Largeness and partial conservation

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Let $\mathcal{P}$ be a decidable property closed under superset

Suppose that every infinite set satisfies $\mathcal{P}$

How large must be a finite set to satisfy $\mathcal{P}$?

Example: the class of sets X such that for every $f : [X]^2 \to 2$, there is an f-homogeneous set $F \subseteq X$ such that $|F| > \min F$.

α-largeness

Generalizes the notion of cardinality

What we want

Let T be a reasonable theory of second-order arithmetic with proof-theoretic ordinal β.

If for some Σ_1^0 property $\mathcal{P}$
$T \vdash$ "Every infinite set satisfies $\mathcal{P}$"

Then there is some $\alpha < \beta$ such that
$T \vdash$ "Every α-large set satisfies $\mathcal{P}$"

Given $\alpha < \epsilon_0$ in short Cantor normal form $\beta + \omega^\gamma$

$$\{\beta + \omega^\gamma\}(x) = \begin{cases} \beta & \text{if } \gamma = 0, \\ \beta + \omega^{\gamma - 1} \cdot x & \text{if } \gamma \text{ is successor }, \\ \beta + \omega^{\{\gamma\}(x)} & \text{if } \gamma \text{ is limit.} \end{cases}$$

$$\{0\}(x) = 0$$

Given a finite set $X = \{x_0 < \cdots < x_s\}$

$$\{\alpha\}(X) = \{\cdots\{\{\alpha\}(x_0)\}(x_1)\cdots\}(x_s)$$

$$X \ \alpha\text{-large iff } \{\alpha\}(X) = 0$$

n-large	$\boxed{n \text{ elements}}$
ω-large	$x_0 \; \boxed{x_0 \text{ elements}}$
$\omega^n \cdot k$-large	$\overbrace{\boxed{\omega^n\text{-large}} \quad \cdots \quad \boxed{\omega^n\text{-large}}}^{k \text{ blocks}}$
ω^{n+1}-large	$x_0 \; \overbrace{\boxed{\omega^n\text{-large}} \quad \cdots \quad \boxed{\omega^n\text{-large}}}^{x_0 \text{ blocks}}$

$$\alpha \gg \beta$$

least exponent of $\alpha \geq$ greatest exponent of β.

$$\omega^5 + \omega^4 \gg \omega^4 + \omega^3 \qquad \omega^4 \cdot 2 \gg \omega^4 \cdot 6$$

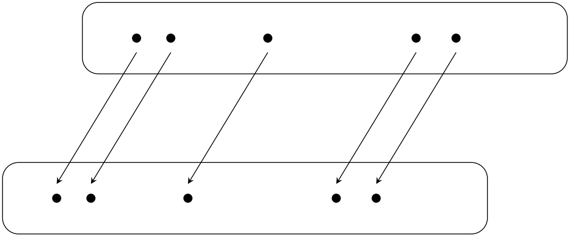
Theorem (Ketonen and Solovay)

Suppose $\alpha, \beta < \epsilon_0$ with $\alpha \gg \beta$. Then

$(\alpha + \beta)$-large iff β-large α-large

Regularity

$A = \{x_0 < x_1 < \cdots < x_n\}$



$B = \{y_0 < y_1 < \cdots < y_n\}$

Theorem (Ketonen and Solovay)

If A is α-large and $y_i \leq x_i$ for every i, then B is α-large.

Partition

X at most α-large

$X \setminus \{\max X\}$ is not α-large

$(\alpha+1)$-large	$\neq$	not at most α-large
$\boxed{\cdot}\ \boxed{\alpha\text{-large}}$		$\boxed{\alpha\text{-large}}\ \boxed{\cdot}$

Theorem (Le Houérou and P.)

Let $\alpha, \beta < \epsilon_0$ and $A, B \subseteq \mathbb{N} \setminus \{0\}$ be at most α-large and β-large, respectively. Then $A \cup B$ is at most $(\alpha \oplus \beta)$-large.

Deconstruction

$$\overbrace{\boxed{\alpha\text{-large}} \quad \cdots \quad \boxed{\alpha\text{-large}}}^{\text{not at most } \beta\text{-large many}}$$

Theorem (Le Houérou and P.)

Let $\alpha, \beta < \epsilon_0$, and let X be $\left(\omega^{\alpha \oplus \beta} + \omega^{\alpha}\right)$-large with $\min X \geq 1$. Then there are ω^{α}-large sets $X_0 < \cdots < X_d \subseteq X$ such that $\{\max X_i : i < d\}$ is ω^{β}-large.

Ketonen, Solovay (1981)

X is always 0-large;

X is $(\alpha+1)$-large iff $X \setminus \{\min X\}$ is α-large;

X is λ-large (λ limit) iff $X \setminus \{\min X\}$ is $\{\lambda\}(\min X)$-large.

Ratajczyk (1988)

X is 0-large* iff $X \neq \emptyset$;

X is $(\alpha+1)$-large* iff $X \setminus \{\min X\}$ is α-large*;

X is λ-large* (λ limit) iff X is $\{\lambda\}(\min X)$-large*.

$$\begin{array}{lcl} (\alpha+1)\text{-large} & \Rightarrow & \alpha\text{-large*} \quad \text{(drop } +1 \text{ if } \alpha \geq \omega) \\ (\alpha+1)\text{-large*} & \Rightarrow & \alpha\text{-large} \end{array}$$

Ramsey's theorem

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

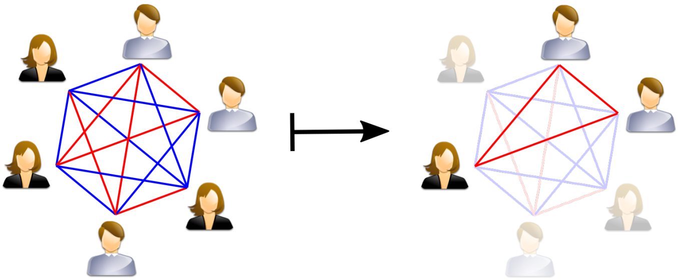
A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



X is RT^n_k-α-large

for every $f : [X]^n \to k$, there is an
α-large homogeneous subset $Y \subseteq X$.

Given α, what is the least β such that
every β-large set is RT^n_k-α-large?

Colorings of pairs, $\alpha < \omega^\omega$

- (Ketonen and Solovay), 1981
  If X is ω^{k+4}-large and $\min X \geq 3$, then X is RT^2_k-ω-large.
- (Bigorajska and Kotlarski), 2002
  If X is $\omega^{\omega^{m} \cdot 2}$-large*, then X is RT^2_2-ω^m-large*.
- (Kotlarski, Piekart and Weiermann), 2007
  If X is RT^2_k-ω^m-large*, then X is ω^{km}-large*.
- (P. and Yokoyama), 2018
  $\forall m \exists k$ such that every ω^k-large set is RT^2_2-ω^m-large.
- (Kołodziejczyk and Yokoyama), 2020
  If X is $\omega^{144(m+1)}$-large and $\min X \geq 3$, then X is RT^2_2-ω^m-large.
- (Le Houérou and P.), 2026
  If X is ω^{km+4}-large and $\min X \geq 2$, then X is RT^2_k-ω^m-large.

Conservation theorems

Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\text{-}\mathsf{CA}_0 \to \mathsf{ATR}_0 \to \mathsf{ACA}_0 \to \mathsf{WKL}_0 \to \mathsf{RCA}_0$$

Mathematics are computationally very structured

$$\begin{tikzcd}
 & \Pi^1_1\text{-}\mathsf{CA}_0 \arrow[d] \\
 & \mathsf{ATR}_0 \arrow[d] \\
 & \mathsf{ACA}_0 \arrow[d] \arrow[dl] \\
\mathsf{RT}^2_2 \arrow[dr] & \mathsf{WKL}_0 \arrow[d] \\
 & \mathsf{RCA}_0
\end{tikzcd}$$

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

The first order-part of a theory T is the set of its theorems in the language of first-order arithmetic.

What is the first-order part of Ramsey's theorem for pairs?

RCA_0

Robinson arithmetics (Q)

$$m + 1 \neq 0$$
$$m + 1 = n + 1 \to m = n$$
$$\neg(m < 0)$$
$$m < n + 1 \leftrightarrow (m < n \vee m = n)$$

$$m + 0 = m$$
$$m + (n + 1) = (m + n) + 1$$
$$m \times 0 = 0$$
$$m \times (n + 1) = (m \times n) + m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \to \forall n \varphi(n)$$

where $\varphi(n)$ is a Σ^0_1 formula

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula where X appears freely, and ψ is a Π^0_1 formula.

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

$$\mathsf{RCA}_0 \equiv \mathsf{Q} + \Delta^0_1\text{-comprehension} + \mathsf{I}\Sigma^0_1$$

Theorem (Hirst)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{B}\Sigma^0_2$.

Theorem (Cholak, Jockusch and Slaman)

$\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

Theorem (Chong, Slaman and Yang)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{I}\Sigma^0_2$.

Is $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

Theorem (Fiori-Carones, Kołodziejczyk, Wong and Yokoyama)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ iff it is $\forall\Pi^0_5$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

$\forall\Pi^0_3$ conservation

(Forcing with ω^a-large sets)

General strategy

How to prove that
T_1 is Γ-conservative over T_0?

- Suppose $T_0 \nvdash \varphi$ with $\varphi \in \Gamma$;
- Take a model $\mathcal{M} \models T_0 \wedge \neg\varphi$;
- Construct another model $\mathcal{N} \models T_1 \wedge \neg\varphi$;
- Thus $T_1 \nvdash \varphi$.

Our strategy

How to prove that

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$-conservative over RCA_0?

Given a first-order structure M and a proper cut I, let

$$\mathrm{Cod}(M/I) = \{F \cap I : F \text{ is } M\text{-coded}\}$$

- Suppose $\mathsf{RCA}_0 \nvdash \varphi$ with φ $\forall\Pi^0_3$;
- Take a model $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 \wedge \neg\varphi$;
- Construct a cut I such that $\mathrm{Cod}(M/I) \models \mathsf{RCA}_0 + \mathsf{RT}^2_2 \wedge \neg\varphi$;
- Thus $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \varphi$.

$\mathsf{WF}(\alpha)$: There is no infinite decreasing sequence of ordinals $< \alpha$

Lemma

$\mathsf{RCA}_0 \vdash \forall \alpha < \epsilon_0$ [$\mathsf{WF}(\alpha) \leftrightarrow$ Every infinite set contains an α-large subset]

Let $\mathcal{M} = (M, S)$ be a countable model of RCA_0.

$$\mathsf{WF}(\omega^{\mathcal{M}}) = \{a \in M : \mathcal{M} \models \mathsf{WF}(\omega^a)\}$$

- $\mathsf{WF}(\omega^{\mathcal{M}})$ is an additive cut
- There is a model $\mathcal{M}$ and some non-standard a such that

$$\mathsf{WF}(\omega^{\mathcal{M}}) = \sup\{a \cdot n : n \in \omega\}$$

Forcing with ω^a-large sets

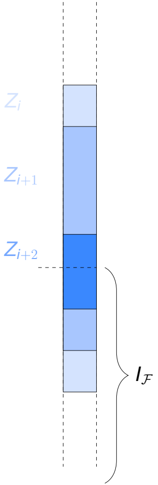
Fix a countable non-standard model $M \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$.

$$(\mathbb{P}, \leq)$$

ω^a-large sets for $a \in M \setminus \omega$ ordered by inclusion.

Every filter $\mathcal{F} \subseteq \mathbb{P}$ induces a cut

$$I_{\mathcal{F}} = \sup\{\min Z : Z \in \mathcal{F}\}$$

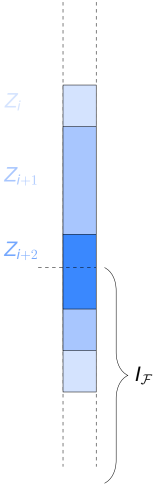


Forcing with ω^a-large sets

Fix a countable non-standard model $M \models \mathsf{Q} + \mathsf{I}\Sigma^0_1$.

- If $Z \in \mathcal{F}$, then $Z \cap I_{\mathcal{F}}$ is unbounded in $I_{\mathcal{F}}$.
- $Z \Vdash (\forall x \in I)\theta(x)$ if $(\forall x < \max Z)\theta(x)$.
- $Z \Vdash (\exists x \in I)\theta(x)$ if $(\exists x < \min Z)\theta(x)$.
- $Z \Vdash (\forall x \in I)(\exists y \in I)\theta(x, y)$ if

  $$(\forall a, b \in Z)[a < b \to (\forall x < a)(\exists y < b)\theta(x, y)]$$



A cut is semi-regular if for every M-coded set $F \subseteq M$ such that $|F| \in I$, $F \cap I$ is bounded in I.

If $M \models \mathrm{PRA}$ and I semi-regular, then $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$.

Lemma (Kirby and Paris)

If $M \models \mathrm{Q} + \mathrm{I}\Sigma^0_1$ and $\mathcal{F}$ is sufficiently generic, then $I_{\mathcal{F}}$ is semi-regular.

- Let $Z \in \mathbb{P}$ be ω^a-large and $F \subseteq M$ be M-coded with $|F| < \min Z$;
- Let $Z_0 < \cdots < Z_{|F|}$ be ω^{a-1}-large subsets of Z ;
- $[\min Z_i, \max Z_i] \cap F = \emptyset$ for some $i \leq |F|$.

Lemma (Kołodziejczyk and Yokoyama)

$\mathsf{Q} + \mathsf{I}\Sigma^0_1$ proves that if X is ω^{300a}-large and $\min X \geq 3$, then for every $f : [X]^2 \to 2$, there is an ω^a-large f-homogeneous set $H \subseteq X$.

Thus if $\mathcal{F}$ is sufficiently generic $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{RT}^2_2$.

Theorem (P. and Yokoyama)

$\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$-conservative over RCA_0.

- Suppose $\mathsf{RCA}_0 \nvdash \forall A \exists x \forall y \psi(A, x, y)$;
- Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \exists A \forall x \exists y \neg\psi(A, x, y)$ be non-standard ;
- Let $A \in S$ and $X = \{b_0 < b_1 < \dots\} \in S$ be such that $(\forall x < b_s)(\exists y < b_{s+1})\neg\psi(A, x, y)$;
- Let $Z \subseteq X$ be ω^a-large for some $a \in M \setminus \omega$;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{WKL}_0 + \mathsf{RT}^2_2 + \exists A \forall x \exists y \neg\psi(A, x, y)$.

$\forall\Pi^0_4$ conservation

(Forcing with ω^a-large(θ) sets)

Theorem (P. and Yokoyama)

$\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$-conservative over RCA_0.

- Suppose $\mathsf{RCA}_0 \nvdash \forall A \exists x \forall y \psi(A, x, y)$;
- Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \exists A \forall x \exists y \neg\psi(A, x, y)$ be non-standard ;
- Let $A \in S$ and $X = \{b_0 < b_1 < \dots\} \in S$ be such that $(\forall x < b_s)(\exists y < b_{s+1})\neg\psi(A, x, y)$;
- Let $Z \subseteq X$ be ω^a-large for some $a \in M \setminus \omega$;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{WKL}_0 + \mathsf{RT}^2_2 + \exists A \forall x \exists y \neg\psi(A, x, y)$.

Theorem (Erroneous Logician)

$\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_4$-conservative over RCA_0.

- Suppose $\mathsf{RCA}_0 \nvdash \forall A \exists x \forall y \exists z \psi(A, x, y, z)$;
- Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \exists A \forall x \exists y \neg\psi(A, x, y)$ be non-standard ;
- Let $A \in S$ and $X = \{b_0 < b_1 < \dots\} \in S$ be such that $(\forall x < b_s)(\exists y < b_{s+1})\forall z \neg\psi(A, x, y, z)$;
- Let $Z \subseteq X$ be ω^a-large for some $a \in M \setminus \omega$;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{WKL}_0 + \mathsf{RT}^2_2 + \exists A \forall x \exists y \forall z \neg\psi(A, x, y, z)$.

$\mathsf{B}\Sigma_2$ is a Π_4-sentence which is not provable over RCA_0.

Let $\theta(x, y, z) \equiv \neg\psi(A, x, y, z)$.

$A < B$ are θ-apart

$$(\forall x < \max A)(\exists y < \min B)(\forall z < \max B)\theta(x, y, z)$$

- X is ω^0-large(θ) if $X \neq \emptyset$.
- X is $\omega^{(n+1)}$-large(θ) if $X \setminus \min X$ is $(\omega^n \cdot \min X)$-large(θ)
- X is $\omega^n \cdot k$-large(θ) if there are k θ-apart ω^n-large(θ) subsets of X

$$X_0 < X_1 < \cdots < X_{k-1}$$

Definition

$\mathsf{WF}_\theta(\omega^a)$: Every infinite set contains an ω^a-large(θ) subset.

Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 + \forall x \exists y \forall z \theta(x, y, z)$.

$$\mathsf{WF}_\theta(\omega^{\mathcal{M}}) = \{a \in M : \mathcal{M} \models \mathsf{WF}_\theta(\omega^a)\}$$

- $\mathsf{WF}_\theta(\omega^{\mathcal{M}})$ is an additive cut
- There is a model $\mathcal{M}$ and some non-standard a such that

$$\mathsf{WF}_\theta(\omega^{\mathcal{M}}) = \sup\{a \cdot n : n \in \omega\}$$

Forcing with ω^a-large(θ) sets

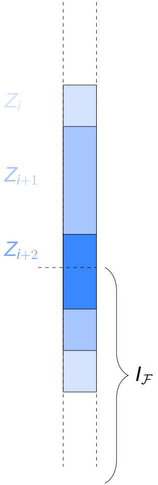
Fix a countable non-standard model
$M \models \mathsf{Q} + \mathsf{I}\Delta_2^0 + \forall x \exists y \forall z \theta(x,y,z)$.

$$(\mathbb{P}, \leq)$$

ω^a-large(θ) sets for $a \in M \setminus \omega$
ordered by inclusion.

Every filter $\mathcal{F} \subseteq \mathbb{P}$ induces a cut

$$I_{\mathcal{F}} = \sup\{\min Z : Z \in \mathcal{F}\}$$



Suppose $M \models \mathrm{Q} + \mathrm{I}\Sigma_1^0 + \forall x \exists y \forall z \theta(x, y, z)$.

Lemma

If $\mathcal{F}$ is sufficiently generic, then $I_{\mathcal{F}}$ is semi-regular.

- Let $Z \in \mathbb{P}$ be ω^a-large(θ) and $F \subseteq M$ be M-coded with $|F| < \min Z$;
- Let $Z_0 < \cdots < Z_{|F|}$ be ω^{a-1}-large(θ) subsets of Z ;
- $[\min Z_i, \max Z_i] \cap F = \emptyset$ for some $i \leq |F|$.

Lemma

If $\mathcal{F}$ is sufficiently generic, then $I_{\mathcal{F}} \models \forall x \exists y \forall z \theta(x, y, z)$.

- Let $Z \in \mathbb{P}$ be ω^a-large(θ) and $x < \min Z$;
- Let $Z_0 < Z_1$ be θ-apart ω^{a-1}-large(θ) subsets of Z ;
- Z_1 forces $\exists y \forall z \theta(x, y, z)$.

Lemma (Le Houérou, P. and Yokoyama)

$\mathsf{Q} + \mathsf{I}\Sigma^0_1$ proves that if X is $\omega^{{a+1}^{16}}$-large(θ) and $\min X \geq 3$, then for every $f : [X]^2 \to 2$, there is an ω^a-large(θ) f-homogeneous set $H \subseteq X$.

Lemma (Le Houérou and P.)

$\mathsf{Q} + \mathsf{I}\Sigma^0_1$ proves that if X is $\omega^{12a(a+1)+\mathcal{O}(1)}$-large$(\theta)$, then for every $f : [X]^2 \to 2$, there is an ω^a-large(θ) f-homogeneous set $H \subseteq X$.

Thus if $\mathcal{F}$ is sufficiently generic $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{RT}^2_2$.

Theorem (Le Houérou, P. and Yokoyama)

$\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_4$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

- Suppose $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \nvdash \forall A \exists x \forall y \exists z \psi(A, x, y, z)$;
- Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0 + \exists A \forall x \exists y \forall z \neg\psi(A, x, y, z)$ be non-standard ;
- Let $A \in S$ be a witness and $\theta(x, y, z) \equiv \neg\psi(A, x, y, z)$;
- Let $Z \subseteq M$ be ω^a-large(θ) for some $a \in M \setminus \omega$;
- Let $\mathcal{F}$ be sufficiently generic filter containing Z ;
- $(I_{\mathcal{F}}, \mathrm{Cod}(M/I_{\mathcal{F}})) \models \mathsf{WKL}_0 + \mathsf{RT}^2_2 + \exists A \forall x \exists y \forall z \neg\psi(A, x, y, z)$.

Proof size

Theorem (Fiori-Carones, Kołodziejczyk, Wong and Yokoyama)

If there is a speed-up for Π^1_1-sentences between $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, then $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is not Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Theorem (Kołodziejczyk, Wong and Yokoyama)

RCA_0 polynomially simulates $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ for $\forall\Pi^0_3$-formulas.

Theorem (Le Houérou and P.)

$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ polynomially simulates $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ for $\forall\Pi^0_4$-formulas.

Open question

$$\text{Is } \mathsf{WKL}_0 + \mathsf{RT}^2_2$$

Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$

or equivalently

$\forall\Pi^0_5$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

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