuts

- A non-empty set $I \subseteq M$ is a cut if it is an initial segment of M closed under successor
- A cut is exponential if it is closed under exponential
- A cut is semi-regular if for every M-coded set $F \subseteq M$ such that $|F| \in I$, $F \cap I$ is bounded in I.

Given a first-order structure M and a proper cut I, let

$$\mathrm{Cod}(M/I) = \{F \cap I : F \text{ is } M\text{-coded}\}$$

If $M \models \mathrm{PRA}$ and I semi-regular, then $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$

WKL_0 is Π_2-conservative over PRA.

The RCA_0-provably total functions are the primitive recursive functions.

If $M \models \mathrm{EFA}$ and I exponential, then $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0^*$

WKL_0^* is Π_2-conservative over EFA.

The RCA_0^*-provably total functions are the elementary functions.

- WKL: Every infinite binary tree admits an infinite path
- $\mathrm{WKL}_0 \equiv \mathrm{RCA}_0 + \mathrm{WKL}$ and $\mathrm{WKL}_0^* \equiv \mathrm{RCA}_0^* + \mathrm{WKL}$

Conservation theorems

Fix a family of formulas Γ.

A theory T_1 is Γ-conservative over T_0 if every Γ-sentence provable over T_1 is provable over T_0.

If T_1 is a Π^1_1-conservative extension of T_0, then they have the same first-order part.

A second-order structure $\mathcal{N} = (N, T)$ is an ω-extension of $\mathcal{M} = (M, S)$ if $N = M$, $T \supseteq S$, $+^{\mathcal{N}} = +^{\mathcal{M}}$ and $<^{\mathcal{N}} = <^{\mathcal{M}}$.

Theorem

If every countable model of $\mathcal{M} \models T_0$ admits an ω-extension $\mathcal{N} \models T_1$, then T_1 is Π^1_1-conservative over T_0.

- Suppose $T_0 \nvDash \forall X \phi(X)$. Let $\mathcal{M} \models T_0 \wedge \exists X \neg \phi(X)$.
- Let $\mathcal{N} \models T_1$ be an ω-extension of $\mathcal{M}$.
- Then $\mathcal{N} \models T_1 \wedge \exists X \neg \phi(X)$. So $T_1 \nvDash \forall X \phi(X)$.

Let $\mathcal{M} = (M, S)$ be a second-order structure, and $G \subseteq M$. $\mathcal{M}[G]$ is the smallest ω-extension containing the $\Delta^0_1(\mathcal{M} \cup \{G\})$ sets.

Theorem

Let P be a Π^1_2-problem and T be a theory. If for every countable model $\mathcal{M} \models T$ and every $X \in \mathcal{M}$ such that $\mathcal{M} \models (X \in \operatorname{dom} \mathsf{P})$, there is a set $Y \subseteq M$ such that $\mathcal{M}[Y] \models T + (Y \in \mathsf{P}(X))$, then $T + \mathsf{P}$ is Π^1_1-conservative over T.

$$\mathcal{M} \subseteq \mathcal{M}[Y_0] \subseteq \mathcal{M}[Y_0][Y_1] \subseteq \dots$$

Preliminary results

Theorem (Hirst)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{B}\Sigma^0_2$.

Theorem (Cholak, Jockusch and Slaman)

For every countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ and every coloring $f : [M]^2 \to 2$ in $\mathcal{M}$, there is an infinite f-homogeneous set $G \subseteq M$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

Thus $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

Theorem (Chong, Slaman and Yang)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{I}\Sigma^0_2$.

Is $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

Question

Given a countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ and a coloring $f : [M]^2 \to 2$ in $\mathcal{M}$, is there an infinite f-homogeneous set $G \subseteq M$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$?

An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \ldots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

COH : Every collection of sets has a cohesive set.

Theorem (Mileti ; Jockusch and Lempp)

$\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \to \mathrm{COH}$.

The following are equivalent over RCA_0:

- $\mathrm{COH} + \mathrm{B}\Sigma^0_2$
- "Every Δ^0_2 infinite binary tree admits an infinite Δ^0_2 path"

The jump of a structure $\mathcal{M} = (M, S)$ is the smallest ω-extension containing the $\Delta^0_2(\mathcal{M})$ sets.

Lemma (Belanger)

Let $\mathcal{M} \models \mathsf{RCA}_0$ and $\mathcal{N}$ be its jump. Then

- $\mathcal{M} \models \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ iff $\mathcal{N} \models \mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$.
- $\mathcal{M} \models \mathsf{B}\Sigma^0_2 + \mathsf{COH} + \neg\mathsf{I}\Sigma^0_2$ iff $\mathcal{N} \models \mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$.

In the jump realm

Theorem (Simpson and Smith)

For every countable model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0^*$ and every infinite tree $T \subseteq 2^{<M}$, there is an infinite path $P \in [T]$ such that $\mathcal{M}[P] \models \mathsf{RCA}_0^*$.

Thus WKL_0^* is Π^1_1-conservative over RCA_0^*.

Theorem (Fiori-Carones, Kołodziejczyk, Wong and Yokoyama)

Les $\mathcal{M}_0 = (M, S_0)$ and $\mathcal{M}_1 = (M, S_1)$ be countable models of WKL_0^* such that $(M, S_0 \cap S_1) \models \neg \mathsf{I}\Sigma^0_1$. Then $\mathcal{M}_0 \cong \mathcal{M}_1$.

A Π^1_2 problem P is Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma^0_1$ iff $\mathsf{WKL}_0^* + \neg \mathsf{I}\Sigma^0_1 \vdash \mathsf{P}$.

In the ground realm

Theorem (Fiori-Carones, Kołodziejczyk, Wong and Yokoyama)

Let $\mathcal{M}_0 = (M, S_0)$ and $\mathcal{M}_1 = (M, S_1)$ be countable models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{COH}$ such that $(M, S_0 \cap S_1) \models \neg\mathsf{I}\Sigma^0_2$. Then their jump models are isomorphic.

Conservation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ can be done without loss of generality by first-jump control.

Theorem (Fiori-Carones, Kołodziejczyk, Wong and Yokoyama)

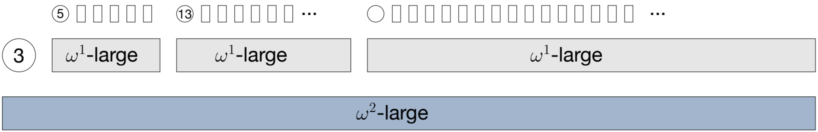
Let P be a $\forall\exists\Pi^0_k$-sentence, where $k \geq 3$. Then P is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ iff it is $\forall\Pi^0_{k+2}$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$.

$\forall\Pi^0_3$ conservation

A finite set $X \subseteq \mathbb{N}$ is

- ω^0-large if $X \neq \emptyset$.
- $\omega^{(n+1)}$-large if $X \setminus \min X$ is $(\omega^n \cdot \min X)$-large
- $\omega^n \cdot k$-large if there are k ω^n-large subsets of X

$$X_0 < X_1 < \cdots < X_{k-1}$$



- $A < B$ means that for all $a \in A$ and $b \in B$, $a < b$.

Lemma

$\mathsf{RCA}_0 \vdash \forall a[\mathrm{WF}(\omega^a) \leftrightarrow$ Every infinite set contains an ω^a-large subset$]$

Let $\mathcal{M} = (M, S)$ be a countable model of RCA_0.

$$\mathrm{WF}(\omega^{\mathcal{M}}) = \{a \in M : \mathcal{M} \models \mathrm{WF}(\omega^a)\}$$

- $\mathrm{WF}(\omega^{\mathcal{M}})$ is an additive cut
- There is a model $\mathcal{M}$ and some non-standard a such that

$$\mathrm{WF}(\omega^{\mathcal{M}}) = \sup\{a \cdot n : n \in \omega\}$$

α-largeness approximates infinity

Theorem (Generalized Parsons theorem)

Let $\psi(F)$ be a Δ_0 formula with only free variable F. Suppose that

$$\mathsf{WKL}_0 \vdash \forall X\, [X \text{ is infinite } \rightarrow (\exists F \subseteq_{\mathtt{fin}} X)\psi(F)]$$

Then there exists some $n \in \omega$ such that

$$\mathsf{Q} + \mathsf{I}\Sigma^0_1 \vdash \forall Z\, [Z \text{ is } \omega^n\text{-large} \rightarrow (\exists F \subseteq Z)\psi(F)]$$

Forcing with ω^a-large sets

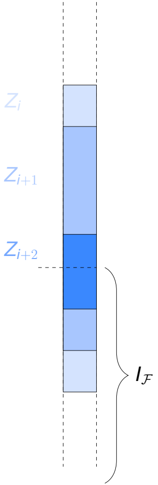
Fix a countable non-standard model $M \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$.

$$(\mathbb{P}, \leq)$$

ω^a-large sets for $a \in M \setminus \omega$ ordered by inclusion.

Every filter $\mathcal{F} \subseteq \mathbb{P}$ induces a cut

$$I_{\mathcal{F}} = \sup\{\min Z : Z \in \mathcal{F}\}$$

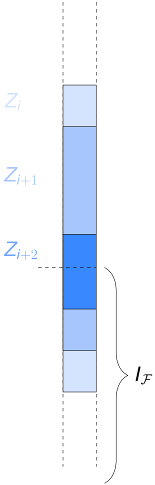


Forcing with ω^a-large sets

Fix a countable non-standard model $M \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$.

- If $Z \in \mathcal{F}$, then $Z \cap I_{\mathcal{F}}$ is unbounded in $I_{\mathcal{F}}$.
- $Z \Vdash (\forall x \in I)\theta(x)$ if $(\forall x < \max Z)\theta(x)$.
- $Z \Vdash (\exists x \in I)\theta(x)$ if $(\exists x < \min Z)\theta(x)$.
- $Z \Vdash (\forall x \in I)(\exists y \in I)\theta(x, y)$ if

$$(\forall a, b \in Z)[a < b \to (\forall x < a)(\exists y < b)\theta(x, y)]$$



A cut is semi-regular if for every M-coded set $F \subseteq M$ such that $|F| \in I$, $F \cap I$ is bounded in I.

If $M \models \mathrm{PRA}$ and I semi-regular, then $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$.

Lemma (Kirby and Paris)

If $M \models \mathrm{Q} + \mathrm{I}\Sigma^0_1$ and $\mathcal{F}$ is sufficiently generic, then $I_{\mathcal{F}}$ is semi-regular.

- Let $Z \in \mathbb{P}$ be ω^a-large and $F \subseteq M$ be M-coded with $|F| < \min Z$;
- Let $Z_0 < \cdots < Z_{|F|}$ be ω^{a-1}-large subsets of Z ;
- $Z_i \cap F = \emptyset$ for some $i \leq |F|$.

Theorem (Hirst)

$\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_2 \leftrightarrow \forall a \mathsf{RT}^1_a$.

X is exp-sparse if $\min X \geq 3$ and $(\forall x, y \in X)(x < y \to 4^x < y)$

Lemma (Kołodziejczyk and Yokoyama)

$\mathsf{Q} + \mathsf{I}\Sigma^0_1$ proves that if X is ω^{a+1}-large and exp-sparse, then for every $f : X \to \min X$, there is an ω^a-large f-homogeneous subset $Y \subseteq X$.

Thus if $\mathcal{F}$ is sufficiently generic $(I_\mathcal{F}, \mathrm{Cod}(M/I_\mathcal{F})) \models \forall a \mathsf{RT}^1_a$.

Theorem (Parsons, Paris and Friedman)

$\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2$ is $\forall\Pi^0_3$-conservative over RCA_0.

- Suppose $\mathrm{RCA}_0 \nvdash \forall A \exists x \forall y \psi(A, x, y)$;
- Let $\mathcal{M} = (M, S) \models \mathrm{RCA}_0 + \exists A \forall x \exists y \neg\psi(A, x, y)$ be non-standard ;
- Let $A \in S$ and $X = \{b_0 < b_1 < \ldots\} \in S$ be such that $(\forall x < b_s)(\exists y < b_{s+1})\neg\psi(A, x, y)$;
- Let $a \in \mathrm{WF}(\omega^{\mathcal{M}}) \setminus \omega$ and let $Z \subseteq X$ be ω^a-large ;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2 + \exists A \forall x \exists y \neg\psi(A, x, y)$.

Lemma (Kołodziejczyk and Yokoyama)

$\mathsf{Q} + \mathsf{I}\Sigma^0_1$ proves that if X is ω^{300a}-large and $\min X \geq 3$, then for every $f : [X]^2 \to 2$, there is an ω^a-large f-homogeneous set $H \subseteq X$.

Thus if $\mathcal{F}$ is sufficiently generic $(I_{\mathcal{F}}, \mathsf{Cod}(M/I_{\mathcal{F}})) \models \mathsf{RT}^2_2$.

Theorem (Patey and Yokoyama)

$\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$-conservative over RCA_0.

$\forall\Pi^0_4$ conservation

Theorem (Parsons, Paris and Friedman)

$\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2$ is $\forall\Pi^0_3$-conservative over RCA_0.

- Suppose $\mathrm{RCA}_0 \nvdash \forall A \exists x \forall y \psi(A, x, y)$;
- Let $\mathcal{M} = (M, S) \models \mathrm{RCA}_0 + \exists A \forall x \exists y \neg\psi(A, x, y)$ be non-standard ;
- Let $A \in S$ and $X = \{b_0 < b_1 < \dots\} \in S$ be such that $(\forall x < b_s)(\exists y < b_{s+1})\neg\psi(A, x, y)$;
- Let $a \in \mathrm{WF}(\omega^{\mathcal{M}}) \setminus \omega$ and let $Z \subseteq X$ be ω^a-large ;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2 + \exists A \forall x \exists y \neg\psi(A, x, y)$.

Theorem (Erroneous Logician)

$\mathsf{WKL}_0 + \mathrm{B}\Sigma^0_2$ is $\forall\Pi^0_4$-conservative over RCA_0.

- Suppose $\mathsf{RCA}_0 \nvdash \forall A \exists x \forall y \exists z \psi(A,x,y,z)$;
- Let $\mathcal{M} = (M,S) \models \mathsf{RCA}_0 + \exists A \forall x \exists y \forall z \neg\psi(A,x,y,z)$ be non-standard ;
- Let $A \in S$ and $X = \{b_0 < b_1 < \dots\} \in S$ be such that $(\forall x < b_s)(\exists y < b_{s+1})\forall z \neg\psi(A,x,y,z)$;
- Let $a \in \mathrm{WF}(\omega^{\mathcal{M}}) \setminus \omega$ and let $Z \subseteq X$ be ω^a-large ;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{WKL}_0 + \mathrm{B}\Sigma^0_2 + \exists A \forall x \exists y \forall z \neg\psi(A,x,y,z)$.

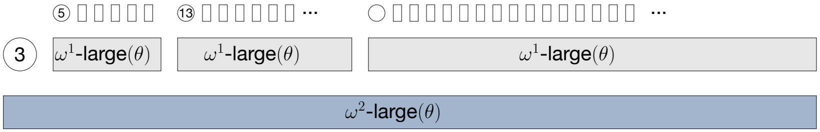
$\mathrm{B}\Sigma_2$ is a Π_4-sentence which is not provable over RCA_0.

Let $\theta(x,y,z) \equiv \neg\psi(A,x,y,z)$.

A finite set $X \subseteq \mathbb{N}$ is

- ω^0-large(θ) if $X \neq \emptyset$.
- $\omega^{(n+1)}$-large(θ) if $X \setminus \min X$ is $(\omega^n \cdot \min X)$-large(θ)
- $\omega^n \cdot k$-large(θ) if there are k θ-apart ω^n-large subsets of X

$$X_0 < X_1 < \cdots < X_{k-1}$$



- $A < B$ are θ-apart if $(\forall x < \max A)(\exists y < \min B)(\forall z < \max B)\theta(x, y, z)$.

Definition

$\mathsf{WF}_\theta(\omega^a)$: Every infinite set contains an ω^a-large(θ) subset.

Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \forall x \exists y \forall z \theta(x, y, z)$.

$$\mathsf{WF}_\theta(\omega^{\mathcal{M}}) = \{a \in M : \mathcal{M} \models \mathsf{WF}_\theta(\omega^a)\}$$

- $\mathsf{WF}_\theta(\omega^{\mathcal{M}})$ is an additive cut
- There is a model $\mathcal{M}$ and some non-standard a such that

$$\mathsf{WF}_\theta(\omega^{\mathcal{M}}) = \sup\{a \cdot n : n \in \omega\}$$

α-largeness(θ) approximates infinity

Theorem (Generalized Parsons theorem)

Let $\psi(F)$ be a Δ_0 formula with only free variable F. Suppose that

$$\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2 + \forall x \exists y \forall z \theta(x,y,z) \vdash \forall X\, [X \text{ is infinite } \rightarrow (\exists F \subseteq_{\mathtt{fin}} X)\psi(F)]$$

Then there exists some $n \in \omega$ such that

$$\mathsf{Q} + \mathsf{I}\Delta^0_2 \vdash \forall Z\, [Z \text{ is } \omega^n\text{-large}(\theta) \rightarrow (\exists F \subseteq Z)\psi(F)]$$

Forcing with ω^a-large(θ) sets

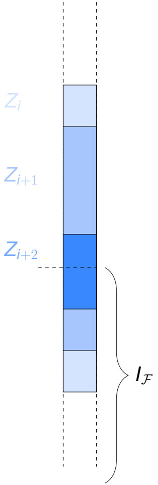
Fix a countable non-standard model
$M \models \mathsf{Q} + \mathsf{I}\Delta^0_2 + \forall x \exists y \forall z \theta(x, y, z)$.

$$(\mathbb{P}, \leq)$$

ω^a-large(θ) sets for $a \in M \setminus \omega$
ordered by inclusion.

Every filter $\mathcal{F} \subseteq \mathbb{P}$ induces a cut

$$I_{\mathcal{F}} = \sup\{\min Z : Z \in \mathcal{F}\}$$



Suppose $M \models \mathsf{Q} + \mathsf{I}\Sigma^0_1 + \forall x \exists y \forall z \theta(x, y, z)$.

Lemma

If $\mathcal{F}$ is sufficiently generic, then $I_{\mathcal{F}}$ is semi-regular.

- Let $Z \in \mathbb{P}$ be ω^a-large(θ) and $F \subseteq M$ be M-coded with $|F| < \min Z$;
- Let $Z_0 < \cdots < Z_{|F|}$ be ω^{a-1}-large(θ) subsets of Z ;
- $Z_i \cap F = \emptyset$ for some $i \leq |F|$.

Lemma

If $\mathcal{F}$ is sufficiently generic, then $I_{\mathcal{F}} \models \forall x \exists y \forall z \theta(x, y, z)$.

- Let $Z \in \mathbb{P}$ be ω^a-large(θ) and $x < \min Z$;
- Let $Z_0 < Z_1$ be θ-apart ω^{a-1}-large(θ) subsets of Z ;
- Z_1 forces $\exists y \forall z \theta(x, y, z)$.

Theorem (Hirst)

$\mathrm{RCA}_0 \vdash \mathrm{B}\Sigma^0_2 \leftrightarrow \forall a \mathrm{RT}^1_a$.

X is exp-sparse if $\min X \geq 3$ and $(\forall x, y \in X)(x < y \to 4^x < y)$

Lemma (Le Houérou, Levy Patey and Yokoyama)

$\mathrm{Q} + \mathrm{I}\Sigma^0_1$ proves that if X is ω^{2a}-large(θ) and exp-sparse, then for every $f : X \to \min X$, there is an ω^a-large(θ) f-homogeneous subset $Y \subseteq X$.

Thus if $\mathcal{F}$ is sufficiently generic $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \forall a \mathrm{RT}^1_a$.

Theorem

$\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2$ is $\forall\Pi^0_4$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

- Suppose $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \nvdash \forall A \exists x \forall y \exists z \psi(A, x, y, z)$;
- Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \exists A \forall x \exists y \forall z \neg\psi(A, x, y, z)$ be non-standard ;
- Let $A \in S$ be a witness and $\theta(x, y, z) \equiv \neg\psi(A, x, y, z)$;
- Let $a \in \mathrm{WF}(\omega^{\mathcal{M}}) \setminus \omega$ and let $Z \subseteq M$ be ω^a-large(θ) ;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2 + \exists A \forall x \exists y \forall z \neg\psi(A, x, y, z)$.

Lemma (Le Houérou, Levy Patey and Yokoyama)

$\mathsf{Q} + \mathsf{I}\Sigma^0_1$ proves that if X is $\omega^{(16^6+1)^a}$-large(θ) and $\min X \geq 3$, then for every $f : [X]^2 \to 2$, there is an ω^a-large(θ) f-homogeneous set $H \subseteq X$.

Thus if $\mathcal{F}$ is sufficiently generic $(I_\mathcal{F}, \mathrm{Cod}(M/I_\mathcal{F})) \models \mathsf{RT}^2_2$.

Theorem (Le Houérou, Levy Patey and Yokoyama)

$\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_4$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Open questions

Is $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

Is $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ $\forall\Pi^0_5$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

Does $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ admit exponential proof-speedup over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

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