Lowness and avoidance

HDR defense

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Computability 101

A set $A \subseteq \mathbb{N}$ is computable if there is a computer program which, on input n, decides whether $n \in A$ or not.

A set $A \subseteq \mathbb{N}$ is computable in B if there is a computer program in an language augmented with the characteristic function of B which, on input n, decides whether $n \in A$ or not.

$$A \leq_T B$$

A is computable in B

$$\Phi_e(x) \downarrow$$

The *e*-th program halts on input *x*.

$$\Phi_e(x)[t] \downarrow$$

The *e*-th program halts on input *x*
in less than *t* steps.

$$\Phi_e^A(x) \downarrow$$

The e-th program with oracle A halts on input x.

$$\Phi_e^A(x)[t] \downarrow$$

The e-th program with oracle A halts on input x in less than t steps.

Arithmetic hierarchy

$$\Sigma_n^0 \quad \varphi(y) \equiv \exists x_1 \forall x_2 \ldots Q x_n \ \psi(y, x_1, \ldots, x_n)$$

$$\Pi_n^0 \quad \varphi(y) \equiv \forall x_1 \exists x_2 \ldots Q x_n \ \psi(y, x_1, \ldots, x_n)$$

where ψ contains only bounded first-order quantifiers

A set is Γ if it is Γ-definable
A set is Δ_n^0 if it is Σ_n^0 and Π_n^0.
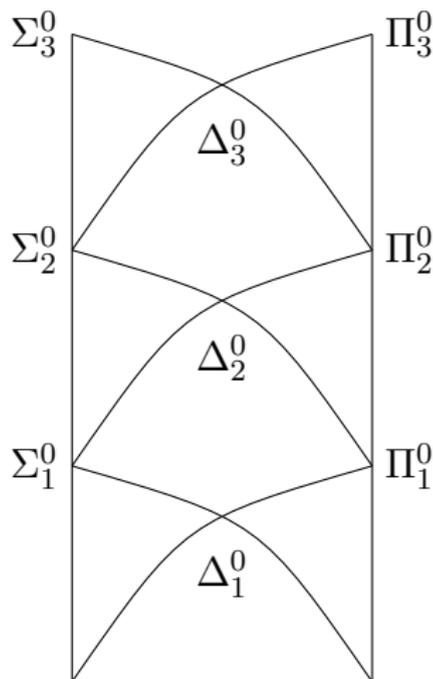
Computability $\equiv$ Definability

Theorem (Gödel)

A set is c.e. iff it is Σ_1^0 and computable iff it is Δ_1^0.

Theorem (Post)

A set is $\emptyset^{(n)}$-c.e. iff it is Σ_{n+1}^0 and $\emptyset^{(n)}$-computable iff it is Δ_{n+1}^0.

where $\emptyset^{(0)} = \emptyset$; $\emptyset^{(n+1)} = \{e : \Phi_e^{\emptyset^{(n)}}(e) \downarrow\}$

Main techniques

Priority method

✓ Definable/effective set
✗ Subtle interdependencies

Forcing

✗ Arbitrary set
✓ Modular requirements

Forcing 101

Partial order
$(\mathbb{P}, \leq)$

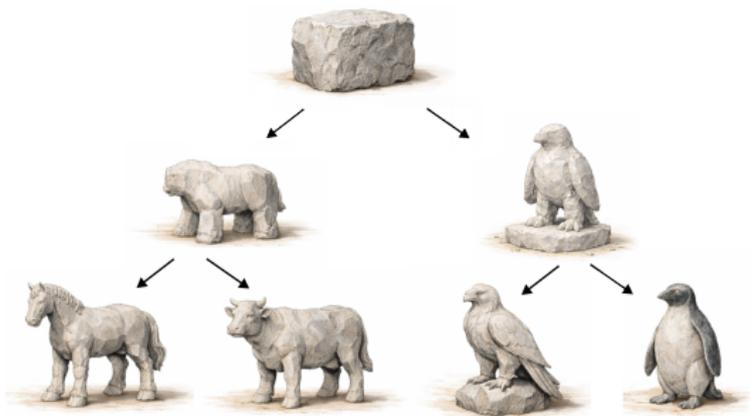**Condition**
$p \in \mathbb{P}$
approximation

Denotation
$[p] \subseteq 2^\omega$
class of candidates

Compatibility
If $q \leq p$ then $[q] \subseteq [p]$

Filter $\mathcal{F} \subseteq \mathbb{P}$

$\forall p \in \mathcal{F} \, \forall q \geq p \; q \in \mathcal{F}$

$\forall p, q \in \mathcal{F}, \, \exists r \in \mathcal{F} \; r \leq p, q$
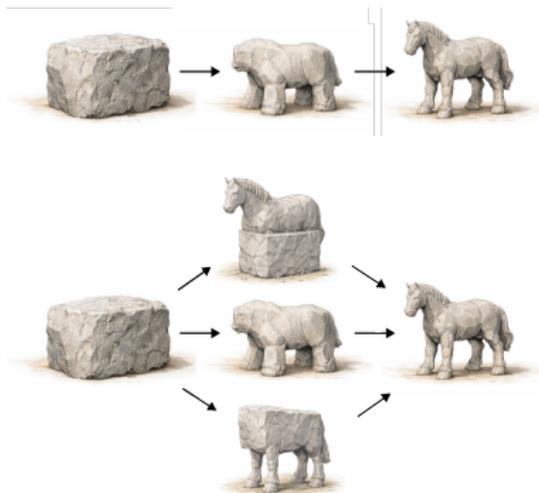
Build a $\searrow$ **sequence**

$p_0 \geq p_1 \geq p_2 \geq \ldots$

Take the upward-closure

$\mathcal{F} = \{q \in \mathbb{P} : \exists i \; q \leq p_i\}$

Denotation

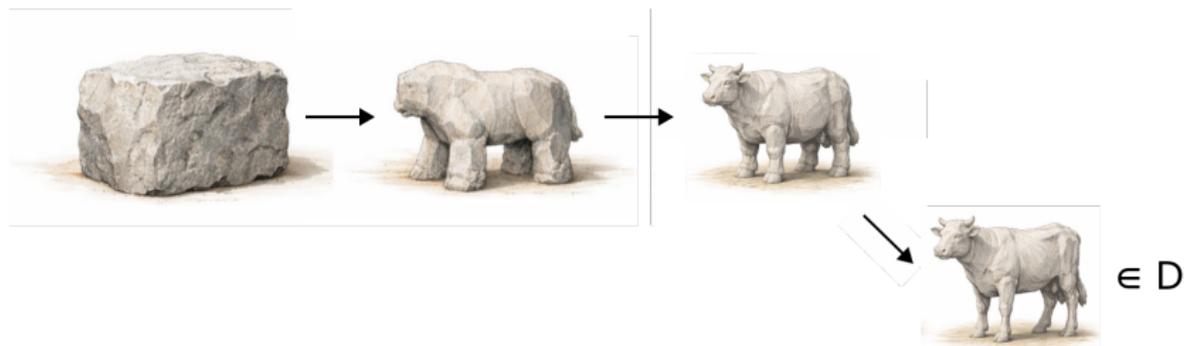$[\mathcal{F}] = \bigcap_{p \in \mathcal{F}} [p]$

Density

$$\forall p \in \mathbb{P} \ \exists q \leq p \ q \in D$$

"It is never too late to intersect *D*"

Ex: Let *D* be the set of underweight scultpures

∈ D

Genericity

Let $\vec{D} = D_0, D_1, \ldots$ be dense sets

$\mathcal{F}$ is $\vec{D}$-generic if it intersects every $D \in \vec{D}$

Lemma

For every countable collection $\vec{D}$ of dense sets, there is a $\vec{D}$-generic filter

Proof: Build a decreasing sequence $p_0 \geq p_1 \geq \ldots$ such that $p_n \in D_n$.

Forcing relation

"Complex properties of the constructed object are already determined during the construction."

$$p \Vdash \varphi(G)$$

where $p \in \mathbb{P}$ and $\varphi(G)$ is arithmetic

$\equiv \varphi(G_{\mathcal{F}})$ holds for every sufficiently generic filter $\mathcal{F}$.

▶ The set $D_\varphi = \{p \in \mathbb{P} : p \Vdash \varphi(G) \text{ or } p \Vdash \neg\varphi(G)\}$ is dense
▶ There is a syntactical, inductive definition of $p \Vdash \varphi(G)$

Effective forcing

Cohen forcing	**Jockusch-Soare forcing**
$\mathbb{P} = 2^{<\mathbb{N}}$ finite binary strings	$\mathbb{P} = \mathcal{T}$ infinite computable binary trees
$\tau \le \sigma$ τ is a suffix of σ	$S \le T$ *S* included in *T*
$[\sigma]$ infinite binary strings with initial segment σ	$[T]$ infinite paths through *T*

Cohen forcing

Theorem (Folklore)

Let $C \not\leq_T \emptyset$. For every sufficiently Cohen generic G, $C \not\leq_T G$.

Lemma

For every $C \not\leq_T \emptyset$ and functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \Phi_e^G \neq C\}$$

Jockusch-Soare forcing

Theorem (Jockusch-Soare)

Let $C \not\leq_T \emptyset$. For every sufficiently Jockusch-Soare generic G, $C \not\leq_T G$.

Lemma

For every $C \not\leq_T \emptyset$ and functional Φ_e, the following set is dense in $(\mathcal{T}, \subseteq)$

$$D = \{T \in \mathcal{T} : T \Vdash \Phi_e^G \neq C\}$$

Cohen forcing

Given $\sigma \in 2^{<\omega}$, define the Σ_1^0 set

$$W = \{(x, v) : \exists \tau \leq \sigma \ \Phi_e^\tau(x) \downarrow = v\}$$

- ▶ Case 1: $\exists x \ (x, 1 - C(x)) \in W$
  Then τ forces $\Phi_e^G \neq C$

- ▶ Case 2: $\exists x \ (x, C(x)) \notin W$
  Then σ forces $\Phi_e^G \neq C$

- ▶ Case 3: W is a Σ_1^0 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Jockusch-Soare forcing

Given $T \in \mathcal{T}$, define the Σ_1^0 set

$$W = \{(x, v) : \exists \ell \forall \sigma \in 2^\ell \cap T \ \Phi_e^\sigma(x) \downarrow = v\}$$

- ▶ Case 1: $\exists x \ (x, 1 - C(x)) \in W$
  Then T forces $\Phi_e^G \neq C$

- ▶ Case 2: $\exists x \ (x, C(x)) \notin W$
  $\{\sigma \in T : \neg(\Phi_e^\sigma(x) \downarrow = v)\}$
  forces $\Phi_e^G \neq C$

- ▶ Case 3: W is a Σ_1^0 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Forcing question

$$p \, ?\!\vdash \varphi(G)$$
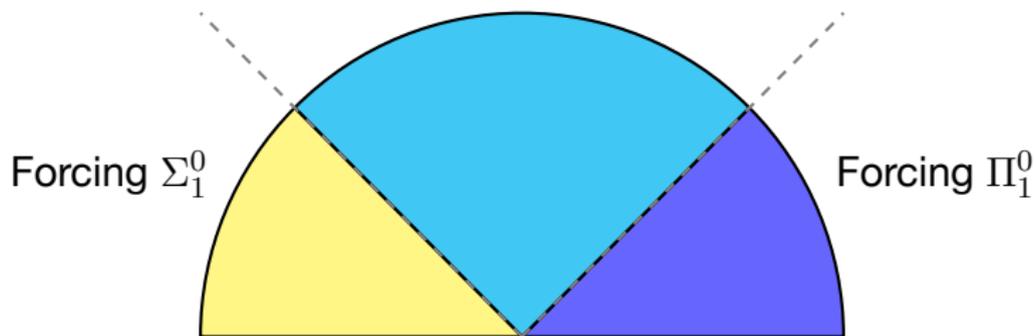
where $p \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_1

Lemma

Let $p \in \mathbb{P}$ and $\varphi(G)$ be a Σ^0_1 formula.
 (a) If $p \, ?\!\vdash \varphi(G)$, then $q \Vdash \varphi(G)$ for some $q \leq p$;
 (b) If $p \, ?\!\nvdash \varphi(G)$, then $q \Vdash \neg\varphi(G)$ for some $q \leq p$.

Jockusch-Soare forcing question

Cohen forcing question

Forcing Σ_1^0

Forcing Π_1^0

Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is Σ_1^0-preserving if for every $p \in \mathbb{P}$ and every Σ_1^0-formula $\varphi(G, x)$, the relation $p \mathbin{?\vdash} \varphi(G, x)$ is Σ_1^0 uniformly in x.

Lemma

Suppose $?\vdash$ is Σ_1^0-preserving. For every non-computable set C and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \Phi_e^G \neq C\}$$

Given $p \in \mathbb{P}$, define the Σ^0_1 set

$$W = \{(x, v) : p \mathbin{?}\vdash \Phi^G_e(x) \downarrow = v\}$$

- ▶ Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then there is an extension forcing $\Phi^G_e \neq C$

- ▶ Case 2: $(x, C(x)) \notin W$ for some x
  Then there is an extension forcing $\Phi^G_e \neq C$

- ▶ Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Forcing question $?\vdash$	Notion of forcing $(\mathbb{P}, \leq)$
Σ^0_1-preserving	cone avoidance
Σ^0_1-pres. and Σ^0_1-compact	pres. of hyperimmunity
Σ^0_1-pres. and Π^0_1-merging	PA avoidance
Σ^0_1-pres. and ω-Π^0_1-merging	DNC avoidance
Σ^0_1-pres. and (Σ^0_1, Π^0_1)-merging	IΣ^0_1 preservation
$\cdots$	$\cdots$

Higher jump control

$$\left(\begin{array}{c} \text{Post's theorem} \\ \Sigma^0_n \equiv \Sigma^0_1(\emptyset^{(n-1)}) \end{array} \right)$$

Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is Γ-preserving if for every $p \in \mathbb{P}$ and every Γ-formula $\varphi(G, x)$, the relation $p\ ?\vdash \varphi(G, x)$ is in Γ uniformly in x.

Lemma

Suppose $?\vdash$ is Σ_n^0-preserving. For every non-$\emptyset^{(n-1)}$-computable set C and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \Phi_e^{G^{(n-1)}} \neq C\}$$

Given $p \in \mathbb{P}$, define the Σ_n^0 set

$$W = \{(x, v) : p \ ?\vdash \Phi_e^{G^{(n-1)}}(x) \downarrow = v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then there is an extension forcing $\Phi_e^{G^{(n-1)}} \neq C$

- Case 2: $(x, C(x)) \notin W$ for some x
  Then there is an extension forcing $\Phi_e^{G^{(n-1)}} \neq C$

- Case 3: W is a Σ_n^0 graph of C
  Impossible, since $C \not\leq_T \emptyset^{(n-1)}$

Let $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ_n^0-formula for $n \geq 1$.

Cohen forcing

$$\sigma \; ? \vdash \varphi(G)$$

$$\begin{cases} \exists x \, \exists \tau \leq \sigma \; \psi(\tau, x) & \text{for } n = 1 \\ \exists x \, \exists \tau \leq \sigma \; \tau \; ? \nvdash \neg\psi(G, x) & \text{for } n > 1 \end{cases}$$

- Σ_n^0-preserving
- Σ_n^0-compact
- Π_n^0-extremal

Jockusch-Soare forcing

$$T \; ? \vdash \varphi(G)$$

$$\begin{cases} \exists x, \ell \; \forall \sigma \in 2^\ell \cap T \; \psi(\sigma, x) & \text{for } n = 1 \\ \exists x, S \leq T \land S \; ? \nvdash \neg\psi(G, x) & \text{for } n > 1 \end{cases}$$

- Σ_n^0-preserving
- Σ_n^0-compact
- Σ_1^0-extremal
- Π_n^0-extremal for $n \geq 2$

Ramsey theory

An infinite set C is cohesive for a sequence $R_0, R_1, \ldots$ if for every i, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$

COH

Cohesiveness principle
Every sequence of sets admits a cohesive set

Cohesiveness is about
jump computation

Mathias condition

$$\left(F, X\right)$$

↗   ↖

Initial segment   Reservoir

F is finite, X is infinite,
$\max F < \min X$

Mathias extension

$$(E, Y) \leq (F, X)$$

$F \subseteq E, Y \subseteq X, E \setminus F \subseteq X$

Cylinder

$$[F, X] = \{G : F \subseteq G \subseteq F \cup X\}$$

Lemma

Let $R_0, R_1, \ldots$ be computable sets. Every sufficiently generic set G for computable Mathias forcing is $\vec{R}$-cohesive

▶ Given (F, X) and R_n, either $(F, X \cap R_n)$ or $(F, X \cap \overline{R}_n)$ is valid

$$(F, X) \; ?\vdash \varphi(G) \equiv \exists E \subseteq X \; \varphi(F \cup E)$$

Lemma

The forcing question for Σ_1^0-formulas is Σ_1^0-preserving

▶ And Σ_1^0-compact, ω-Π_1^0-merging, (Σ_1^0, Π_1^0)-merging

A function $g : \mathbb{N} \to \mathbb{N}$ dominates $f : \mathbb{N} \to \mathbb{N}$ if $\forall^\infty x \, g(x) \geq f(x)$.

The principal function of an infinite set $X = \{x_0 < x_1 < \dots\}$ is the function $p_X : n \mapsto x_n$.

A Turing degree $\mathbf{d}$ is high if $\mathbf{d}' \geq \mathbf{0}''$.

Theorem (Martin domination)

A degree is high iff it computes a function dominating every computable function

Lemma

If G is sufficiently Mathias generic, then p_G dominates every computable function

- ▶ Let $f : \mathbb{N} \to \mathbb{N}$ be a total computable function and (F, X) be a Mathias condition
- ▶ Let $Y \subseteq X$ be such that $p_{F \cup Y}$ dominates f
- ▶ The extension (F, Y) forces p_G to dominate f

Mathias forcing produces sparse sets
which computes fast-growing functions
even when using computable reservoirs

Solution: restrict reservoirs

Let $R_0, R_1, \ldots$ be an infinite sequence of sets

Given $\sigma \in 2^{<\mathbb{N}}$, let

$$\vec{R}_\sigma = \bigcap_{\sigma(i)=0} \overline{R}_i \bigcap_{\sigma(i)=1} R_i$$

Let $T(\vec{R})$ be the Σ_1^0 tree of all σ such that card $\vec{R}_\sigma > |\sigma|$

(F, σ) denotes $(F, R_\sigma \setminus [0, max(F)])$

(F, σ) denotes a Mathias condition iff σ is extensible in $T(\vec{R})$

Cohesiveness

A condition is a tuple (F, σ, T) such that
- (a) F is a finite set
- (b) T is an infinite, $\emptyset'$-p.r. subtree of $T(\vec{R})$
- (c) $\sigma \in 2^{<\omega}$ is a stem of T

A condition (E, τ, S) extends (F, σ, T) iff
- (i) $F \subseteq E$, $E \setminus F \subseteq R_\sigma \setminus [0, max(F)]$
- (ii) τ suffix of σ
- (iii) $S \subseteq T$

Σ_1^0 case

$$(F, \sigma) \; ?\vdash \varphi(G)$$

$$\equiv$$

$$\exists E \subseteq R_\sigma \setminus [0, \max F] \; \varphi(F \cup E)$$

Lemma

The forcing question for Σ_1^0-formulas is Σ_1^0-preserving

► And Σ_1^0-compact, $\omega\text{-}\Pi_1^0$-merging, (Σ_1^0, Π_1^0)-merging

Σ_2^0 case

$$(F, \sigma, T) \;?\vdash \exists x \varphi(G, x)$$

$$\equiv$$

$$\exists E \subseteq R_\sigma \setminus [0, \max F] \; \exists \ell, x \in \mathbb{N} \; \forall \tau \in 2^\ell \cap T \; (F \cup E, \tau) \;?\nvdash \neg\varphi(G, x)$$

Lemma

The forcing question for Σ_1^0-formulas is Σ_1^0-preserving

- And Σ_2^0-compact, (Σ_2^0, Π_2^0)-merging

Σ_n^0 case, $n \geq 3$

$$(F, \sigma, T) \; ?\vdash \varphi(G)$$

$$\equiv$$

$$\exists (E, \tau, S) \leq (F, \sigma, T) \; \exists x \in \mathbb{N} \; (E, \tau, S) \; ?\nvdash \neg\varphi(G, x)$$
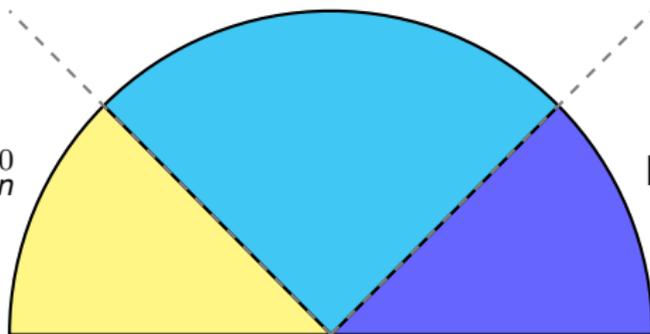
Lemma

The forcing question for Σ_1^0-formulas is Σ_1^0-preserving

▶ And Σ_n^0-compact, ω-Π_n^0-merging, (Σ_n^0, Π_n^0)-merging

Cohesive forcing question for Σ_2^0

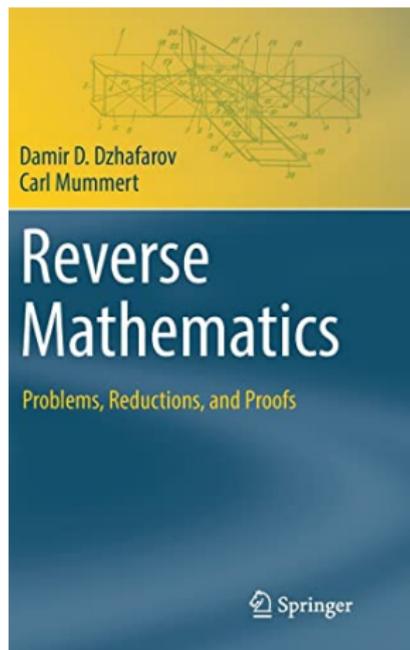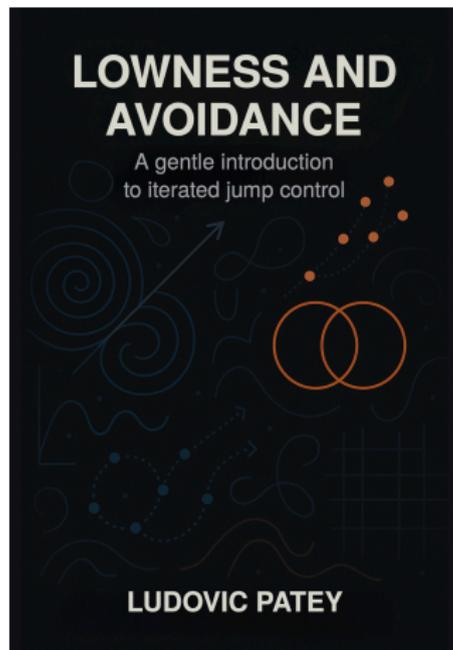Cohesive forcing question for Σ_1^0 and Σ_n^0, $n \geq 3$

Forcing Σ_n^0

Forcing Π_n^0

Conclusion

The computability-theoretic properties
of forcing notions are consequences of
combinatorial and definitional features
of their forcing questions

Reverse Mathematics, 2022

Lowness and avoidance, 2025